

Mathematical Methods Units 1 & 2

Chapter 5 — Functions, relations and graphs

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Chapter 5 Functions, relations and graphs — 5A Sets and set notation

Theory

A **set** is a collection of objects, called **elements**. If x is an element of a set A we write $x \in A$; if not, $x \notin A$.

Sets of numbers. Four standard sets are used throughout this course:

- $\mathbb{N}$ — the **natural numbers** $\{1, 2, 3, \dots\}$;
- $\mathbb{Z}$ — the **integers** $\{\dots, -2, -1, 0, 1, 2, \dots\}$;
- $\mathbb{Q}$ — the **rational numbers** (those that can be written as a fraction $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$);
- $\mathbb{R}$ — the **real numbers** (all numbers on the number line).

Each is a **subset** of the next: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. We write $A \subseteq B$ (“ A is a subset of B ”) when every element of A is also an element of B . A number that is real but not rational (such as π or $\sqrt{2}$) is called **irrational**.

Set-builder notation. A set can be described by a defining property, using a **colon** (the Australian convention):

$\{x : x > 3\}$ means *the set of all x such that $x > 3$* .

Interval notation describes a continuous piece of the number line. A **square** bracket includes the endpoint; a **round** bracket excludes it. Infinity is always excluded.

Interval	Inequality	Set-builder
$[a, b]$	$a \leq x \leq b$	$\{x : a \leq x \leq b\}$
(a, b)	$a < x < b$	$\{x : a < x < b\}$
$[a, b)$	$a \leq x < b$	$\{x : a \leq x < b\}$
$(-\infty, b]$	$x \leq b$	$\{x : x \leq b\}$

Combining sets.

- **Union** $A \cup B$: elements in A **or** B (or both).
- **Intersection** $A \cap B$: elements in **both** A and B .
- **Set difference** $A \setminus B$: elements in A but **not** in B (use a backslash, not a forward slash).
- The **empty set** (or $\{\}$) has no elements; e.g. $\mathbb{R} \setminus \mathbb{R} = \{\}$.

Worked examples

Example 1 — scaffolded

Classify each of $\sqrt{9}$, -3 , $2/5$, π by listing which of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ it belongs to.

Working	Comment
$\sqrt{9} = 3$, so $\sqrt{9} \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.	Simplify first — $\sqrt{9}$ is a whole number.
$-3 \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ (but $-3 \notin \mathbb{N}$).	The natural numbers are positive, so -3 is not one.
$2/5 \in \mathbb{Q}, \mathbb{R}$.	A fraction of integers is rational but not an integer.
$\pi \in \mathbb{R}$ only.	π is irrational — it cannot be written as a fraction.

Example 2 — scaffolded

Write $\{x : -2 \leq x < 5\}$ in interval notation.

Working	Comment
The values run from -2 up to 5 .	Identify the endpoints.
-2 is included ($\leq$) $\rightarrow$ square bracket; 5 is excluded ($<$) $\rightarrow$ round bracket.	Match each bracket to its inequality.
$\{x: -2 \leq x < 5\} = [-2, 5)$.	Write the interval.

Example 3 — unscaffolded

Let $A = [-3, 4)$ and $B = (1, 6]$. Find $A \cup B$, $A \cap B$ and $A \setminus B$.

$A \cup B$ covers everything in either interval: from -3 (included) to 6 (included), so $A \cup B = [-3, 6]$.

$A \cap B$ is where they overlap: between 1 (excluded, since B is open there) and 4 (excluded, since A is open there), so $A \cap B = (1, 4)$.

$A \setminus B$ is the part of A not in B : from -3 (included) up to 1 (included, since $1 \notin B$), so $A \setminus B = [-3, 1]$.

$\therefore A \cup B = [-3, 6]$, $A \cap B = (1, 4)$, $A \setminus B = [-3, 1]$.

Example 4 — unscaffolded

Express $(-\infty, 2] \cup (5, \infty)$ in set-builder notation.

The first interval is $\{x: x \leq 2\}$ and the second is $\{x: x > 5\}$.

$\therefore (-\infty, 2] \cup (5, \infty) = \{x: x \leq 2\} \cup \{x: x > 5\} = \{x: x \leq 2 \text{ or } x > 5\}$.

Practice questions

Scaffolded

Q1. Classify each of $\sqrt{16}$, -7 , 0.25 , $\sqrt{2}$ by listing which of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ it belongs to.

Q2. Write the inequality $-1 < x \leq 6$ in interval notation.

Q3. Write the interval $[-4, 2)$ in set-builder notation.

Q4. Find $[-1, 5] \cup [2, 8]$.

Q5. Find $(0, 6) \cap [3, 9]$.

Q6. Find $[-4, 4] \setminus [0, 2]$.

Unscaffolded

Q7. Classify each of $\frac{22}{7}$, $\sqrt{25}$, $-\sqrt{3}$, 0 into the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

Q8. True or false (give a reason): every integer is a rational number.

Q9. Write $-3 \leq x < 7$ in interval notation.

Q10. Write $(-\infty, 4]$ in set-builder notation and as an inequality.

Q11. Write $\{x: x \geq -2\}$ in interval notation.

Q12. Write $[1, 9)$ as an inequality and in set-builder notation.

Q13. For $A = [-2, 3]$ and $B = [1, 5]$, find $A \cup B$ and $A \cap B$.

Q14. For $A = (-\infty, 2)$ and $B = [0, \infty)$, find $A \cup B$ and $A \cap B$.

Q15. Find $(-1, 4] \cap [2, 7)$.

Q16. Find $[0, 10] \setminus (3, 6)$.

Q17. True or false (give a reason): $\mathbb{Q} \subseteq \mathbb{Z}$.

Q18. Describe the set of real numbers between -5 (excluded) and 3 (included) in three ways: as an inequality, in interval notation, and in set-builder notation.

Answers

Q1. $\sqrt{16}=4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$; $-7 \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$; $0.25 \in \mathbb{Q}, \mathbb{R}$; $\sqrt{2} \in \mathbb{R}$ only. **Q2.** $(-1, 6]$. **Q3.** $\{x: -4 \leq x < 2\}$. **Q4.** $[-1, 8]$. **Q5.** $[3, 6)$. **Q6.** $[-4, 0) \cup (2, 4]$. **Q7.** $22/7 \in \mathbb{Q}, \mathbb{R}$; $\sqrt{25}=5 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$; $-\sqrt{3} \in \mathbb{R}$ only; $0 \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. **Q8.** True. **Q9.** $[-3, 7)$. **Q10.** $\{x: x \leq 4\}$; $x \leq 4$. **Q11.** $[-2, \infty)$. **Q12.** $1 \leq x < 9$; $\{x: 1 \leq x < 9\}$. **Q13.** $A \cup B = [-2, 5]$, $A \cap B = [1, 3]$. **Q14.** $A \cup B = \mathbb{R}$, $A \cap B = [0, 2)$. **Q15.** $[2, 4]$. **Q16.** $[0, 3] \cup [6, 10]$. **Q17.** False. **Q18.** $-5 < x \leq 3$; $(-5, 3]$; $\{x: -5 < x \leq 3\}$.

Fully-worked solutions

Q1. $\sqrt{16}=4$, a natural number, so it is in all four sets. -7 is a negative integer, so $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ (not $\mathbb{N}$). $0.25 = 1/4$ is rational, so $\mathbb{Q}, \mathbb{R}$. $\sqrt{2}$ is irrational, so $\mathbb{R}$ only.

Q2. -1 is excluded (round), 6 is included (square): $(-1, 6]$.

Q3. $[-4, 2)$ means $-4 \leq x < 2$, i.e. $\{x: -4 \leq x < 2\}$.

Q4. $[-1, 5] \cup [2, 8]$: the intervals overlap, so the union runs from -1 to 8 : $[-1, 8]$.

Q5. $(0, 6) \cap [3, 9]$: the overlap runs from 3 (included, from $[3, 9]$) to 6 (excluded, from $(0, 6)$): $[3, 6)$.

Q6. $[-4, 4] \setminus [0, 2]$: remove $[0, 2]$ from $[-4, 4]$, leaving $[-4, 0)$ (up to but not including 0) and $(2, 4]$ (after but not including 2): $[-4, 0) \cup (2, 4]$.

Q7. $22/7$ is a fraction of integers $\rightarrow$ rational $\rightarrow \mathbb{Q}, \mathbb{R}$. $\sqrt{25}=5 \rightarrow$ natural $\rightarrow$ all four sets. $-\sqrt{3}$ is irrational $\rightarrow \mathbb{R}$ only. 0 is an integer (but not natural) $\rightarrow \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Q8. True. Any integer n can be written as $\frac{n}{1}$, a ratio of integers, so it satisfies the definition of a rational number. Hence $\mathbb{Z} \subseteq \mathbb{Q}$.

Q9. -3 included, 7 excluded: $[-3, 7)$.

Q10. $(-\infty, 4]$ includes all numbers up to and including 4 : $\{x: x \leq 4\}$, i.e. $x \leq 4$.

Q11. $\{x: x \geq -2\}$ starts at -2 (included) and continues without bound: $[-2, \infty)$.

Q12. $[1, 9)$: 1 included, 9 excluded, so $1 \leq x < 9$, i.e. $\{x: 1 \leq x < 9\}$.

Q13. $A = [-2, 3]$, $B = [1, 5]$. Union: from -2 to 5 , $A \cup B = [-2, 5]$. Intersection (overlap): from 1 to 3 , $A \cap B = [1, 3]$.

Q14. $A = (-\infty, 2)$, $B = [0, \infty)$. Together they cover the whole line: $A \cup B = \mathbb{R}$. The overlap is from 0 (included, from B) to 2 (excluded, from A): $A \cap B = [0, 2)$.

Q15. $(-1, 4] \cap [2, 7)$: the overlap runs from 2 (included) to 4 (included): $[2, 4]$.

Q16. $[0, 10] \setminus (3, 6)$: remove the open interval $(3, 6)$. Since 3 and 6 are not in $(3, 6)$, they remain: $[0, 3] \cup [6, 10]$.

Q17. False. $\mathbb{Q} \subseteq \mathbb{Z}$ would mean every rational is an integer, but e.g. $1/2 \in \mathbb{Q}$ and $1/2 \notin \mathbb{Z}$. (The correct inclusion is $\mathbb{Z} \subseteq \mathbb{Q}$.)

Q18. -5 excluded, 3 included. Inequality: $-5 < x \leq 3$. Interval: $(-5, 3]$. Set-builder: $\{x: -5 < x \leq 3\}$.

Chapter 5 Functions, relations and graphs — 5B Domain and range of relations

Theory

A **relation** is a set of ordered pairs (x, y) . It may be described by a list of ordered pairs, by a rule (with a stated domain), or by a graph.

- The **domain** of a relation is the set of all **first** coordinates (x -values).
- The **range** of a relation is the set of all **second** coordinates (y -values).

Discrete or continuous. A relation may be **discrete** — a finite set of separate points — or **continuous** — an unbroken line or curve. For a discrete relation the domain and range are written as sets, e.g. $\{-1, 1, 3\}$; a value is listed once even if it is repeated in the pairs.

Reading from a graph. The domain is the extent of the graph along the x -axis; the range is its extent along the y -axis. Watch the endpoints: a **closed dot** $\bullet$ means the point is **included**, an **open dot** $\circ$ means it is **excluded**. Closed endpoints give a square bracket $[]$; open endpoints give a round bracket $()$.

Notation. Use set notation with a colon, $\{x : -2 \leq x < 3\}$, or interval notation, $[-2, 3)$. For graphs with a break (such as a hyperbola with asymptote $x = h$) use the set $\mathbb{R} \setminus \{h\}$ ("all real numbers except h ").

Worked examples

Example 1 — scaffolded

State the domain and range of the relation $\{(1, 2), (3, 4), (-1, 5), (3, 7)\}$.

Working

First coordinates: $1, 3, -1, 3$.

Domain = $\{-1, 1, 3\}$.

Second coordinates: $2, 4, 5, 7$.

Range = $\{2, 4, 5, 7\}$.

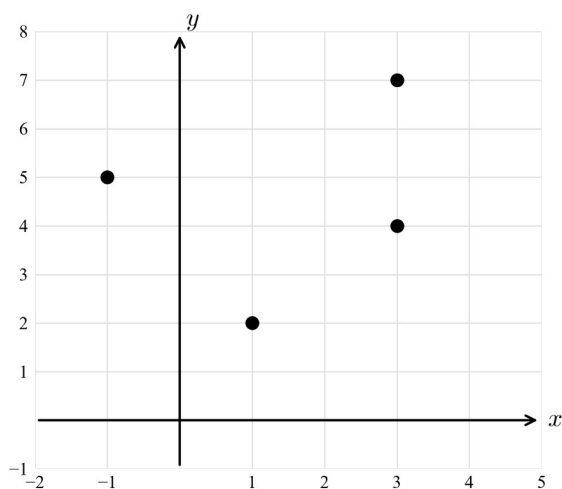
Comment

The domain is the set of x -values.

List each value once, in order; 3 appears twice but is written once.

The range is the set of y -values.

This is a **discrete** relation (separate points).



Example 2 — scaffolded

State the domain and range of the relation graphed below (a line segment from $(-2, -1)$, included, to $(3, 4)$, excluded).

Working

The x -values run from -2 (closed) up to 3 (open).

Domain $= [-2, 3)$.

The y -values run from -1 (closed) up to 4 (open).

Range $= [-1, 4)$.

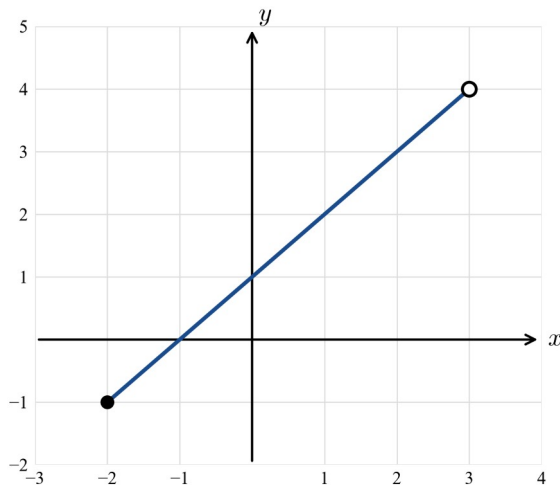
Comment

Read the horizontal extent.

Closed at $-2 \rightarrow [$; open at $3 \rightarrow)$.

Read the vertical extent.

The excluded endpoint $(3, 4)$ removes $y = 4$.



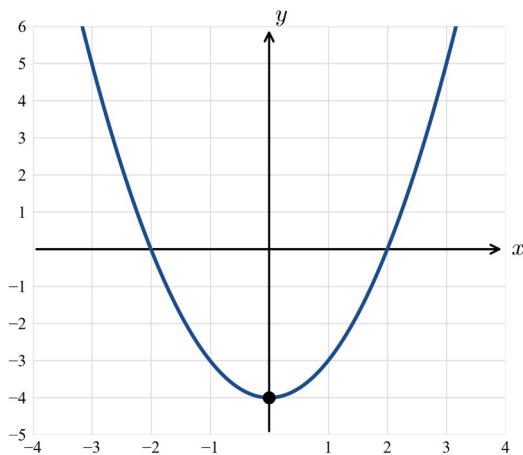
Example 3 — unscaffolded

State the domain and range of $y = x^2 - 4$.

The parabola is defined for every real x , so the domain is $\mathbb{R}$.

Its minimum is the turning point $(0, -4)$, and it opens upwards, so $y \geq -4$.

$\therefore$ domain $= \mathbb{R}$, range $= [-4, \infty)$.

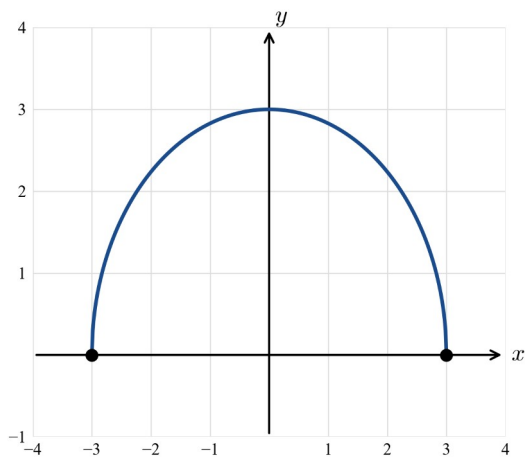


Example 4 — unscaffolded

State the domain and range of $y = \sqrt{9 - x^2}$.

This is the **upper** semicircle of $x^2 + y^2 = 9$ (radius 3). It is defined when $9 - x^2 \geq 0$, i.e. $-3 \leq x \leq 3$, and the y -values run from 0 up to 3.

$\therefore$ domain $= [-3, 3]$, range $= [0, 3]$.



Practice questions

Scaffolded

Q1. For $\{(0, 1), (2, 3), (4, 5), (2, 6)\}$: (a) state the domain; (b) state the range; (c) is the relation discrete or continuous?

Q2. The graph is the segment from $(-1, 2)$ (included) to $(3, -2)$ (excluded). (a) State the domain. (b) State the range.

Q3. For $y = x^2 - 1$: (a) state the domain; (b) find the turning point; (c) hence state the range.

Q4. For $y = \frac{1}{x}$: (a) which value of x is not allowed? (b) state the domain; (c) state the range.

Q5. For $y = \sqrt{x}$: (a) state the domain; (b) state the range.

Q6. For the upper semicircle $y = \sqrt{4 - x^2}$: (a) state the domain; (b) state the range.

Un scaffolded

State the domain and range of each relation.

Q7. $\{(2, 1), (4, 3), (6, 5), (2, 7)\}$

Q8. $\{(-3, 0), (-1, 0), (1, 2), (3, 4)\}$

Q9. The segment from $(-2, 3)$ (included) to $(4, -1)$ (excluded).

Q10. The segment from $(1, -2)$ (excluded) to $(5, 2)$ (included).

Q11. $y = x^2 + 1$

Q12. $y = -(x - 1)^2 + 4$

Q13. $y = \frac{1}{x - 2}$

Q14. $y = \sqrt{x + 1}$

Q15. $y = -\sqrt{16 - x^2}$

Q16. $y = 2x - 1$, with domain $[0, 3]$.

Q17. Let $A = \{(1, 2), (2, 4), (3, 6)\}$ and $B = \{(2, 1), (4, 2), (6, 3)\}$. State the domain and range of each, and decide whether $A = B$.

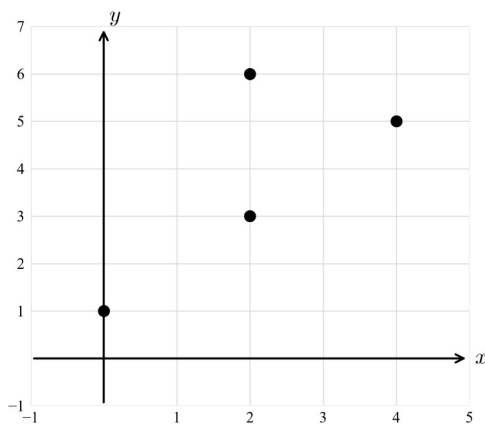
Q18. $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$

Answers

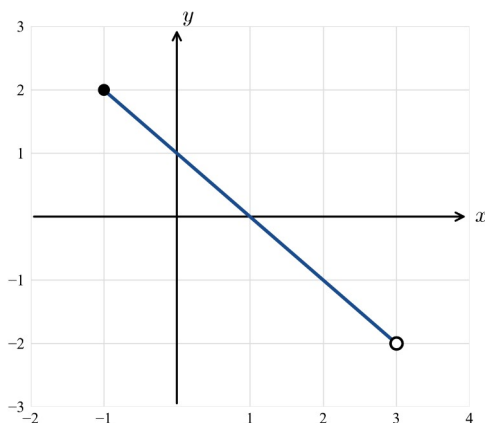
Q1. Domain $\{0, 2, 4\}$; range $\{1, 3, 5, 6\}$; discrete. **Q2.** Domain $[-1, 3)$; range $(-2, 2]$. **Q3.** Domain $\mathbb{R}$; TP $(0, -1)$; range $[-1, \infty)$. **Q4.** $x=0$; domain $\mathbb{R} \setminus \{0\}$; range $\mathbb{R} \setminus \{0\}$. **Q5.** Domain $[0, \infty)$; range $[0, \infty)$. **Q6.** Domain $[-2, 2]$; range $[0, 2]$. **Q7.** Domain $\{2, 4, 6\}$; range $\{1, 3, 5, 7\}$. **Q8.** Domain $\{-3, -1, 1, 3\}$; range $\{0, 2, 4\}$. **Q9.** Domain $[-2, 4]$; range $(-1, 3]$. **Q10.** Domain $(1, 5]$; range $(-2, 2]$. **Q11.** Domain $\mathbb{R}$; range $[1, \infty)$. **Q12.** Domain $\mathbb{R}$; range $(-\infty, 4]$. **Q13.** Domain $\mathbb{R} \setminus \{2\}$; range $\mathbb{R} \setminus \{0\}$. **Q14.** Domain $[-1, \infty)$; range $[0, \infty)$. **Q15.** Domain $[-4, 4]$; range $[-4, 0]$. **Q16.** Domain $[0, 3]$; range $[-1, 5]$. **Q17.** A: domain $\{1, 2, 3\}$, range $\{2, 4, 6\}$; B: domain $\{2, 4, 6\}$, range $\{1, 2, 3\}$; $A \neq B$ (the ordered pairs differ — B swaps the coordinates of A). **Q18.** Domain $\{1, 2, 3, 4\}$; range $\{1, 4, 9, 16\}$.

Fully-worked solutions

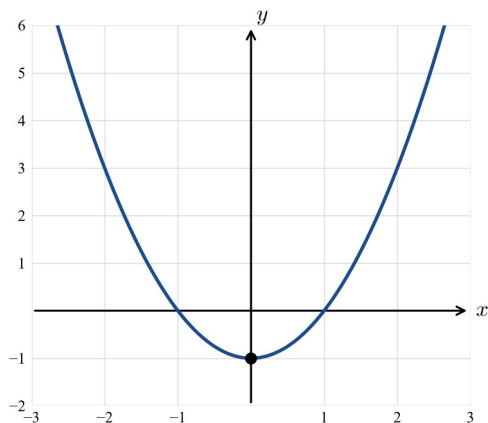
Q1. First coordinates $0, 2, 4, 2 \rightarrow$ domain $\{0, 2, 4\}$. Second coordinates $1, 3, 5, 6 \rightarrow$ range $\{1, 3, 5, 6\}$. The points are separate, so the relation is **discrete**.



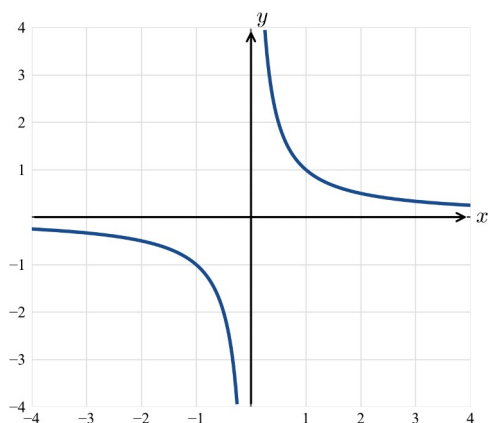
Q2. The segment runs from $x = -1$ (closed) to $x = 3$ (open): domain $[-1, 3)$. The y -values run from 2 (closed) down to -2 (open): range $(-2, 2]$.



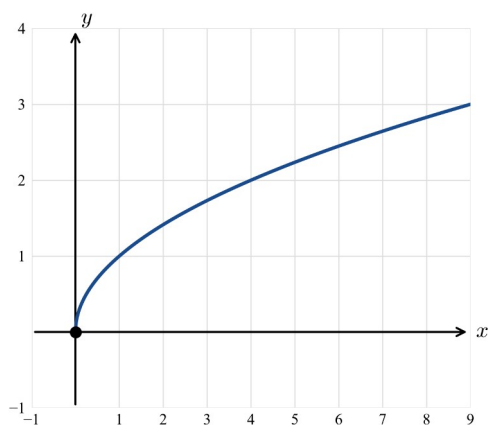
Q3. (a) A parabola is defined for all real x : domain $\mathbb{R}$. (b) Turning point $(0, -1)$. (c) It opens upwards from $y = -1$: range $[-1, \infty)$.



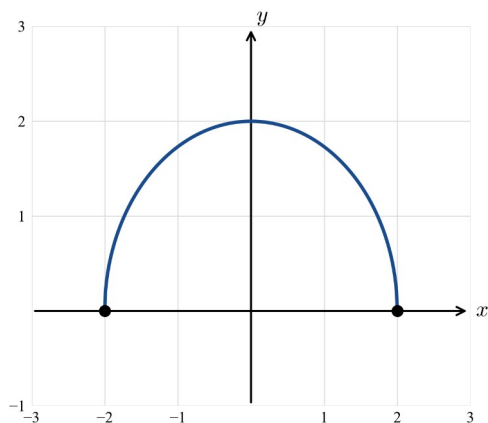
Q4. (a) $x=0$ makes the denominator zero. (b) Domain $\mathbb{R} \setminus \{0\}$. (c) y can be any value except 0 (the horizontal asymptote): range $\mathbb{R} \setminus \{0\}$.



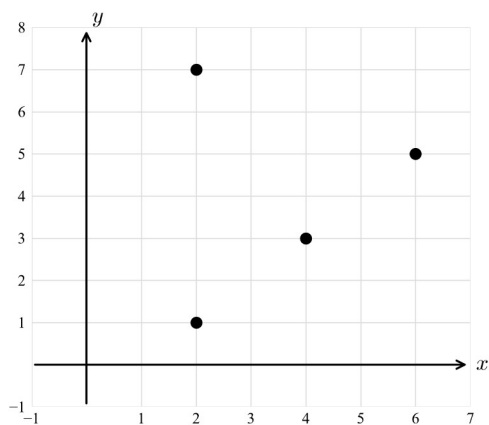
Q5. (a) $\sqrt{x}$ requires $x \geq 0$: domain $[0, \infty)$. (b) The output is never negative: range $[0, \infty)$.



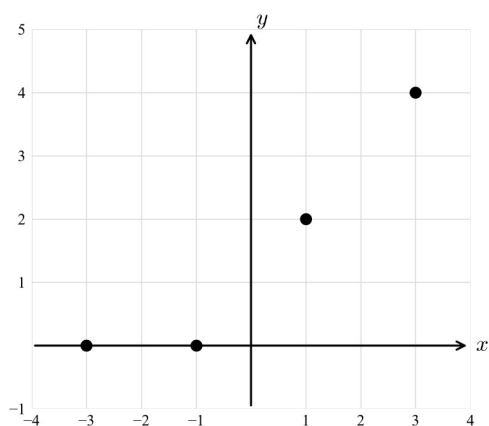
Q6. (a) Defined when $4 - x^2 \geq 0$, i.e. $-2 \leq x \leq 2$: domain $[-2, 2]$. (b) The upper semicircle of radius 2: range $[0, 2]$.



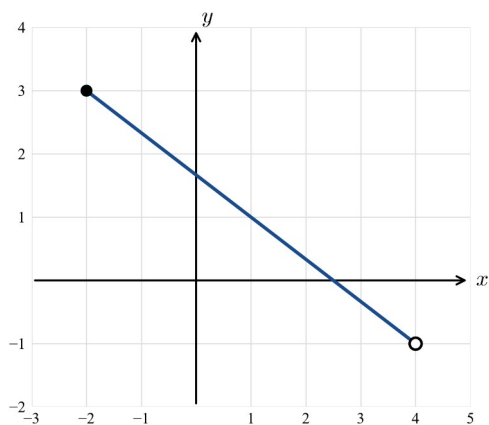
Q7. Domain $\{2, 4, 6\}$ (the value 2 is listed once); range $\{1, 3, 5, 7\}$.



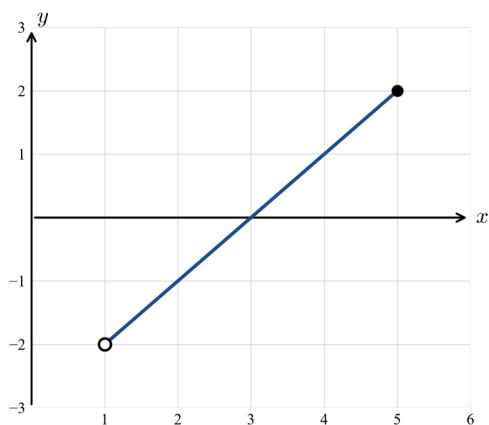
Q8. Domain $\{-3, -1, 1, 3\}$; range $\{0, 2, 4\}$ ($y=0$ is listed once).



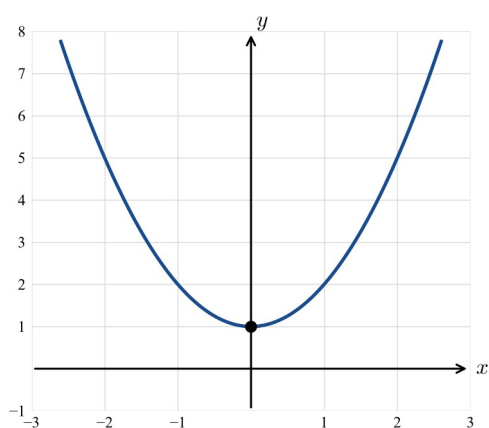
Q9. x runs from -2 (closed) to 4 (open): domain $[-2, 4)$. y runs from 3 (closed, at $x = -2$) down to -1 (open, at $x = 4$): range $(-1, 3]$.



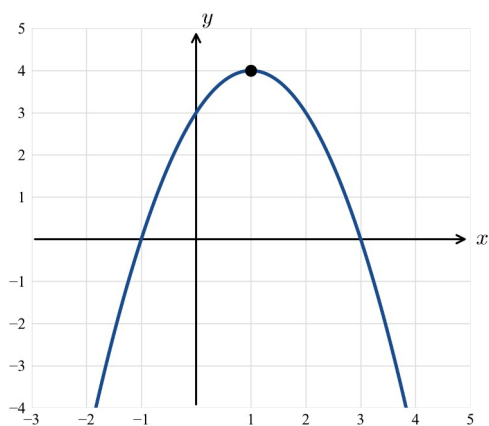
Q10. x runs from 1 (open) to 5 (closed): domain $(1, 5]$. y runs from -2 (open) to 2 (closed): range $(-2, 2]$.



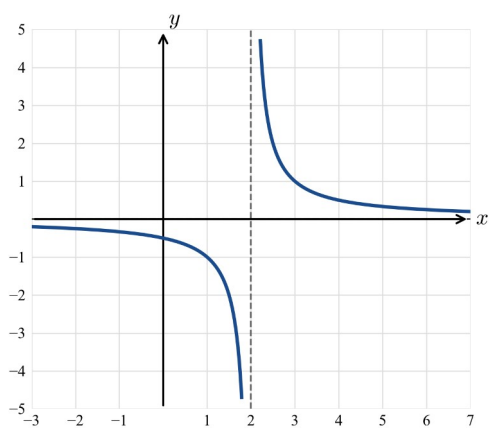
Q11. Domain $\mathbb{R}$; minimum at $(0, 1)$, opening upwards: range $[1, \infty)$.



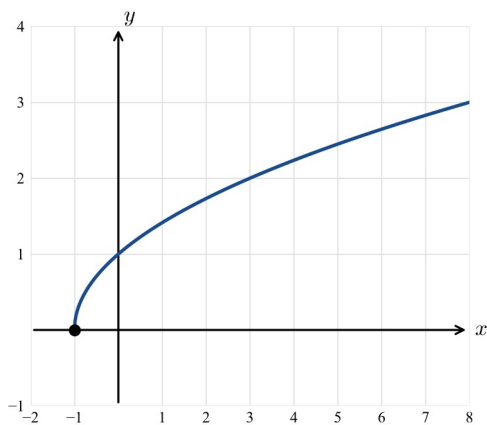
Q12. Domain $\mathbb{R}$; maximum turning point $(1, 4)$, opening downwards: range $(-\infty, 4]$.



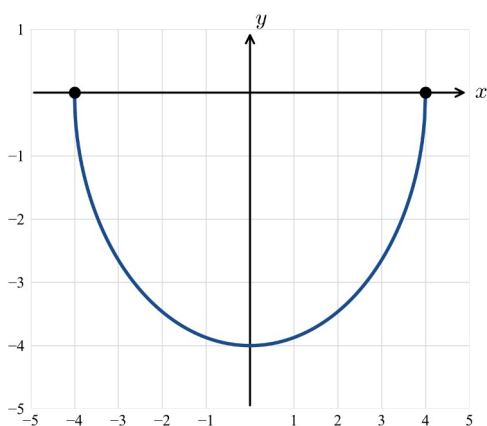
Q13. Undefined at $x=2$: domain $\mathbb{R} \setminus \{2\}$. Horizontal asymptote $y=0$, never attained: range $\mathbb{R} \setminus \{0\}$.



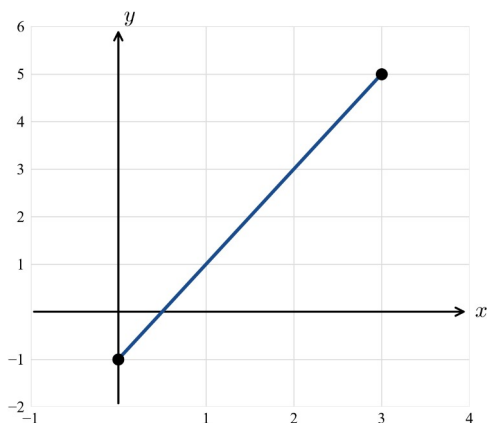
Q14. $\sqrt{x+1}$ requires $x+1 \geq 0$, i.e. $x \geq -1$: domain $[-1, \infty)$; output never negative: range $[0, \infty)$.



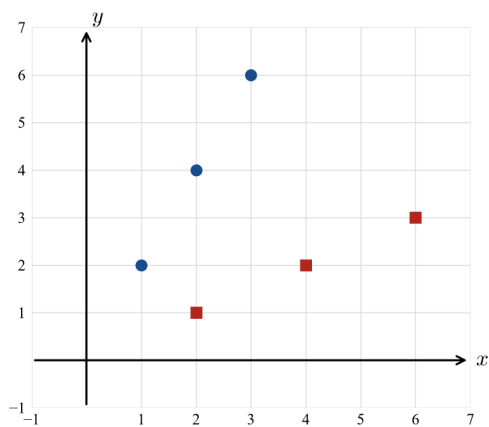
Q15. The **lower** semicircle of $x^2 + y^2 = 16$: defined for $-4 \leq x \leq 4$, domain $[-4, 4]$; y runs from -4 up to 0 : range $[-4, 0]$.



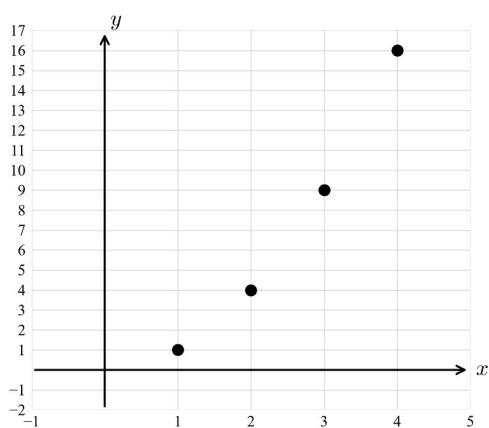
Q16. With domain $[0, 3]$, the endpoints give $y(0) = -1$ and $y(3) = 5$; as the line is increasing, range $[-1, 5]$.



Q17. A : domain $\{1, 2, 3\}$, range $\{2, 4, 6\}$. B : domain $\{2, 4, 6\}$, range $\{1, 2, 3\}$. The relations are **not equal**: B reverses each ordered pair of A (e.g. $(1, 2) \in A$ but $(1, 2) \notin B$).



Q18. Domain $\{1, 2, 3, 4\}$; range $\{1, 4, 9, 16\}$ (the squares).



Chapter 5 Functions, relations and graphs — 5C Functions

Theory

A **relation** is any set of ordered pairs (x, y) . A **function** is a special relation in which **each x -value is paired with exactly one y -value**.

The vertical line test. A graph represents a function if and only if **no vertical line meets the graph more than once**. If some vertical line cuts the graph twice (or more), the relation is **not** a function — that x -value would have two y -values.

Function notation. We write $f(x)$ for the rule of a function f applied to x . For example, if $f(x) = 2x + 3$ then $f(5) = 2(5) + 3 = 13$. To **evaluate** $f(a)$, substitute $x = a$.

Formal notation. A function is fully described by its rule **and** its domain:

$$f : D \rightarrow \mathbb{R}, f(x) = \dots$$

where D is the domain (the set of allowed x -values). The set after the arrow — here $\mathbb{R}$ — is the **co-domain**: the declared target set from which the outputs are drawn (often $\mathbb{R}$), which need not equal the **range** (the actual set of outputs), so in general $\text{range} \subseteq \text{co-domain}$. If no domain is stated, take the **maximal (implied) domain** — all x for which the rule is defined.

Solving $f(x) = k$. Set the rule equal to k and solve the resulting equation for x .

Range over a restricted domain. The **range** is the set of y -values produced. Over a restricted domain, check the endpoints and any turning point of the rule.

Worked examples

Example 1 — scaffolded

For $f(x) = 3x - 2$, find (a) $f(4)$, (b) $f(-1)$, and (c) the value of x for which $f(x) = 13$.

Working

Comment

(a) $f(4) = 3(4) - 2 = 12 - 2 = 10$.

Substitute $x = 4$ into the rule.

(b) $f(-1) = 3(-1) - 2 = -3 - 2 = -5$.

Substitute $x = -1$; take care with signs.

(c) $3x - 2 = 13 \Rightarrow 3x = 15 \Rightarrow x = 5$.

Set the rule equal to 13 and solve for x .

Example 2 — scaffolded

Use the vertical line test to decide whether (a) $y = x^2$ and (b) $x = y^2$ are functions.

Working

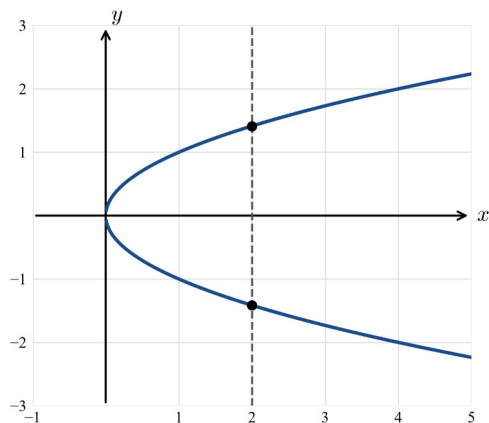
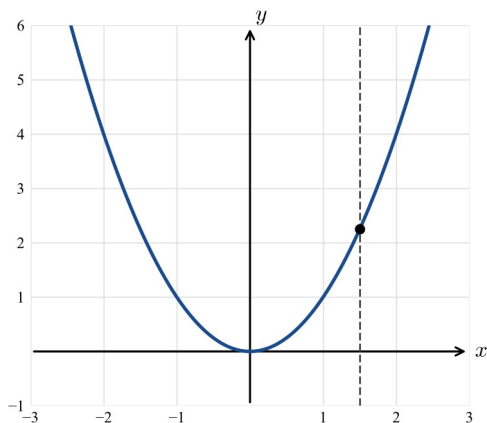
Comment

(a) Every vertical line meets $y = x^2$ **once**, so it is a **function**.

See the graph below — the dashed test line cuts it at one point.

(b) A vertical line such as $x = 2$ meets $x = y^2$ at $(2, \sqrt{2})$ and $(2, -\sqrt{2})$ — **twice** — so it is **not a function**.

One x -value gives two y -values.



Example 3 — unscaffolded

For $g(x) = x^2 - 4x$, find $g(3)$ and $g(-2)$, and solve $g(x) = 0$.

$$g(3) = 3^2 - 4(3) = 9 - 12 = -3. \quad g(-2) = (-2)^2 - 4(-2) = 4 + 8 = 12.$$

$$g(x) = 0 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x - 4) = 0, \text{ so } x = 0 \text{ or } x = 4.$$

$$\therefore g(3) = -3, \quad g(-2) = 12, \text{ and } g(x) = 0 \text{ when } x = 0 \text{ or } x = 4.$$

Example 4 — unscaffolded

Find the range of $f: [0, 4] \rightarrow \mathbb{R}, f(x) = 2x - 1$.

f is linear with positive gradient, so it is increasing: the least and greatest values occur at the endpoints. $f(0) = -1$ and $f(4) = 2(4) - 1 = 7$.

$$\therefore \text{the range is } [-1, 7].$$

Practice questions

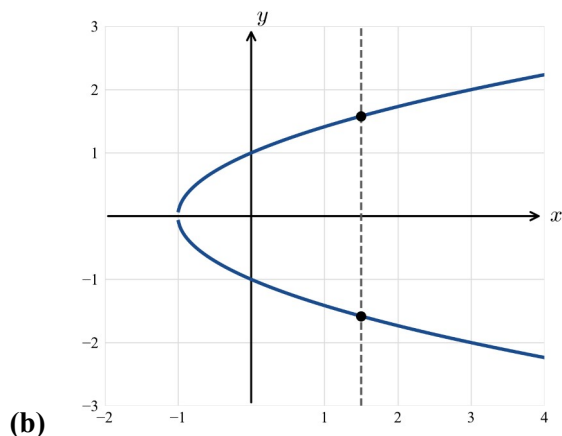
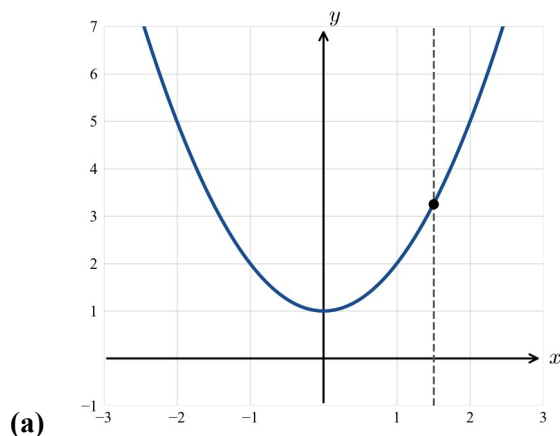
Scaffolded

Q1. For $f(x) = 4x + 1$, find $f(0)$, $f(3)$ and $f(-2)$.

Q2. For $g(x) = x^2 + 1$, find $g(2)$, $g(-3)$ and $g(0)$.

Q3. For $h(x) = 5 - 2x$: (a) solve $h(x) = 1$; (b) solve $h(x) = -3$.

Q4. Use the vertical line test to decide whether each graph is a function.



Q5. For $f(x) = x^2 - 9$: (a) solve $f(x) = 0$; (b) find $f(4)$.

Q6. Find the range of $f: [-1, 3] \rightarrow \mathbb{R}, f(x) = 2x + 1$.

Unscaffolded

Q7. For $f(x) = 3x - 5$, find $f(2)$ and $f(-4)$, and find x when $f(x) = 7$.

Q8. For $g(x) = x^2 - 2x$, find $g(4)$ and $g(-1)$, and solve $g(x) = 0$.

Q9. Is the relation $\{(1, 2), (2, 3), (1, 4)\}$ a function? Explain.

Q10. Use the vertical line test to decide whether $x^2 + y^2 = 9$ is a function.

Q11. Use the vertical line test to decide whether $y = x^3$ is a function.

Q12. For $f(x) = 2x^2 + 1$, find $f(0)$ and $f(-2)$, and solve $f(x) = 9$.

Q13. For $h(x) = (x - 1)(x + 3)$, find $h(2)$ and solve $h(x) = 0$.

Q14. Find the range of $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 - 4$.

Q15. Find the range of $f: [0, 3] \rightarrow \mathbb{R}, f(x) = x^2$.

Q16. Find the range of $f: [-2, 1] \rightarrow \mathbb{R}, f(x) = 1 - x^2$.

Q17. A function has the rule $f(x) = ax + b$. Given $f(1) = 5$ and $f(3) = 11$, find a and b .

Q18. Use the vertical line test to decide whether $y^2 = x + 1$ is a function.

Q19. State the domain, co-domain and range of $f: [0, 4] \rightarrow \mathbb{R}, f(x) = x^2$.

Answers

Q1. 1, 13, -7. **Q2.** 5, 10, 1. **Q3.** (a) $x=2$; (b) $x=4$. **Q4.** (a) function; (b) not a function. **Q5.** (a) $x=\pm 3$; (b) 7. **Q6.** $[-1, 7]$. **Q7.** $f(2)=1$, $f(-4)=-17$, $x=4$. **Q8.** $g(4)=8$, $g(-1)=3$, $x=0$ or 2. **Q9.** Not a function. **Q10.** Not a function. **Q11.** Function. **Q12.** $f(0)=1$, $f(-2)=9$, $x=\pm 2$. **Q13.** $h(2)=5$, $x=1$ or -3 . **Q14.** $[-4, \infty)$. **Q15.** $[0, 9]$. **Q16.** $[-3, 1]$. **Q17.** $a=3$, $b=2$. **Q18.** Not a function. **Q19.** domain $[0, 4]$, co-domain $\mathbb{R}$, range $[0, 16]$.

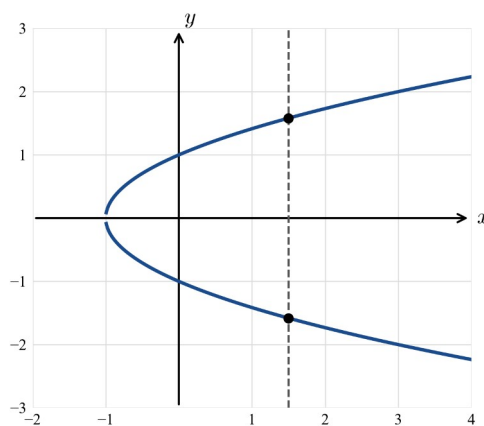
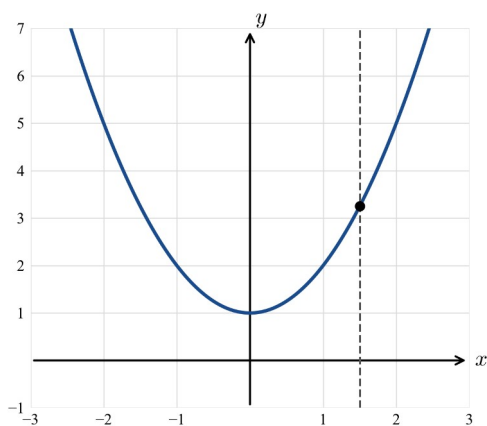
Fully-worked solutions

Q1. $f(0)=4(0)+1=1$; $f(3)=4(3)+1=13$; $f(-2)=4(-2)+1=-7$.

Q2. $g(2)=2^2+1=5$; $g(-3)=(-3)^2+1=10$; $g(0)=0+1=1$.

Q3. (a) $5-2x=1 \Rightarrow -2x=-4 \Rightarrow x=2$. (b) $5-2x=-3 \Rightarrow -2x=-8 \Rightarrow x=4$.

Q4. (a) Every vertical line meets the graph once — it **is a function**. (b) A vertical line meets the graph twice — it is **not a function**.



Q5. (a) $x^2-9=0 \Rightarrow x^2=9 \Rightarrow x=\pm 3$. (b) $f(4)=4^2-9=16-9=7$.

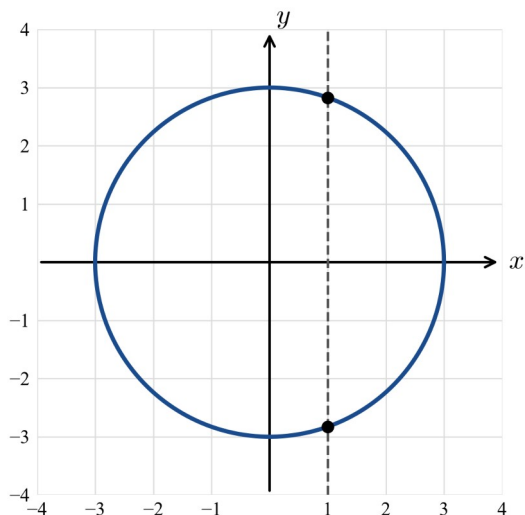
Q6. f is increasing; $f(-1)=2(-1)+1=-1$ and $f(3)=2(3)+1=7$, so the range is $[-1, 7]$.

Q7. $f(2)=3(2)-5=1$; $f(-4)=3(-4)-5=-17$; $3x-5=7 \Rightarrow 3x=12 \Rightarrow x=4$.

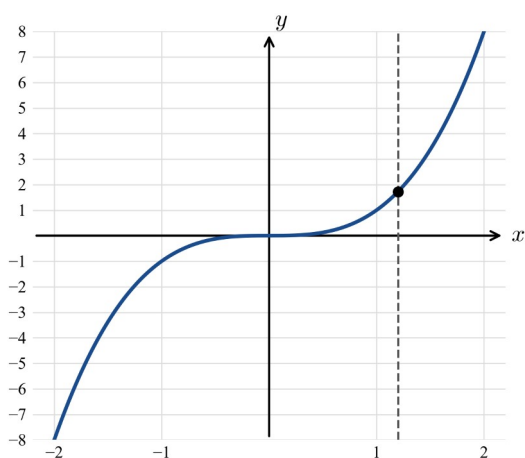
Q8. $g(4)=4^2-2(4)=8$; $g(-1)=(-1)^2-2(-1)=3$; $g(x)=0 \Rightarrow x(x-2)=0$, so $x=0$ or $x=2$.

Q9. No. The value $x=1$ is paired with both $y=2$ and $y=4$, so it is **not a function**.

Q10. The vertical line $x=1$ meets the circle at $(1, \sqrt{8})$ and $(1, -\sqrt{8})$ — twice — so $x^2+y^2=9$ is **not a function**.



Q11. Every vertical line meets $y = x^3$ exactly once, so it **is a function**.



Q12. $f(0) = 2(0)^2 + 1 = 1$; $f(-2) = 2(-2)^2 + 1 = 9$; $2x^2 + 1 = 9 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

Q13. $h(2) = (2-1)(2+3) = (1)(5) = 5$; $h(x) = 0 \Rightarrow x = 1$ or $x = -3$.

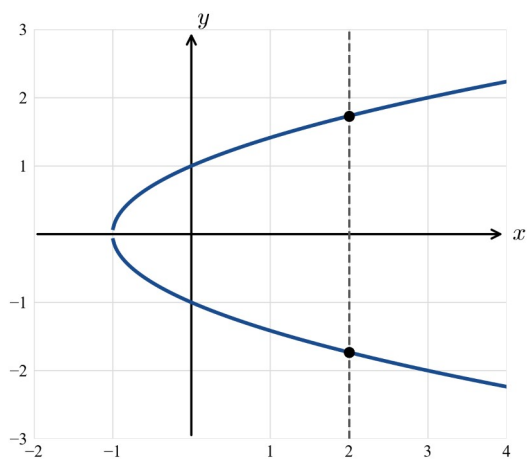
Q14. The least value of $x^2 - 4$ is -4 (at $x = 0$) and it increases without bound, so the range is $[-4, \infty)$.

Q15. On $[0, 3]$, x^2 increases from $0^2 = 0$ to $3^2 = 9$, so the range is $[0, 9]$.

Q16. $f(x) = 1 - x^2$ has a maximum of 1 at $x = 0$ (which lies in $[-2, 1]$) and its least value on the interval is at $x = -2$: $f(-2) = 1 - 4 = -3$. Range $[-3, 1]$.

Q17. $f(1) = a + b = 5$ and $f(3) = 3a + b = 11$. Subtracting: $2a = 6 \Rightarrow a = 3$, then $b = 5 - 3 = 2$. $\therefore a = 3, b = 2$.

Q18. The vertical line $x = 2$ meets $y^2 = x + 1$ at $(2, \sqrt{3})$ and $(2, -\sqrt{3})$ — twice — so $y^2 = x + 1$ is **not a function**.



Q19. The rule is written $f:[0, 4] \rightarrow \mathbb{R}$, so the **domain** is the set before the arrow, $[0, 4]$, and the **co-domain** is the set after the arrow, $\mathbb{R}$. For the **range**, x^2 is increasing on $[0, 4]$, running from $0^2=0$ up to $4^2=16$, so the range is $[0, 16]$. (Here range $[0, 16] \subseteq \mathbb{R}$, the co-domain.)

Chapter 5 Functions, relations and graphs — 5D One-to-one functions and maximal domains

Theory

One-to-one functions. A function is **one-to-one** if each y -value comes from *exactly one* x -value — that is, no two different inputs give the same output.

- **The horizontal line test.** A function is one-to-one if and only if **every horizontal line meets its graph at most once**. If some horizontal line cuts the graph more than once, the function is **not** one-to-one.
- (Compare: the *vertical* line test decides whether a graph is a function at all; the *horizontal* line test decides whether that function is one-to-one.)
- For example, $y = x$ and $y = x^3$ are one-to-one, but $y = x^2$ is not, because the line $y = 4$ meets it at both $(-2, 4)$ and $(2, 4)$.

Implied (maximal) domain. When a rule is written without a stated domain, its **implied domain** (or *maximal domain*) is the largest set of real x -values for which the rule gives a real number. Two restrictions arise most often:

- **Square roots:** $\sqrt{g(x)}$ is real only when the radicand $g(x) \geq 0$.
- **Denominators:** $\frac{1}{g(x)}$ is defined only when $g(x) \neq 0$.

When a rule combines both — for example a square root in a denominator, $\frac{1}{\sqrt{g(x)}}$ — **both** conditions apply *and* the denominator cannot be zero, so we need $g(x) > 0$ (strictly). For a rule built from several parts, find each part's restriction and take the **intersection**.

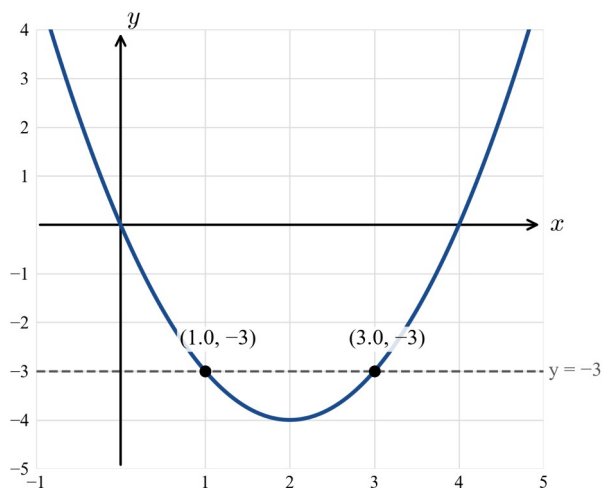
State the implied domain in interval or set notation, e.g. $[3, \infty)$ or $\mathbb{R} \setminus \{-3, 3\}$.

Worked examples

Example 1 — scaffolded

Determine whether $f(x) = x^2 - 4x$ is one-to-one.

Working	Comment
$f(x) = x^2 - 4x = x(x - 4)$: a parabola with x -intercepts 0 and 4 and turning point $(2, -4)$.	Identify the graph.
The horizontal line $y = -3$ meets the graph where $x^2 - 4x = -3 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0$, i.e. at $(1, -3)$ and $(3, -3)$.	Apply the horizontal line test.
A horizontal line meets the graph at two points, so f is not one-to-one .	One input-per-output fails.



Example 2 — scaffolded

Find the implied domain of $f(x) = \sqrt{x-3}$.

Working

The rule contains $\sqrt{x-3}$, which is real only when the radicand is non-negative.

$$x-3 \geq 0 \Rightarrow x \geq 3.$$

Implied domain $[3, \infty)$.

Comment

Square roots need $g(x) \geq 0$.

Solve the inequality.

State in interval notation.

Example 3 — unscaffolded

Find the implied domain of $f(x) = \frac{1}{x^2-9}$.

The rule is defined provided its denominator is non-zero: $x^2-9 \neq 0 \Rightarrow x^2 \neq 9 \Rightarrow x \neq \pm 3$.

$\therefore$ implied domain $\mathbb{R} \setminus \{-3, 3\}$.

Example 4 — unscaffolded

Find the implied domain of $f(x) = \frac{\sqrt{x+1}}{x-2}$.

Two conditions must hold. The radicand: $x+1 \geq 0 \Rightarrow x \geq -1$. The denominator: $x-2 \neq 0 \Rightarrow x \neq 2$.

Taking the intersection ($x \geq -1$ and $x \neq 2$):

$\therefore$ implied domain $[-1, 2) \cup (2, \infty)$.

Practice questions

Scaffolded

Q1. Use the horizontal line test to determine whether $f(x) = 2x+3$ is one-to-one.

Q2. Use the horizontal line test to determine whether $f(x) = 4-x^2$ is one-to-one.

Q3. Find the implied domain of $f(x) = \sqrt{x+5}$.

Q4. Find the implied domain of $f(x) = \frac{1}{x-4}$.

Q5. Find the implied domain of $f(x) = \sqrt{2x - 6}$.

Q6. Use the horizontal line test to determine whether $f(x) = \frac{1}{x}$ is one-to-one.

Unscaffolded

Q7. Is $f(x) = x^2 + 1$ one-to-one? Justify using the horizontal line test.

Q8. Is $f(x) = x^3$ one-to-one? Justify your answer.

Q9. Find the implied domain of $f(x) = \sqrt{x - 7}$.

Q10. Find the implied domain of $f(x) = \sqrt{4 - x}$.

Q11. Find the implied domain of $f(x) = \frac{1}{x + 3}$.

Q12. Find the implied domain of $f(x) = \frac{1}{x^2 - 16}$.

Q13. Find the implied domain of $f(x) = \sqrt{x^2 - 9}$.

Q14. Find the implied domain of $f(x) = \frac{1}{\sqrt{x - 2}}$.

Q15. Find the implied domain of $f(x) = \frac{\sqrt{x + 4}}{x - 1}$.

Q16. Find the implied domain of $f(x) = \sqrt{x} + \frac{1}{x - 3}$.

Q17. For $f(x) = \sqrt{x}$, state the implied domain and range, and determine whether it is one-to-one.

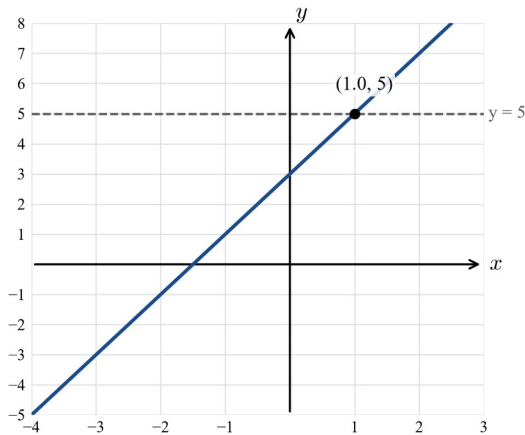
Q18. Is $f(x) = |x|$ one-to-one? State its implied domain.

Answers

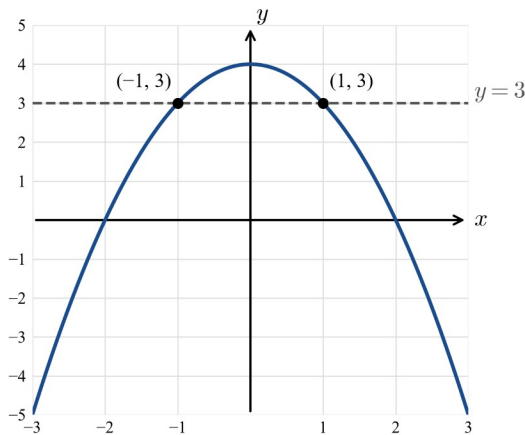
Q1. One-to-one. **Q2.** Not one-to-one. **Q3.** $[-5, \infty)$. **Q4.** $\mathbb{R} \setminus \{4\}$. **Q5.** $[3, \infty)$. **Q6.** One-to-one. **Q7.** Not one-to-one. **Q8.** One-to-one. **Q9.** $[7, \infty)$. **Q10.** $(-\infty, 4]$. **Q11.** $\mathbb{R} \setminus \{-3\}$. **Q12.** $\mathbb{R} \setminus \{-4, 4\}$. **Q13.** $(-\infty, -3] \cup [3, \infty)$. **Q14.** $(2, \infty)$. **Q15.** $[-4, 1) \cup (1, \infty)$. **Q16.** $[0, 3) \cup (3, \infty)$. **Q17.** Domain $[0, \infty)$, range $[0, \infty)$; one-to-one. **Q18.** Not one-to-one; implied domain $\mathbb{R}$.

Fully-worked solutions

Q1. $f(x) = 2x + 3$ is a straight line with gradient 2. Every horizontal line meets it exactly once, so f is **one-to-one**.



Q2. The line $y = 3$ meets $y = 4 - x^2$ at $(-1, 3)$ and $(1, 3)$ — two points — so f is **not one-to-one**.

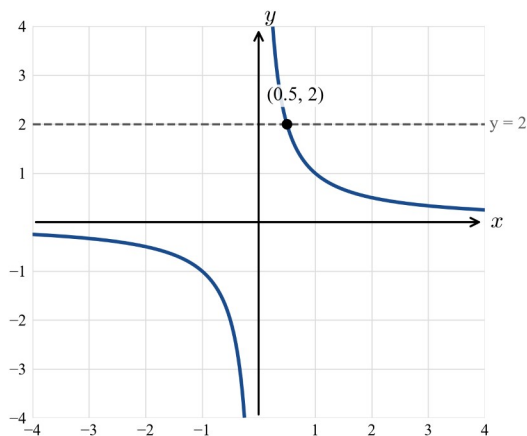


Q3. Need $x + 5 \geq 0 \Rightarrow x \geq -5$. Implied domain $[-5, \infty)$.

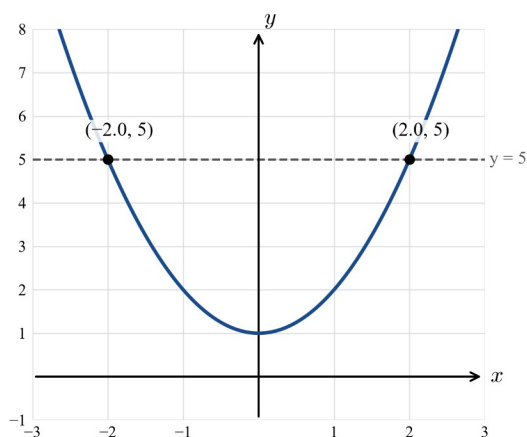
Q4. Need $x - 4 \neq 0 \Rightarrow x \neq 4$. Implied domain $\mathbb{R} \setminus \{4\}$.

Q5. Need $2x - 6 \geq 0 \Rightarrow 2x \geq 6 \Rightarrow x \geq 3$. Implied domain $[3, \infty)$.

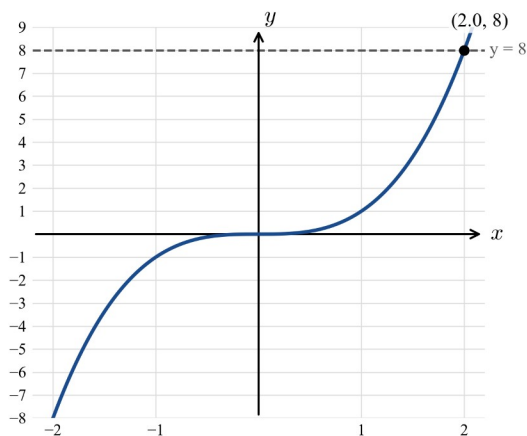
Q6. For $f(x) = \frac{1}{x}$, every horizontal line ($y \neq 0$) meets the graph exactly once (e.g. $y = 2$ only at $x = 1/2$), and $y = 0$ meets it not at all. So f is **one-to-one**.



Q7. The line $y=5$ meets $y=x^2+1$ where $x^2+1=5 \Rightarrow x^2=4 \Rightarrow x=\pm 2$, i.e. at $(-2, 5)$ and $(2, 5)$. A horizontal line meets the graph twice, so f is **not one-to-one**.



Q8. $y=x^3$ is strictly increasing, so every horizontal line meets it **exactly once** (e.g. $y=8$ only at $x=2$). $\therefore f$ is **one-to-one**.



Q9. Need $x-7 \geq 0 \Rightarrow x \geq 7$. Implied domain $[7, \infty)$.

Q10. Need $4-x \geq 0 \Rightarrow x \leq 4$. Implied domain $(-\infty, 4]$.

Q11. Need $x+3 \neq 0 \Rightarrow x \neq -3$. Implied domain $\mathbb{R} \setminus \{-3\}$.

Q12. Need $x^2-16 \neq 0 \Rightarrow x \neq \pm 4$. Implied domain $\mathbb{R} \setminus \{-4, 4\}$.

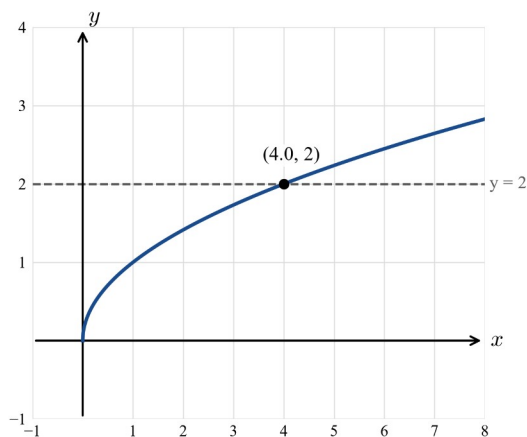
Q13. Need $x^2-9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow x \leq -3$ or $x \geq 3$. Implied domain $(-\infty, -3] \cup [3, \infty)$.

Q14. Here $\sqrt{x-2}$ is in the denominator, so the radicand must be **strictly** positive: $x-2 > 0 \Rightarrow x > 2$. Implied domain $(2, \infty)$.

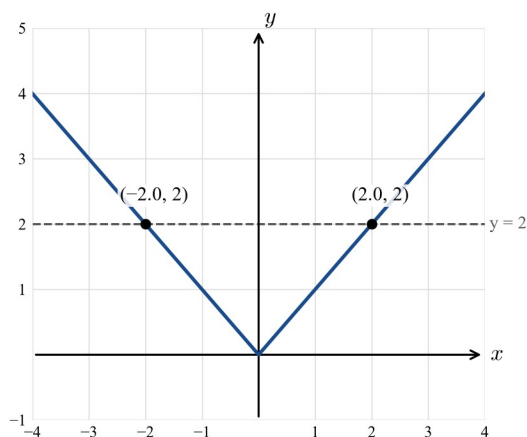
Q15. Radicand: $x + 4 \geq 0 \Rightarrow x \geq -4$. Denominator: $x - 1 \neq 0 \Rightarrow x \neq 1$. Intersecting: $x \geq -4$ and $x \neq 1$. Implied domain $[-4, 1) \cup (1, \infty)$.

Q16. The $\sqrt{x}$ term needs $x \geq 0$; the $\frac{1}{x-3}$ term needs $x \neq 3$. Intersecting: $x \geq 0$ and $x \neq 3$. Implied domain $[0, 3) \cup (3, \infty)$.

Q17. For $f(x) = \sqrt{x}$: the radicand needs $x \geq 0$, so the implied domain is $[0, \infty)$; the outputs are all ≥ 0 , so the range is $[0, \infty)$. Every horizontal line $y = c$ ($c \geq 0$) meets the graph once (at $x = c^2$), so f is **one-to-one**.



Q18. The line $y = 2$ meets $y = |x|$ at $(-2, 2)$ and $(2, 2)$ — two points — so f is **not one-to-one**. Its rule is defined for every real number, so the implied domain is $\mathbb{R}$.



Chapter 5 Functions, relations and graphs — 5E Hybrid functions

Theory

A **hybrid** (or **piecewise-defined**) function is defined by different rules on different parts of its domain. For example,

$$f(x) = \begin{cases} 3x - 1 & x \leq 2 \\ x^2 - 4 & x > 2 \end{cases}$$

uses the rule $3x - 1$ when $x \leq 2$, and the rule $x^2 - 4$ when $x > 2$.

Evaluating. To find $f(a)$, choose the rule whose condition contains a . For the function above, $f(0)$ uses $3x - 1$ (since $0 \leq 2$), giving -1 ; and $f(4)$ uses $x^2 - 4$ (since $4 > 2$), giving 12 .

Sketching. Sketch each piece over its own part of the domain *only*. At an endpoint use a **closed** dot $\bullet$ if that endpoint is **included** ($\leq$ or $\geq$), and an **open** dot $\circ$ if it is **excluded** ($<$ or $>$).

Continuity. A piecewise function is **continuous** at a join $x = a$ if the pieces meet there — the value approached from the left equals the value approached from the right. If they differ, the graph has a **jump** (a discontinuity) at $x = a$.

Domain and range. The domain is the union of the sub-domains. The range is the union of the ranges of the individual pieces over their sub-domains.

Worked examples

Example 1 — scaffolded

For $f(x) = \begin{cases} 3x - 1 & x \leq 2 \\ x^2 - 4 & x > 2 \end{cases}$, find $f(0)$, $f(2)$ and $f(4)$.

Working

Comment

$f(0)$: since $0 \leq 2$, use $3x - 1$: $f(0) = 3(0) - 1 = -1$.

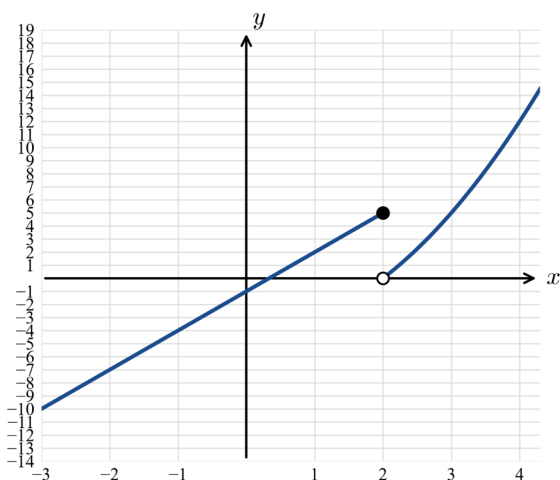
Choose the rule whose condition holds.

$f(2)$: since $2 \leq 2$, use $3x - 1$: $f(2) = 3(2) - 1 = 5$.

The endpoint $x = 2$ belongs to the first rule ($\leq$).

$f(4)$: since $4 > 2$, use $x^2 - 4$: $f(4) = 4^2 - 4 = 12$.

Use the second rule for $x > 2$.



Example 2 — scaffolded

Sketch $f(x) = \begin{cases} x + 1 & x < 2 \\ 6 - x & x \geq 2 \end{cases}$ and state its domain and range, and whether it is continuous.

Piece 1: $y = x + 1$ for $x < 2$. As $x \rightarrow 2^-$, $y \rightarrow 3$, so an **open** dot at $(2, 3)$.

The endpoint $x = 2$ is excluded ($<$).

Piece 2: $y = 6 - x$ for $x \geq 2$. At $x = 2$, $y = 4$, so a **closed** dot at $(2, 4)$.

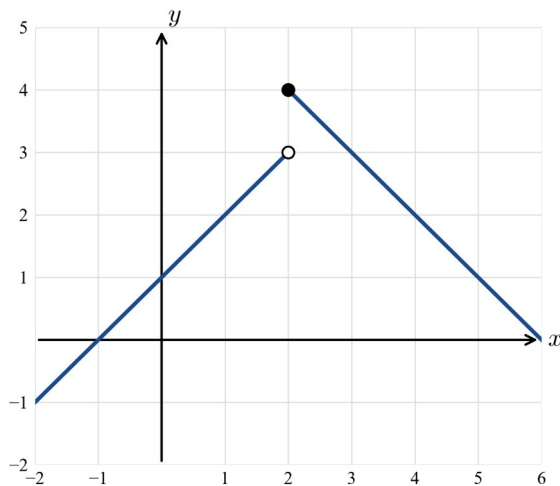
The endpoint $x = 2$ is included ($\geq$).

At $x = 2$ the left value $3 \neq$ the right value 4 , so there is a **jump** — f is **not continuous** at $x = 2$.

Compare the two one-sided values.

Domain $\mathbb{R}$; range $(-\infty, 4]$.

Piece 1 gives $(-\infty, 3)$, piece 2 gives $(-\infty, 4]$.



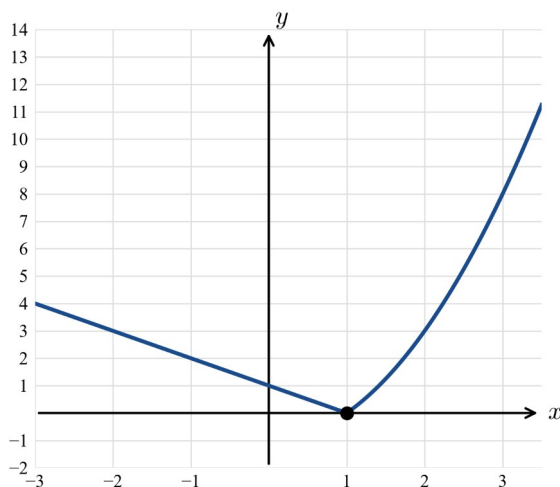
Example 3 — unscaffolded

Sketch $f(x) = \begin{cases} 1 - x & x < 1 \\ x^2 - 1 & x \geq 1 \end{cases}$, and state its domain, range and continuity.

Piece 1 ($y = 1 - x$, $x < 1$) approaches $(1, 0)$; piece 2 ($y = x^2 - 1$, $x \geq 1$) has the value 0 at $x = 1$. The two values agree, so f is **continuous** at $x = 1$ (closed dot at $(1, 0)$).

Domain $\mathbb{R}$. Piece 1 gives $(0, \infty)$ and piece 2 gives $[0, \infty)$, so the range is $[0, \infty)$.

$\therefore f$ is continuous with domain $\mathbb{R}$ and range $[0, \infty)$.



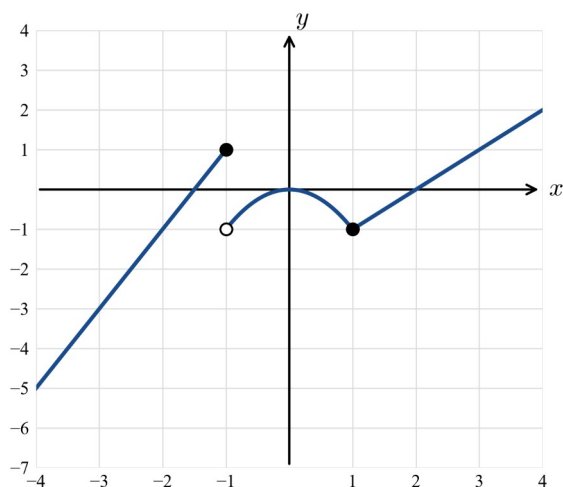
Example 4 — unscaffolded

Sketch $f(x) = \begin{cases} 2x + 3 & x \leq -1 \\ -x^2 & -1 < x \leq 1 \\ x - 2 & x > 1 \end{cases}$, and state its domain, range and continuity.

At $x = -1$: left value $2(-1) + 3 = 1$ (closed), right value $-(-1)^2 = -1$ — a **jump**, so f is discontinuous at $x = -1$. At $x = 1$: left value $- (1)^2 = -1$ (closed), right value $1 - 2 = -1$ — these agree, so f is **continuous** at $x = 1$.

Domain $\mathbb{R}$. Piece 1 gives $(-\infty, 1]$, piece 2 gives $[-1, 0]$, piece 3 gives $(-1, \infty)$; together the range is $\mathbb{R}$.

$\therefore f$ has a single discontinuity at $x = -1$, domain $\mathbb{R}$ and range $\mathbb{R}$.



Practice questions

Scaffolded

Q1. For $f(x) = \begin{cases} x^2 + 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$: (a) find $f(-2)$; (b) find $f(0)$; (c) find $f(3)$; (d) is f continuous at $x = 0$?

Q2. For $f(x) = \begin{cases} 5 & x \leq -2 \\ 1 - x & -2 < x < 3 \\ x^2 & x \geq 3 \end{cases}$, find $f(-3)$, $f(0)$, $f(3)$ and $f(5)$.

Q3. For $f(x) = \begin{cases} x + 2 & x \leq 1 \\ 3 & x > 1 \end{cases}$: (a) sketch the graph; (b) state the domain and range; (c) is f continuous?

Q4. For $f(x) = \begin{cases} -x & x < 0 \\ x + 1 & x \geq 0 \end{cases}$: (a) sketch the graph, marking open/closed endpoints; (b) state the domain and range; (c) state where f is discontinuous.

Q5. For $f(x) = \begin{cases} x^2 & x \leq 1 \\ 2 - x & x > 1 \end{cases}$: (a) sketch the graph; (b) show that f is continuous at $x = 1$; (c) state the range.

Q6. For $f(x) = \begin{cases} 2x + k & x \leq 2 \\ x^2 - 1 & x > 2 \end{cases}$, find the value of k for which f is continuous at $x = 2$.

Un scaffolded

For each function, sketch the graph (marking open/closed endpoints), and state the domain, the range, and whether f is continuous (naming any points of discontinuity).

Q7. $f(x) = \begin{cases} x + 3 & x \leq -1 \\ x^2 & -1 < x \leq 2 \\ 8 - 2x & x > 2 \end{cases}$

$$\text{Q8. } f(x) = \begin{cases} 1-x & x < 0 \\ 1+x & x \geq 0 \end{cases}$$

$$\text{Q9. } f(x) = \begin{cases} x+1 & x \leq 0 \\ 1 & 0 < x \leq 2 \\ 5-2x & x > 2 \end{cases}$$

$$\text{Q10. } f(x) = \begin{cases} x^2-4 & x \leq 2 \\ 2x-4 & x > 2 \end{cases}$$

$$\text{Q11. } f(x) = \begin{cases} 3 & x < 1 \\ -x+1 & x \geq 1 \end{cases}$$

Q12. Australia Post charges the following for a parcel of weight w grams:

$$C(w) = \begin{cases} 2 & 0 < w \leq 50 \\ 3 & 50 < w \leq 100 \\ 5 & 100 < w \leq 250 \end{cases}$$

where C is the cost in dollars. Sketch the graph, state the domain and range, and state where C is discontinuous.

$$\text{Q13. } f(x) = \begin{cases} (x+2)^2 & x \leq 0 \\ 4-x & x > 0 \end{cases}$$

$$\text{Q14. } f(x) = \begin{cases} 2x+1 & x \leq -1 \\ -1 & -1 < x < 2 \\ x-3 & x \geq 2 \end{cases}$$

$$\text{Q15. } f(x) = \begin{cases} x+2 & x < 0 \\ x^2+1 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

$$\text{Q16. } f(x) = \begin{cases} ax+1 & x < 3 \\ x^2-5 & x \geq 3 \end{cases}. \text{ Find the value of } a \text{ for which } f \text{ is continuous at } x=3.$$

$$\text{Q17. } f(x) = \begin{cases} -(x+1)^2+4 & x \leq 1 \\ 2x-4 & x > 1 \end{cases}$$

Q18. A mobile data plan costs \$30 per month for up to 10 GB, then \$2 for each additional GB:

$$C(x) = \begin{cases} 30 & 0 \leq x \leq 10 \\ 30+2(x-10) & x > 10 \end{cases}$$

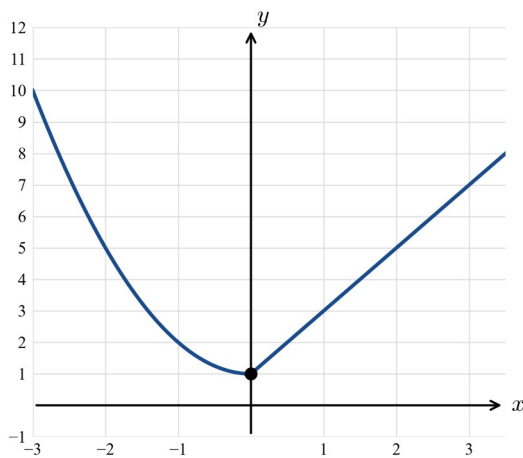
where x is the data used in GB. Sketch the graph, find the cost of using 25 GB, state the domain and range, and state whether C is continuous.

Answers

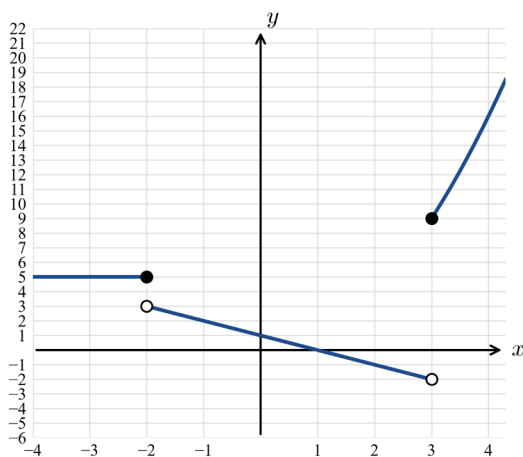
Q1. (a) 5 (b) 1 (c) 7 (d) yes (both sides give 1). **Q2.** $f(-3)=5$, $f(0)=1$, $f(3)=9$, $f(5)=25$. **Q3.** Domain $\mathbb{R}$; range $(-\infty, 3]$; continuous. **Q4.** Domain $\mathbb{R}$; range $(0, \infty)$; discontinuous at $x=0$ (jump). **Q5.** Range $\mathbb{R}$ (continuous at $x=1$). **Q6.** $k=-1$. **Q7.** Domain $\mathbb{R}$; range $(-\infty, 4]$; discontinuous at $x=-1$. **Q8.** Domain $\mathbb{R}$; range $[1, \infty)$; continuous. **Q9.** Domain $\mathbb{R}$; range $(-\infty, 1]$; continuous. **Q10.** Domain $\mathbb{R}$; range $[-4, \infty)$; continuous. **Q11.** Domain $\mathbb{R}$; range $(-\infty, 0] \cup \{3\}$; discontinuous at $x=1$. **Q12.** Domain $(0, 250]$; range $\{2, 3, 5\}$; discontinuous at $w=50$ and $w=100$. **Q13.** Domain $\mathbb{R}$; range $\mathbb{R}$; continuous. **Q14.** Domain $\mathbb{R}$; range $\mathbb{R}$; continuous. **Q15.** Domain $\mathbb{R}$; range $(-\infty, 5]$; discontinuous at $x=0$ and $x=2$. **Q16.** $a=1$. **Q17.** Domain $\mathbb{R}$; range $\mathbb{R}$; discontinuous at $x=1$. **Q18.** $C(25)=\$60$; domain $[0, \infty)$; range $[30, \infty)$; continuous.

Fully-worked solutions

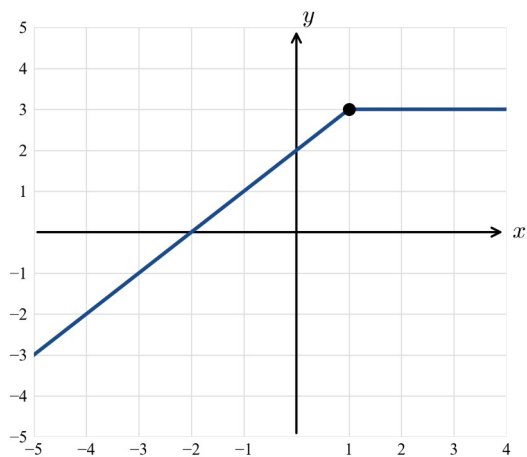
Q1. (a) $x=-2 < 0$, use x^2+1 : $(-2)^2+1=5$. (b) $x=0 \geq 0$, use $2x+1$: 1. (c) $x=3 \geq 0$: $2(3)+1=7$. (d) Left of 0: $x^2+1 \rightarrow 1$; right: $2x+1 \rightarrow 1$. The values agree, so f is **continuous** at $x=0$.



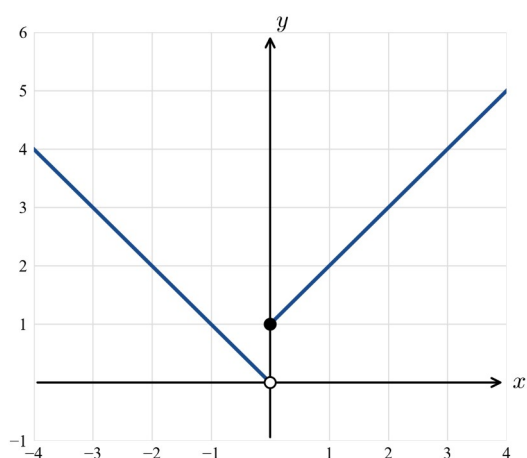
Q2. $f(-3)$: $-3 \leq -2$, use 5, so 5. $f(0)$: $-2 < 0 < 3$, use $1-x$, so 1. $f(3)$: $3 \geq 3$, use x^2 , so 9. $f(5)$: use x^2 , so 25.



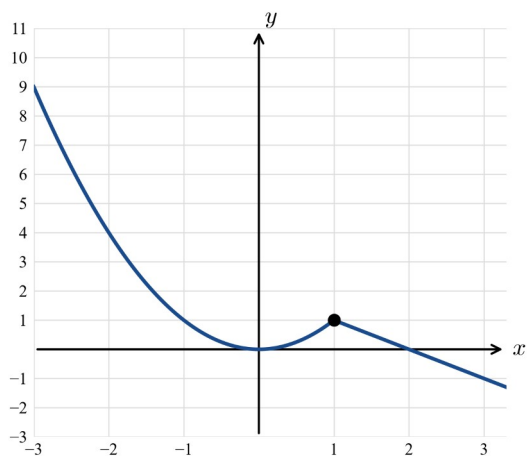
Q3. (a) Line $y=x+2$ up to a closed dot at $(1, 3)$, then the horizontal line $y=3$ for $x>1$ (open dot at $(1, 3)$, which coincides). (b) Domain $\mathbb{R}$; piece 1 gives $(-\infty, 3]$ and piece 2 gives $\{3\}$, so range $(-\infty, 3]$. (c) The pieces meet at $(1, 3)$, so f is **continuous**.



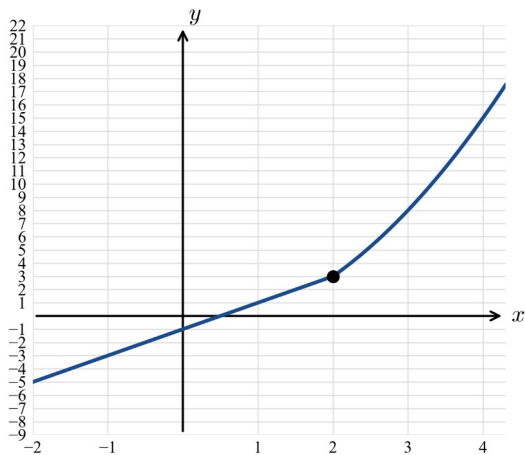
Q4. (a) See graph — open dot at $(0, 0)$ (from $-x$), closed dot at $(0, 1)$ (from $x+1$). (b) Domain $\mathbb{R}$; piece 1 gives $(0, \infty)$, piece 2 gives $[1, \infty)$, so range $(0, \infty)$. (c) Left value $0 \neq$ right value 1 , so f is **discontinuous** at $x=0$ (a jump).



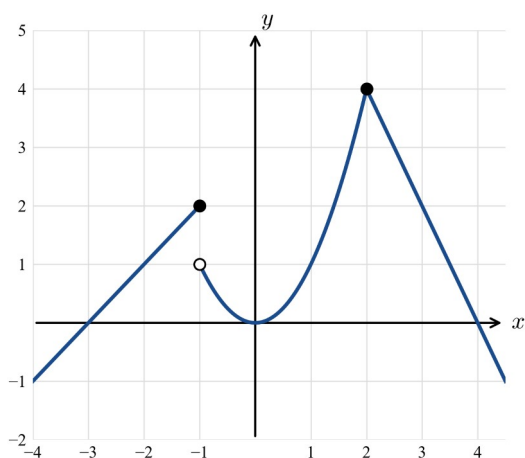
Q5. (a) Parabola $y=x^2$ for $x \leq 1$ meeting the line $y=2-x$ for $x > 1$. (b) At $x=1$: $x^2=1$ and $2-x=1$; the values agree, so f is continuous at $x=1$. (c) Piece 1 gives $[0, \infty)$, piece 2 gives $(-\infty, 1)$; together the range is $\mathbb{R}$.



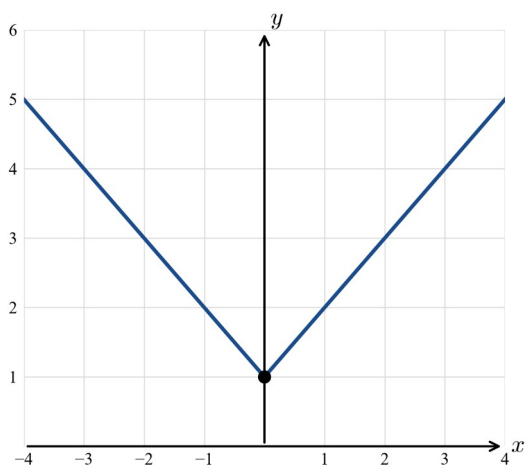
Q6. For continuity at $x=2$ the pieces must agree there: $2(2)+k=2^2-1$, i.e. $4+k=3$, so $k=-1$.



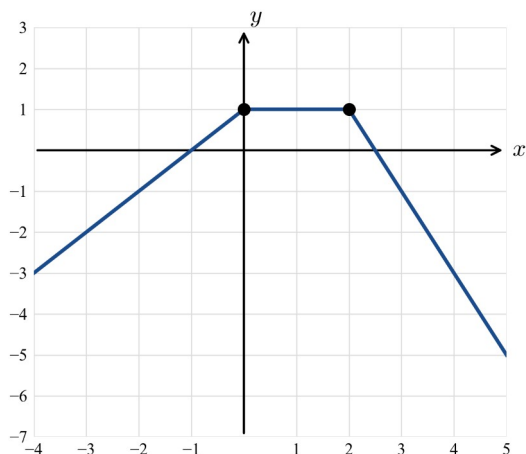
Q7. At $x = -1$: left $-1 + 3 = 2$ (closed), right $(-1)^2 = 1$ — a jump, so discontinuous at $x = -1$. At $x = 2$: left $2^2 = 4$, right $8 - 2(2) = 4$ — continuous. Domain $\mathbb{R}$; the pieces give $(-\infty, 2]$, $[0, 4]$ and $(-\infty, 4)$, so range $(-\infty, 4]$.



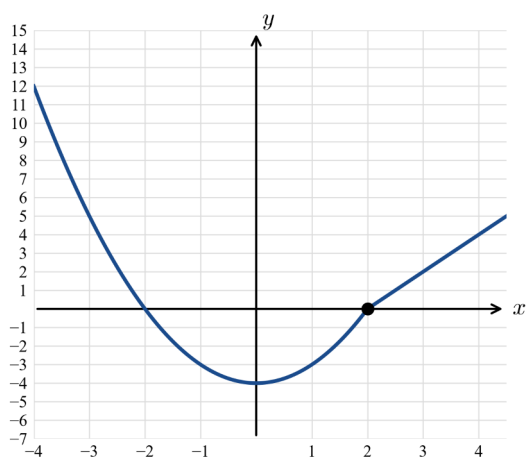
Q8. This is a “V”. At $x = 0$: $1 - x \rightarrow 1$ and $1 + x \rightarrow 1$, so continuous. Domain $\mathbb{R}$; both pieces give values ≥ 1 , so range $[1, \infty)$.



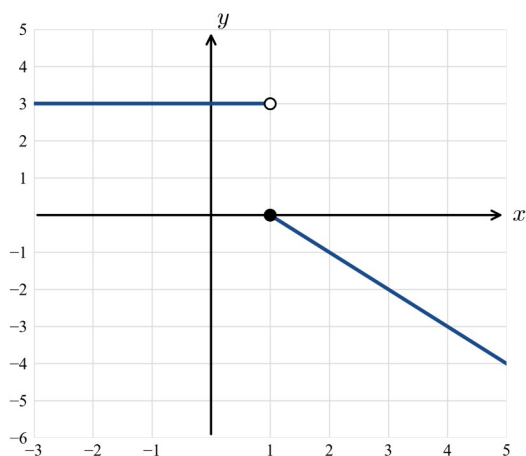
Q9. At $x = 0$: $x + 1 \rightarrow 1$ and the middle rule is 1 — continuous. At $x = 2$: middle rule 1 and $5 - 2x \rightarrow 1$ — continuous. Domain $\mathbb{R}$; range $(-\infty, 1]$.



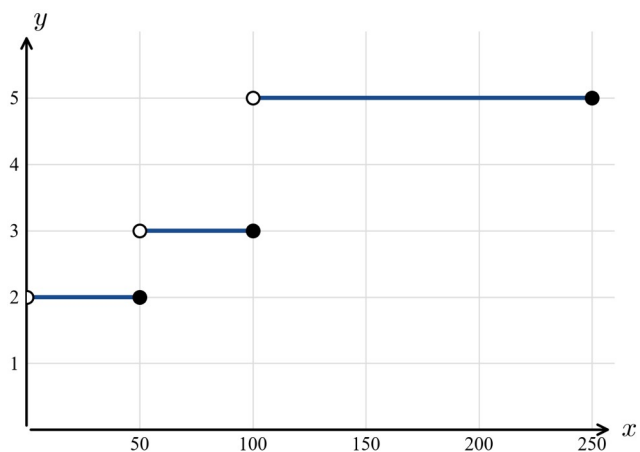
Q10. At $x=2$: $x^2 - 4 = 0$ and $2x - 4 = 0$ — continuous. Domain $\mathbb{R}$; piece 1 ($x \leq 2$) has minimum -4 at $x=0$ giving $[-4, \infty)$, piece 2 gives $(0, \infty)$, so range $[-4, \infty)$.



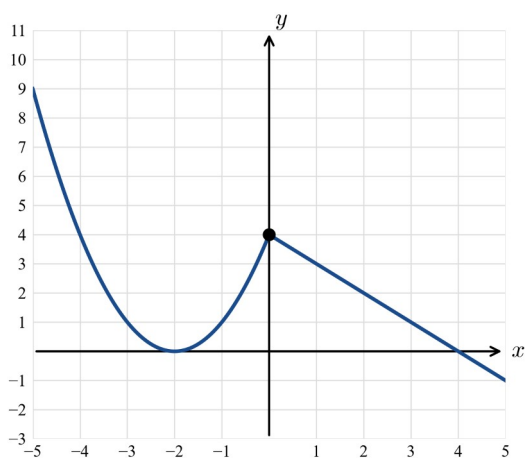
Q11. At $x=1$: left value 3, right value $-1 + 1 = 0$ — a jump, so discontinuous at $x=1$. Domain $\mathbb{R}$; piece 1 gives $\{3\}$, piece 2 ($x \geq 1$) gives $(-\infty, 0]$, so range $(-\infty, 0] \cup \{3\}$.



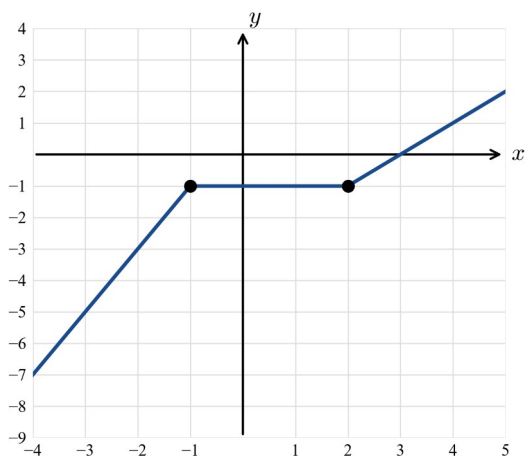
Q12. The graph is three horizontal steps: 2 on $(0, 50]$, 3 on $(50, 100]$, 5 on $(100, 250]$ — each with an open dot on the left and a closed dot on the right. Domain $(0, 250]$; range $\{2, 3, 5\}$; **discontinuous at $w=50$ and $w=100$** (jumps).



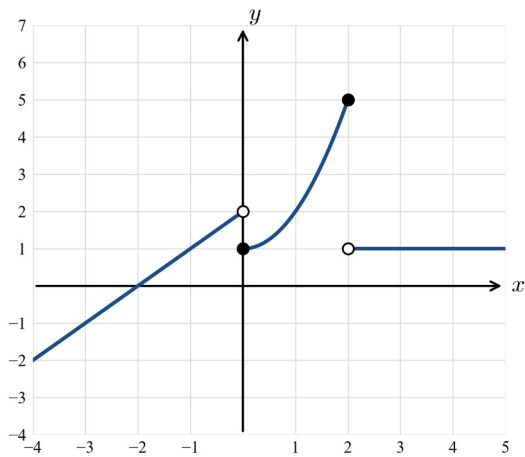
Q13. At $x=0$: $(0+2)^2=4$ and $4-0=4$ — continuous. Domain $\mathbb{R}$; piece 1 gives $[0, \infty)$, piece 2 gives $(-\infty, 4)$, so range $\mathbb{R}$.



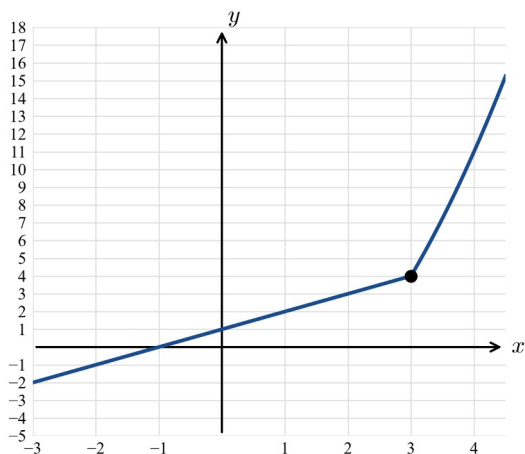
Q14. At $x=-1$: $2(-1)+1=-1$ and the middle rule is -1 — continuous. At $x=2$: middle rule -1 and $2-3=-1$ — continuous. Domain $\mathbb{R}$; range $\mathbb{R}$.



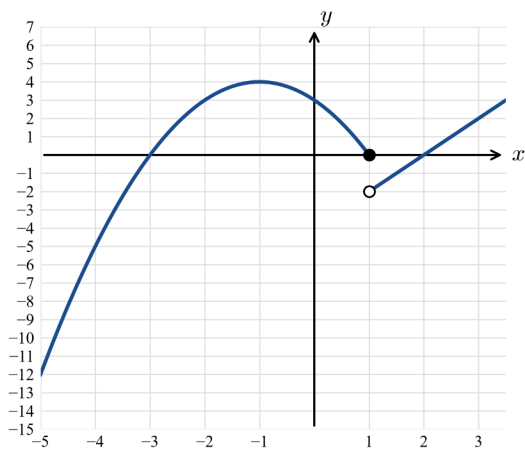
Q15. At $x=0$: left $x+2 \rightarrow 2$, right $0^2+1=1$ — a jump. At $x=2$: left $2^2+1=5$, right 1 — a jump. So discontinuous at $x=0$ and $x=2$. Domain $\mathbb{R}$; pieces give $(-\infty, 2)$, $[1, 5]$ and $\{1\}$, so range $(-\infty, 5]$.



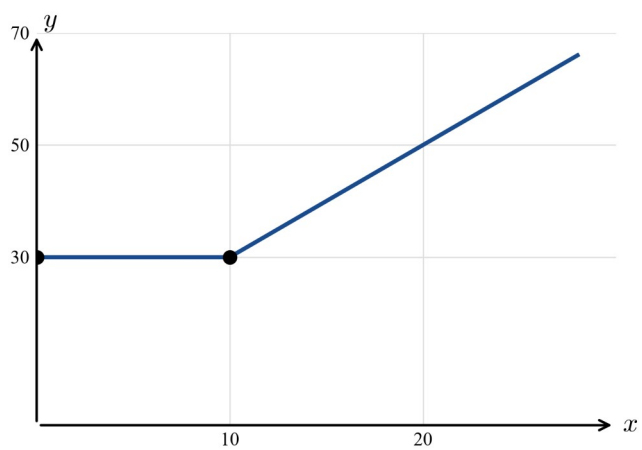
Q16. For continuity at $x=3$: $a(3)+1=3^2-5$, i.e. $3a+1=4$, so $a=1$.



Q17. At $x=1$: left $-(1+1)^2+4=0$ (closed), right $2(1)-4=-2$ — a jump, so discontinuous at $x=1$. Domain $\mathbb{R}$; piece 1 has maximum 4 (at $x=-1$) giving $(-\infty, 4]$, piece 2 gives $(-2, \infty)$, so range $\mathbb{R}$.



Q18. For 25 GB, $x=25>10$: $C(25)=30+2(25-10)=30+30=\60 . At $x=10$: the first rule gives 30 and $30+2(10-10)=30$ — the pieces meet, so C is **continuous**. Domain $[0, \infty)$; range $[30, \infty)$. The gradient of 2 for $x>10$ means each extra GB costs \$2.



Theory

A **function** f is a rule that assigns exactly one output to each input. The notation $f(x)$ means “the value of the function f at x ”. We evaluate a function by **substituting** the input in place of x .

Evaluating at a number. For $f(x) = x^2 - 3$, $f(4) = 4^2 - 3 = 13$.

Evaluating at an expression. The input can itself be an expression — substitute it in full and simplify. For $f(x) = x^2 - 3$: $f(a+1) = (a+1)^2 - 3 = a^2 + 2a - 2$.

$f(x+c)$ is **not** $f(x) + c$. These are different operations:

- $f(x+c)$ replaces the **input** x by $x+c$ (a change *inside* the function);
- $f(x) + c$ adds c to the **output** (a change *after* the function).

For $f(x) = x^2$: $f(x+3) = (x+3)^2 = x^2 + 6x + 9$, but $f(x) + 3 = x^2 + 3$.

The difference quotient. The expression

$$\frac{f(x+h) - f(x)}{h}$$

measures the average rate of change of f over the interval from x to $x+h$. Expand $f(x+h)$, subtract $f(x)$, then cancel the common factor h . (This expression is the foundation of calculus — see Chapter 14.)

Solving equations. To solve $f(x) = k$ (or $f(x) = g(x)$), write out the rule and solve the resulting equation for x .

Composite evaluation. $f(g(a))$ means “apply g first, then apply f to the result”: work from the inside out.

Worked examples

Example 1 — scaffolded

Given $f(x) = 2x^2 - 3x + 1$, find $f(4)$ and $f(-2)$.

Working	Comment
$f(4) = 2(4)^2 - 3(4) + 1 = 32 - 12 + 1$	Substitute $x = 4$ everywhere.
$= 21$	Evaluate.
$f(-2) = 2(-2)^2 - 3(-2) + 1 = 8 + 6 + 1$	Substitute $x = -2$; note $(-2)^2 = 4$ and $-3(-2) = +6$.
$= 15$	Evaluate.

Example 2 — scaffolded

Given $f(x) = x^2 + 5$, find and simplify $f(x+2)$ and $f(x) + 2$, and comment.

Working	Comment
$f(x+2) = (x+2)^2 + 5$	Replace the <i>input</i> x by $x+2$.
$= x^2 + 4x + 4 + 5 = x^2 + 4x + 9$	Expand and simplify.
$f(x) + 2 = (x^2 + 5) + 2 = x^2 + 7$	Add 2 to the <i>output</i> .
The results differ: $f(x+2) \neq f(x) + 2$.	Changing the input is not the same as changing the output.

Example 3 — unscaffolded

Given $f(x) = 3x - 1$, simplify $\frac{f(x+h) - f(x)}{h}$.

$$f(x+h) = 3(x+h) - 1 = 3x + 3h - 1.$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(3x + 3h - 1) - (3x - 1)}{h} = \frac{3h}{h}.$$

$$\therefore \frac{f(x+h) - f(x)}{h} = 3.$$

Example 4 — unscaffolded

Given $f(x) = x^2 - 4x$ and $g(x) = 2x + 1$, find $f(g(2))$.

$$g(2) = 2(2) + 1 = 5. \text{ Then } f(5) = 5^2 - 4(5) = 25 - 20 = 5.$$

$$\therefore f(g(2)) = 5.$$

Practice questions**Scaffolded**

Q1. For $f(x) = 5x - 2$, find (a) $f(3)$, (b) $f(0)$, (c) $f(-1)$.

Q2. For $f(x) = x^2 - 2x + 4$, find (a) $f(2)$, (b) $f(-3)$.

Q3. For $f(x) = x^2 + 1$, find and simplify (a) $f(a)$, (b) $f(2a)$, (c) $f(a+1)$.

Q4. For $f(x) = 4 - x^2$, find and simplify $f(x+1)$ and $f(x) + 1$. Are they equal?

Q5. For $f(x) = 2x + 3$, simplify $\frac{f(x+h) - f(x)}{h}$.

Q6. For $f(x) = x^2$ and $g(x) = x - 3$, find (a) $f(g(4))$, (b) $g(f(4))$.

Unscaffolded

Q7. For $f(x) = 3x^2 - x$, find $f(2)$ and $f(-1)$.

Q8. For $f(x) = \frac{x-1}{x+2}$, find $f(3)$ and $f(0)$.

Q9. For $f(x) = x^2 - 5$, find $f(\sqrt{2})$ and simplify $f(a-1)$.

Q10. For $f(x) = 2x^2 + 3$, find and simplify $f(x+3)$ and $f(x) + 3$.

Q11. For $f(x) = x^2$, simplify $\frac{f(x+h) - f(x)}{h}$.

Q12. For $f(x) = x^2 - 3x$, simplify $\frac{f(x+h) - f(x)}{h}$.

Q13. For $f(x) = 5x + 1$, solve $f(x) = 16$.

Q14. For $f(x) = x^2 - 2x$, solve $f(x) = 8$.

Q15. For $f(x) = 2x - 7$ and $g(x) = x^2 + 1$, find $f(g(3))$ and $g(f(3))$.

Q16. For $f(x) = ax + 3$, it is known that $f(2) = 11$. Find a .

Q17. For $f(x) = x^2 + bx - 5$, it is known that $f(3) = 13$. Find b .

Q18. For $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, solve $f(x) = g(x)$.

Answers

Q1. 13, -2, -7. **Q2.** 4, 19. **Q3.** a^2+1 ; $4a^2+1$; a^2+2a+2 . **Q4.** $f(x+1)=3-2x-x^2$; $f(x)+1=5-x^2$; not equal.
Q5. 2. **Q6.** 1; 13. **Q7.** 10, 4. **Q8.** $2/5$, $-1/2$. **Q9.** -3; a^2-2a-4 . **Q10.** $2x^2+12x+21$; $2x^2+6$. **Q11.** $2x+h$.
Q12. $2x+h-3$. **Q13.** $x=3$. **Q14.** $x=4$ or $x=-2$. **Q15.** 13; 2. **Q16.** $a=4$. **Q17.** $b=3$. **Q18.** No real solutions.

Fully-worked solutions

Q1. $f(3)=5(3)-2=13$; $f(0)=-2$; $f(-1)=5(-1)-2=-7$.

Q2. $f(2)=4-4+4=4$; $f(-3)=9+6+4=19$.

Q3. (a) $f(a)=a^2+1$. (b) $f(2a)=(2a)^2+1=4a^2+1$. (c) $f(a+1)=(a+1)^2+1=a^2+2a+2$.

Q4. $f(x+1)=4-(x+1)^2=4-(x^2+2x+1)=3-2x-x^2$; $f(x)+1=(4-x^2)+1=5-x^2$. They are **not** equal.

Q5. $f(x+h)=2(x+h)+3=2x+2h+3$, so $\frac{(2x+2h+3)-(2x+3)}{h}=\frac{2h}{h}=2$.

Q6. (a) $g(4)=1$, so $f(g(4))=f(1)=1$. (b) $f(4)=16$, so $g(f(4))=g(16)=13$.

Q7. $f(2)=3(4)-2=10$; $f(-1)=3(1)-(-1)=4$.

Q8. $f(3)=\frac{3-1}{3+2}=\frac{2}{5}$; $f(0)=\frac{-1}{2}=-\frac{1}{2}$.

Q9. $f(\sqrt{2})=(\sqrt{2})^2-5=2-5=-3$; $f(a-1)=(a-1)^2-5=a^2-2a+1-5=a^2-2a-4$.

Q10. $f(x+3)=2(x+3)^2+3=2(x^2+6x+9)+3=2x^2+12x+21$; $f(x)+3=(2x^2+3)+3=2x^2+6$.

Q11. $f(x+h)=(x+h)^2=x^2+2xh+h^2$, so $\frac{(x^2+2xh+h^2)-x^2}{h}=\frac{2xh+h^2}{h}=2x+h$.

Q12. $f(x+h)=(x+h)^2-3(x+h)=x^2+2xh+h^2-3x-3h$, so $\frac{f(x+h)-f(x)}{h}=\frac{2xh+h^2-3h}{h}=2x+h-3$.

Q13. $5x+1=16 \Rightarrow 5x=15 \Rightarrow x=3$.

Q14. $x^2-2x=8 \Rightarrow x^2-2x-8=0 \Rightarrow (x-4)(x+2)=0$, so $x=4$ or $x=-2$.

Q15. $g(3)=3^2+1=10$, so $f(g(3))=f(10)=2(10)-7=13$. And $f(3)=2(3)-7=-1$, so $g(f(3))=g(-1)=(-1)^2+1=2$.

Q16. $f(2)=2a+3=11 \Rightarrow 2a=8 \Rightarrow a=4$.

Q17. $f(3)=9+3b-5=13 \Rightarrow 3b+4=13 \Rightarrow 3b=9 \Rightarrow b=3$.

Q18. $x^2+1=2x-1 \Rightarrow x^2-2x+2=0$. The discriminant is $\Delta=(-2)^2-4(1)(2)=4-8=-4<0$, so there are **no** real solutions.

Chapter 5 Functions, relations and graphs — 5G Inverse functions

Theory

The **inverse** of a function “undoes” it: if f maps a to b , then f^{-1} maps b back to a .

When does an inverse function exist? Only when f is **one-to-one** — that is, no two different x -values give the same y -value. Graphically, f is one-to-one when it passes the **horizontal line test** (every horizontal line cuts the graph at most once). If f is not one-to-one, we **restrict the domain** to a piece on which it is (for example $f(x) = x^2$ is not one-to-one on $\mathbb{R}$, but it is on $[0, \infty)$).

Finding the rule of f^{-1} :

1. Write $y = f(x)$.
2. **Swap** x and y (the inverse reverses the roles of input and output).
3. Make y the subject. This rule is $f^{-1}(x)$.

Domain and range swap. The domain of f^{-1} equals the **range** of f , and the range of f^{-1} equals the **domain** of f .

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Check: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Note: $f^{-1}(x)$ means the inverse function — it does **not** mean $\frac{1}{f(x)}$.

Worked examples

Example 1 — scaffolded

Find the inverse of $f(x) = 2x - 3$, and sketch f and f^{-1} on the one set of axes.

Working

Comment

f is linear (gradient $2 \neq 0$), so it is one-to-one and the inverse exists.

A non-constant linear function is always one-to-one.

Let $y = 2x - 3$.

Write y for $f(x)$.

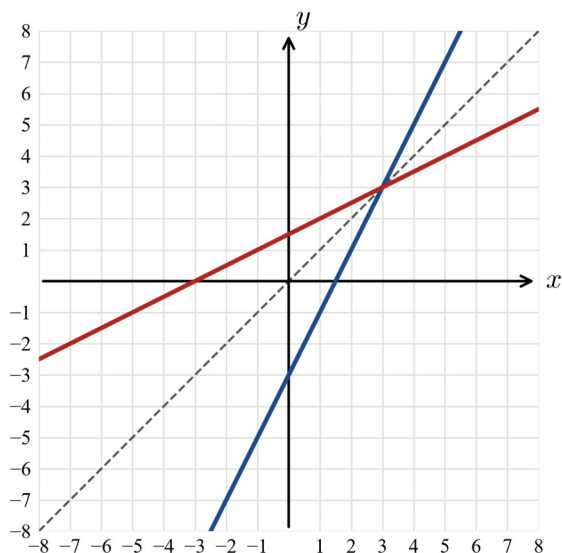
Swap x and y : $x = 2y - 3$.

The inverse reverses input and output.

Solve for y : $2y = x + 3$, so $y = \frac{x+3}{2}$.

Make y the subject.

$\therefore f^{-1}(x) = \frac{x+3}{2}$; domain and range are both $\mathbb{R}$.



Example 2 — scaffolded

For $f(x) = \sqrt{x-1}$, $x \geq 1$, find f^{-1} , state its domain, and sketch.

Working

Domain of f is $[1, \infty)$; range of f is $[0, \infty)$.

Let $y = \sqrt{x-1}$. Swap: $x = \sqrt{y-1}$ (with $x \geq 0$).

Square both sides: $x^2 = y - 1$, so $y = x^2 + 1$.

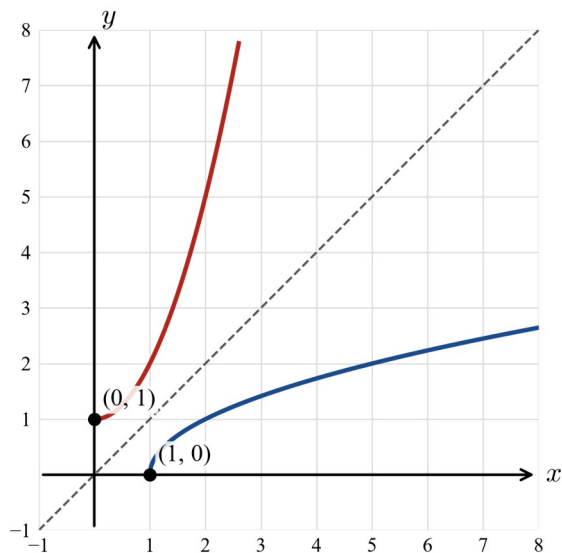
$\therefore f^{-1}(x) = x^2 + 1$, with domain $[0, \infty)$ (the range of f) and range $[1, \infty)$.

Comment

The square-root function returns values ≥ 0 .

Valid because $x \geq 0$.

Domain of $f^{-1} = \text{range of } f$.

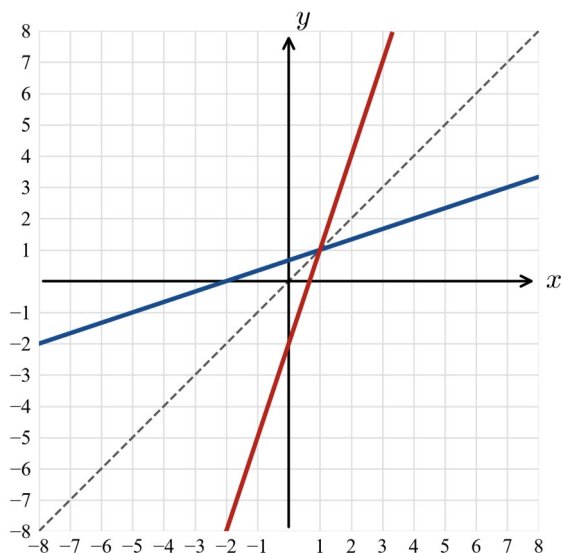


Example 3 — unscaffolded

Find the inverse of $f(x) = \frac{x+2}{3}$.

Let $y = \frac{x+2}{3}$. Swap: $x = \frac{y+2}{3} \Rightarrow 3x = y + 2 \Rightarrow y = 3x - 2$.

$\therefore f^{-1}(x) = 3x - 2$, with domain and range $\mathbb{R}$.

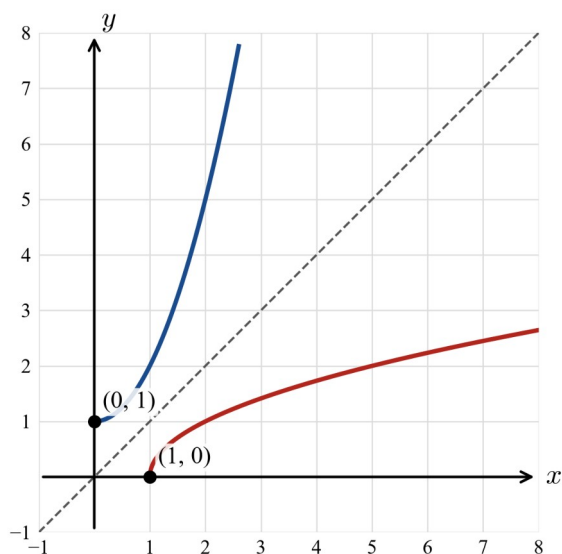


Example 4 — unscaffolded

For $f(x) = x^2 + 1$, $x \geq 0$, find f^{-1} and state its domain and range.

The range of f is $[1, \infty)$. Let $y = x^2 + 1$. Swap: $x = y^2 + 1 \Rightarrow y^2 = x - 1 \Rightarrow y = \sqrt{x - 1}$ (the **positive** root, since the range of f^{-1} is the domain of f , namely $[0, \infty)$).

$\therefore f^{-1}(x) = \sqrt{x - 1}$, domain $[1, \infty)$, range $[0, \infty)$. Check: $f(f^{-1}(x)) = (\sqrt{x - 1})^2 + 1 = x - 1 + 1 = x$.



Practice questions

Scaffolded

Q1. For $f(x) = x + 5$: (a) explain why the inverse exists; (b) find $f^{-1}(x)$; (c) sketch f , f^{-1} and $y = x$.

Q2. For $f(x) = 4x$: (a) find $f^{-1}(x)$; (b) state its domain and range; (c) sketch.

Q3. For $f(x) = 3x - 1$: (a) find $f^{-1}(x)$; (b) verify $f(f^{-1}(x)) = x$; (c) sketch.

Q4. For $f(x) = \frac{x-4}{2}$: (a) find $f^{-1}(x)$; (b) sketch f and f^{-1} .

Q5. For $f(x) = \sqrt{x+2}$, $x \geq -2$: (a) state the range of f ; (b) find $f^{-1}(x)$ and its domain; (c) sketch.

Q6. For $f(x) = x^2 - 3$, $x \geq 0$: (a) state the range of f ; (b) find $f^{-1}(x)$ and its domain; (c) sketch.

Unscaffolded

Find the inverse function f^{-1} , stating its domain. (Its graph is the reflection of f in the line $y = x$.)

Q7. $f(x) = 2x + 7$

Q8. $f(x) = 5 - x$

Q9. $f(x) = \frac{x+1}{4}$

Q10. $f(x) = -3x + 2$

Q11. $f(x) = \frac{x}{2} + 3$

Q12. $f(x) = \sqrt{x}$, $x \geq 0$

Q13. $f(x) = \sqrt{x-3}$, $x \geq 3$

Q14. $f(x) = x^2 + 2$, $x \geq 0$

Q15. $f(x) = (x-1)^2$, $x \geq 1$

Q16. $f(x) = \frac{1}{x}$, $x \neq 0$

Q17. Explain why $f(x) = x^2$, $x \in \mathbb{R}$, has no inverse function. Give a restricted domain on which the inverse exists, and find it.

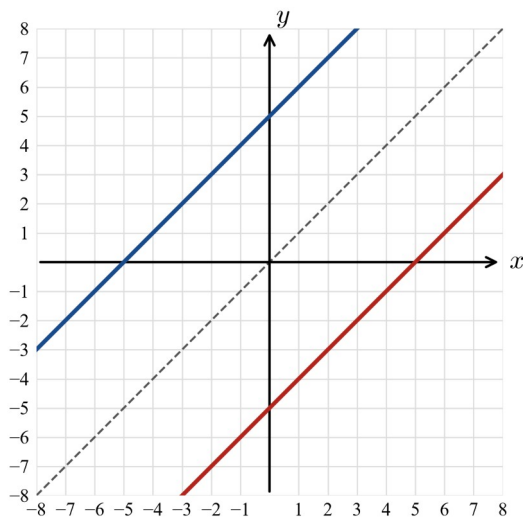
Q18. $f(x) = \frac{2x-1}{3}$

Answers

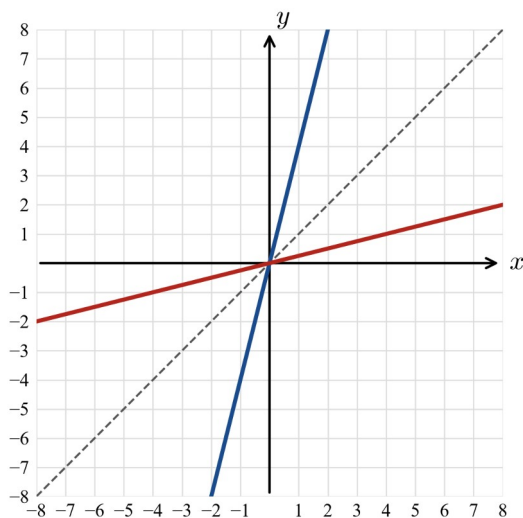
Q1. $f^{-1}(x) = x - 5$. **Q2.** $f^{-1}(x) = \frac{x}{4}$; domain and range $\mathbb{R}$. **Q3.** $f^{-1}(x) = \frac{x+1}{3}$. **Q4.** $f^{-1}(x) = 2x + 4$. **Q5.** range of f is $[0, \infty)$; $f^{-1}(x) = x^2 - 2$, domain $[0, \infty)$. **Q6.** range of f is $[-3, \infty)$; $f^{-1}(x) = \sqrt{x+3}$, domain $[-3, \infty)$. **Q7.** $f^{-1}(x) = \frac{x-7}{2}$. **Q8.** $f^{-1}(x) = 5 - x$ (self-inverse). **Q9.** $f^{-1}(x) = 4x - 1$. **Q10.** $f^{-1}(x) = \frac{2-x}{3}$. **Q11.** $f^{-1}(x) = 2x - 6$. **Q12.** $f^{-1}(x) = x^2$, domain $[0, \infty)$. **Q13.** $f^{-1}(x) = x^2 + 3$, domain $[0, \infty)$. **Q14.** $f^{-1}(x) = \sqrt{x-2}$, domain $[2, \infty)$. **Q15.** $f^{-1}(x) = 1 + \sqrt{x}$, domain $[0, \infty)$. **Q16.** $f^{-1}(x) = \frac{1}{x}$ (self-inverse), domain $\mathbb{R} \setminus \{0\}$. **Q17.** Not one-to-one on $\mathbb{R}$ (fails the horizontal line test); on $[0, \infty)$ the inverse is $f^{-1}(x) = \sqrt{x}$. **Q18.** $f^{-1}(x) = \frac{3x+1}{2}$.

Fully-worked solutions

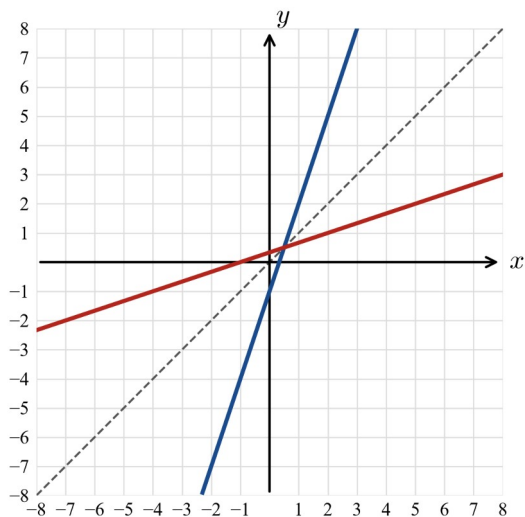
Q1. (a) f is a non-constant linear function, so it is one-to-one and the inverse exists. (b) $y = x + 5 \Rightarrow x = y + 5 \Rightarrow y = x - 5$, so $f^{-1}(x) = x - 5$. (c)



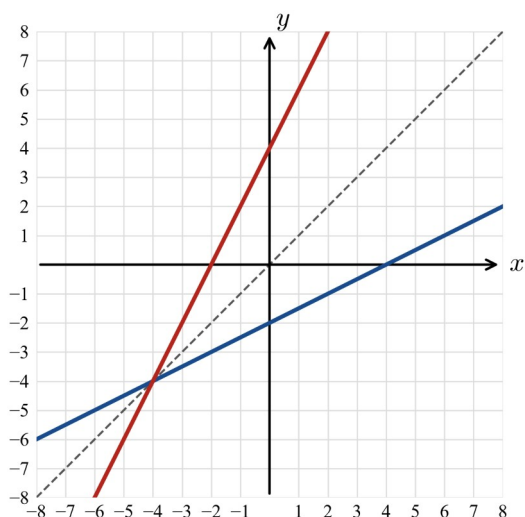
Q2. (a) $y = 4x \Rightarrow x = \frac{y}{4} \Rightarrow y = \frac{x}{4}$, so $f^{-1}(x) = \frac{x}{4}$. (b) Domain and range both $\mathbb{R}$. (c)



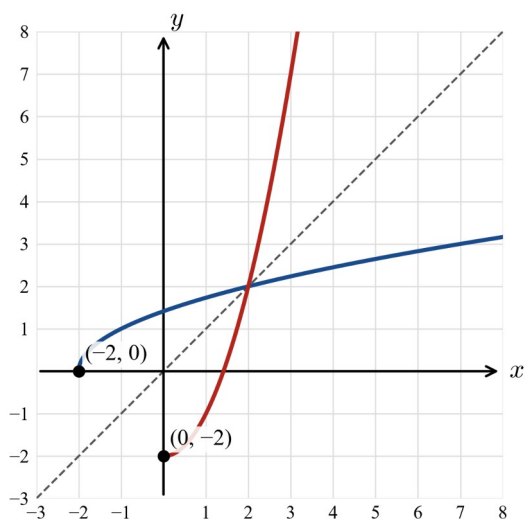
Q3. (a) $y = 3x - 1 \Rightarrow x = \frac{y+1}{3} \Rightarrow y = \frac{x+1}{3}$, so $f^{-1}(x) = \frac{x+1}{3}$. (b) $f(f^{-1}(x)) = 3 \cdot \frac{x+1}{3} - 1 = (x+1) - 1 = x$. ✓ (c)



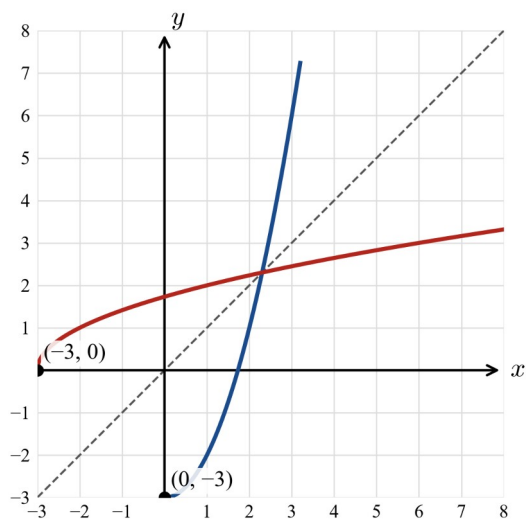
Q4. $y = \frac{x-4}{2} \Rightarrow x = \frac{y-4}{2} \Rightarrow 2x = y-4 \Rightarrow y = 2x+4$, so $f^{-1}(x) = 2x+4$.



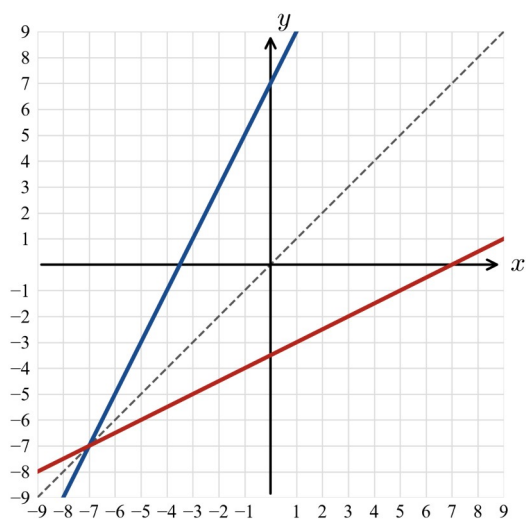
Q5. (a) Range of f is $[0, \infty)$. (b) $y = \sqrt{x+2} \Rightarrow x = \sqrt{y+2} \Rightarrow x^2 = y+2 \Rightarrow y = x^2 - 2$, so $f^{-1}(x) = x^2 - 2$, domain $[0, \infty)$.



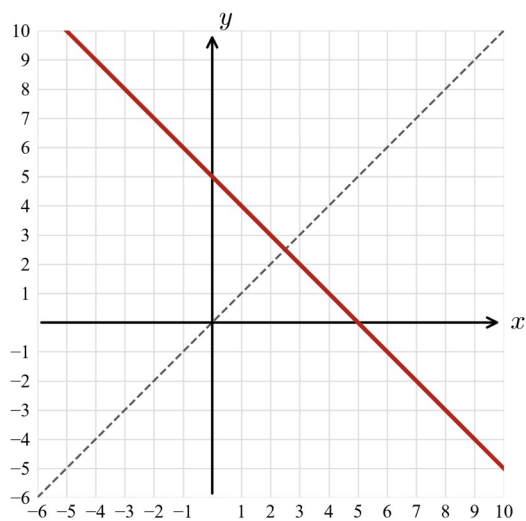
Q6. (a) Range of f is $[-3, \infty)$. (b) $y = x^2 - 3 \Rightarrow x = \sqrt{y+3} \Rightarrow y = \sqrt{x+3}$ (positive root, since range of $f^{-1} = [0, \infty)$), so $f^{-1}(x) = \sqrt{x+3}$, domain $[-3, \infty)$.



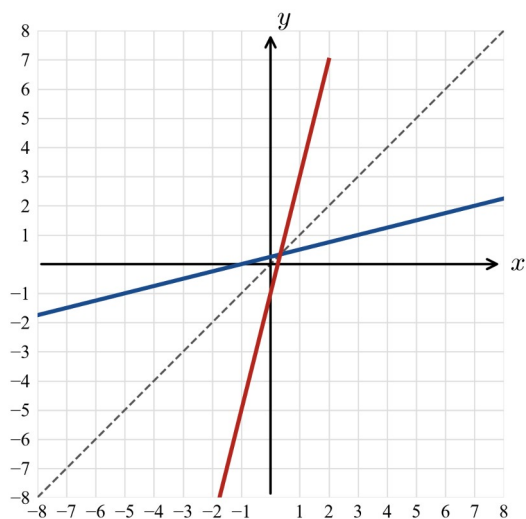
Q7. $y = 2x + 7 \Rightarrow x = 2y + 7 \Rightarrow y = \frac{x-7}{2}$, so $f^{-1}(x) = \frac{x-7}{2}$, domain $\mathbb{R}$.



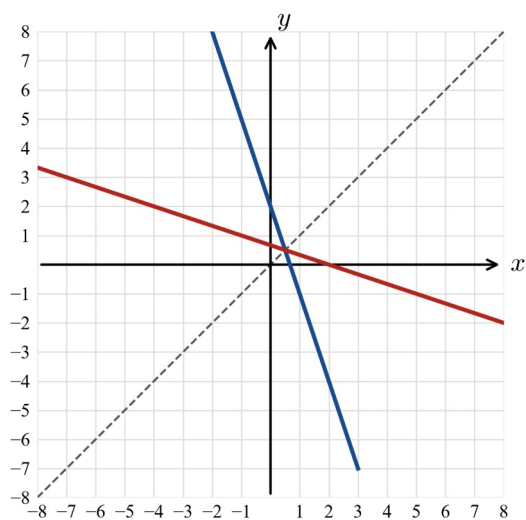
Q8. $y = 5 - x \Rightarrow x = 5 - y \Rightarrow y = 5 - x$, so $f^{-1}(x) = 5 - x$. The rule is its **own inverse**, so its graph is symmetric about $y = x$.



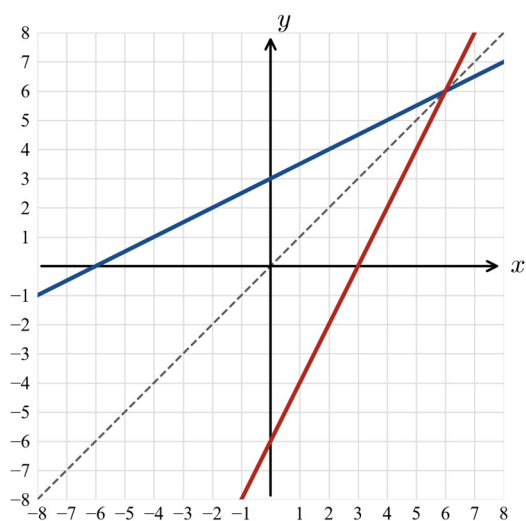
Q9. $y = \frac{x+1}{4} \Rightarrow x = \frac{y+1}{4} \Rightarrow 4x = y+1 \Rightarrow y = 4x - 1$, so $f^{-1}(x) = 4x - 1$.



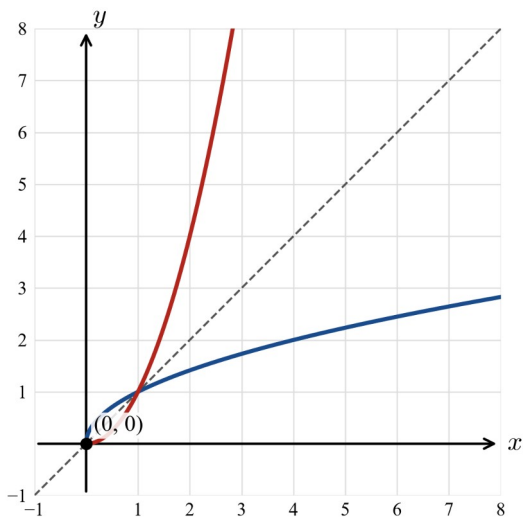
Q10. $y = -3x + 2 \Rightarrow x = -3y + 2 \Rightarrow 3y = 2 - x \Rightarrow y = \frac{2-x}{3}$, so $f^{-1}(x) = \frac{2-x}{3}$.



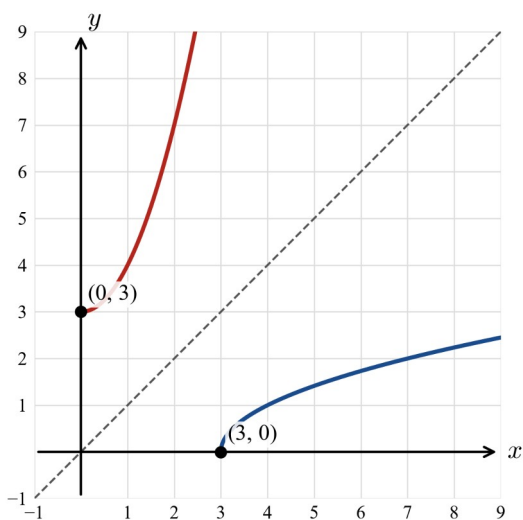
Q11. $y = \frac{x}{2} + 3 \Rightarrow x = \frac{y}{2} + 3 \Rightarrow \frac{y}{2} = x - 3 \Rightarrow y = 2x - 6$, so $f^{-1}(x) = 2x - 6$.



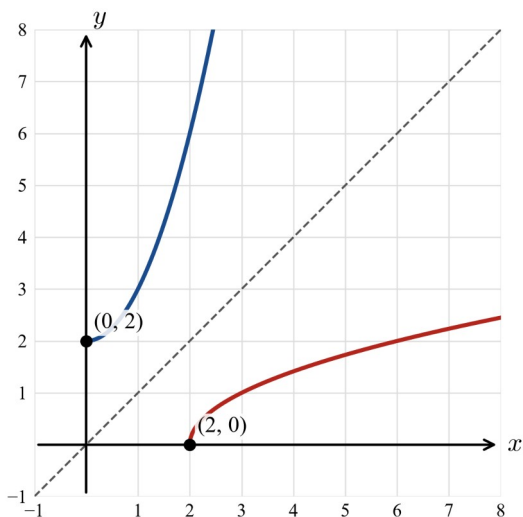
Q12. $y = \sqrt{x} \Rightarrow x = \sqrt{y} \Rightarrow y = x^2$, so $f^{-1}(x) = x^2$ with domain $[0, \infty)$ (the range of f).



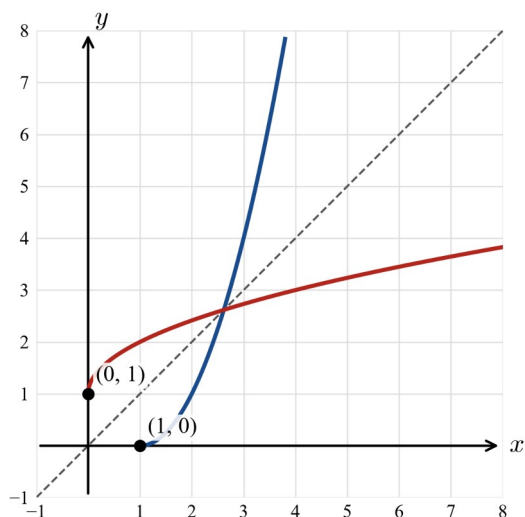
Q13. $y = \sqrt{x-3} \Rightarrow x = \sqrt{y-3} \Rightarrow x^2 = y-3 \Rightarrow y = x^2+3$, so $f^{-1}(x) = \sqrt{x-3}$, domain $[3, \infty)$.



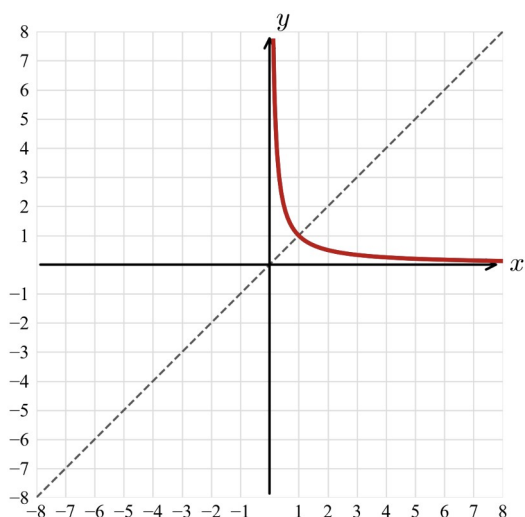
Q14. $y = x^2+2 \Rightarrow x = \sqrt{y-2}$ (positive root), so $f^{-1}(x) = \sqrt{x-2}$, domain $[2, \infty)$.



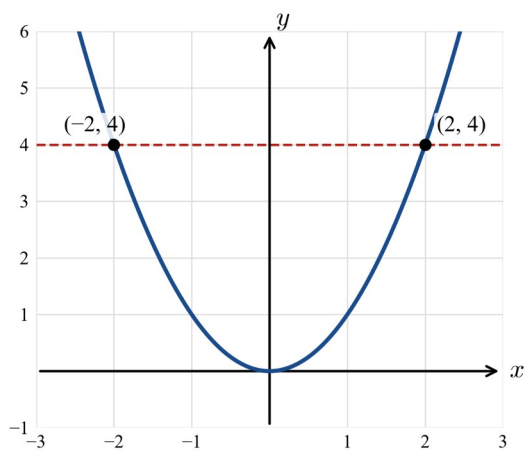
Q15. $y = (x-1)^2 \Rightarrow x = (y-1)^2 \Rightarrow \sqrt{x} = y-1$ (positive root, since $y \geq 1$) $\Rightarrow y = 1 + \sqrt{x}$, so $f^{-1}(x) = 1 + \sqrt{x}$, domain $[0, \infty)$.



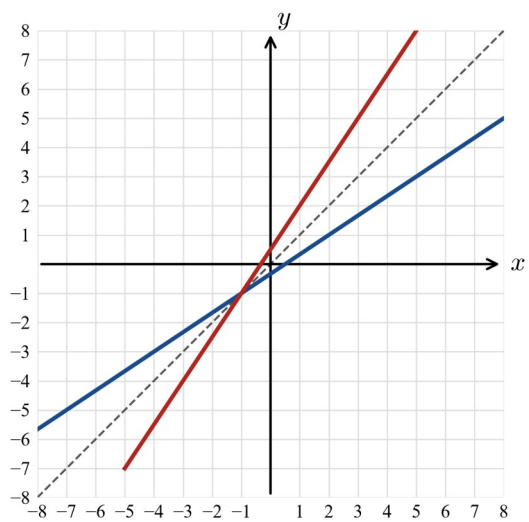
Q16. $y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$, so $f^{-1}(x) = \frac{1}{x}$ — the rule is its own inverse, domain $\mathbb{R} \setminus \{0\}$.



Q17. $f(x) = x^2$ on $\mathbb{R}$ is **not one-to-one**: for example $f(-2) = f(2) = 4$, so the horizontal line $y = 4$ meets the graph twice. It therefore has no inverse function. Restricting the domain to $[0, \infty)$ makes f one-to-one, and then $y = x^2 \Rightarrow x = \sqrt{y}$ gives $f^{-1}(x) = \sqrt{x}$, domain $[0, \infty)$.



Q18. $y = \frac{2x-1}{3} \Rightarrow x = \frac{2y-1}{3} \Rightarrow 3x = 2y-1 \Rightarrow y = \frac{3x+1}{2}$, so $f^{-1}(x) = \frac{3x+1}{2}$, domain $\mathbb{R}$.



Theory

A **model** uses a function to describe a real relationship between two quantities. Modelling a situation well involves four ideas.

- **Choose the function type from the situation.** A constant rate of change gives a **linear** model $y = mx + c$; an area or a projectile height gives a **quadratic** model; a volume built from a length gives a **cubic** model.
- **Restrict the domain to the context.** The mathematical rule may be defined for all real numbers, but the model only makes sense for some values — e.g. a time $t \geq 0$, a length $x > 0$, or a side that cannot exceed the material available. Always state the **sensible (natural) domain**.
- **Interpret in context.** The value $f(a)$, the domain and the range all carry meaning and **units**. The gradient of a linear model is a *rate*; the y -intercept is often an *initial* or *fixed* value; the turning point of a quadratic model gives a *maximum* or *minimum*.
- **Piecewise models** describe situations with a changing rule — overtime pay, postage by weight, or tax by income bracket — using a different expression on each part of the domain.

Worked examples

Example 1 — scaffolded

A farmer has 40 m of fencing to make a rectangular paddock. Let one side be x m. Find a rule for the area A , state a sensible domain, and find the dimensions giving the greatest area.

Working

Comment

Two sides have length x , using $2x$ m; the other two sides share the remaining $40 - 2x$ m, so each is

The perimeter is 40 m.

$$\frac{40 - 2x}{2} = 20 - x \text{ m.}$$

$$A(x) = x(20 - x).$$

Area = length $\times$ width.

Domain: $0 < x < 20$.

Both x and $20 - x$ must be positive.

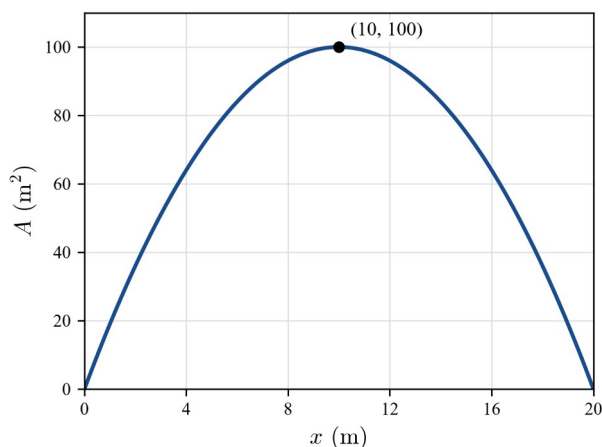
$A(x) = 20x - x^2$ is a parabola with maximum at

Maximum at the turning point.

$$x = -\frac{b}{2a} = \frac{20}{2} = 10.$$

$A(10) = 10(10) = 100$. $\therefore$ the greatest area is 100 m^2 , a $10 \text{ m} \times 10 \text{ m}$ square.

Interpret in context.



Example 2 — scaffolded

A worker is paid \$22 per hour for the first 38 hours worked in a week, and \$33 per hour for each hour beyond that. Let h be the hours worked and $P(h)$ the weekly pay in dollars. Find $P(h)$ and evaluate $P(45)$.

Working

Comment

For $0 \leq h \leq 38$: $P(h) = 22h$.

Ordinary rate.

At $h = 38$, $P = 22 \times 38 = 836$.

The pay for a full ordinary week is \$836.

For $h > 38$: \$836 plus \$33 for each of the $(h - 38)$ overtime hours: $P(h) = 836 + 33(h - 38)$.

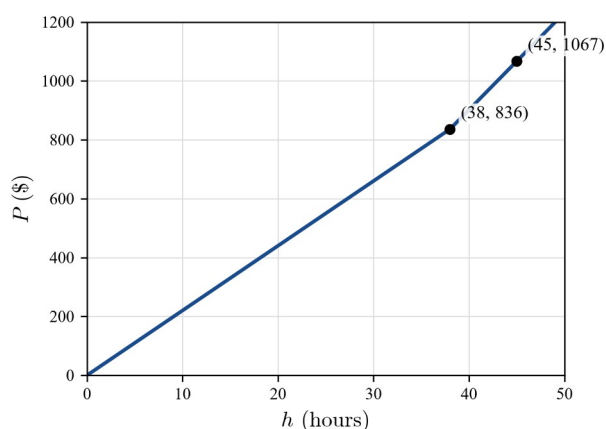
Overtime is added on top.

$$P(h) = \begin{cases} 22h & 0 \leq h \leq 38 \\ 836 + 33(h - 38) & h > 38 \end{cases}$$

A piecewise model.

$P(45) = 836 + 33(7) = 836 + 231 = 1067$. $\therefore$ the pay is \$1067.

$45 > 38$, so use the second rule.



Example 3 — unscaffolded

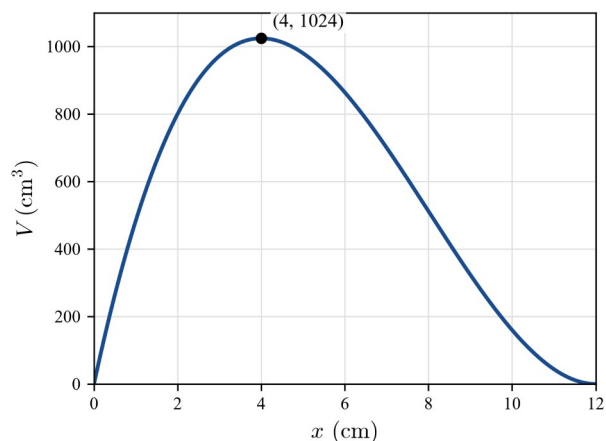
An open box is made from a 24 cm square of cardboard by cutting a square of side x cm from each corner and folding up the sides. Find a rule for the volume V , state a sensible domain, and find $V(4)$.

The base is a square of side $24 - 2x$ cm and the height is x cm, so

$$V(x) = x(24 - 2x)^2.$$

Domain: $0 < x < 12$ (the side $24 - 2x$ must be positive).

$$V(4) = 4(24 - 8)^2 = 4(16)^2 = 4 \times 256 = 1024. \therefore V(4) = 1024 \text{ cm}^3.$$

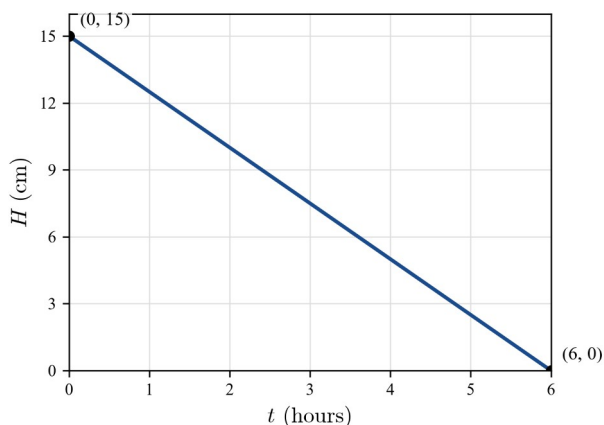


Example 4 — unscaffolded

A candle is 15 cm tall and burns down at a constant 2.5 cm per hour. Let $H(t)$ be its height after t hours. Find $H(t)$, a sensible domain, and the height after 4 hours.

$H(t) = 15 - 2.5t$. The candle burns out when $H = 0$: $15 - 2.5t = 0 \Rightarrow t = 6$, so the domain is $0 \leq t \leq 6$.

$H(4) = 15 - 2.5(4) = 15 - 10 = 5$. $\therefore$ after 4 hours the candle is 5 cm tall.



Practice questions

Scaffolded

Q1. A rectangular garden is to be enclosed with 100 m of fencing. Let one side be x m. (a) Show that the area is $A(x) = x(50 - x)$. (b) State a sensible domain. (c) Find the dimensions giving the greatest area and that area. (d) Sketch A against x .

Q2. A courier charges a fixed \$8 plus \$1.50 per kilogram. Let w be the mass in kg and $C(w)$ the cost in dollars. (a) Write a rule for $C(w)$ and state the domain. (b) Find $C(12)$. (c) Interpret the numbers 8 and 1.5 in the context.

Q3. A ball is thrown and its height (m) after t seconds is $h(t) = 25t - 5t^2$. (a) State a sensible domain and the time the ball lands. (b) Find the maximum height and when it occurs. (c) Sketch the graph.

Q4. A phone plan costs \$20 per month plus \$0.10 per call minute. Let m be the minutes used and $C(m)$ the monthly cost in dollars. (a) Write a rule for $C(m)$. (b) Find $C(300)$. (c) How many minutes give a cost of \$35?

Q5. Postage is \$2 for parcels up to 250 g and \$4 for parcels over 250 g and up to 500 g. (a) Write $P(w)$ as a piecewise function. (b) Find $P(200)$ and $P(400)$. (c) Explain why P is not a smooth (continuous) function.

Q6. The area of a circle is a function of its radius r cm. (a) Write the rule $A(r)$. (b) State the domain. (c) Find $A(4)$ as an exact value.

Unscaffolded

For each situation, form the model, state a sensible domain, and answer the question.

Q7. A rectangular pen against no walls is enclosed by 60 m of fencing, with one side x m. Find the dimensions giving the greatest area, and that area.

Q8. An open box is made from a 30 cm square by cutting a square of side x cm from each corner. Find the volume rule and evaluate the volume when $x = 5$ cm.

Q9. A stone is thrown so its height (m) after t seconds is $h(t) = 30t - 5t^2$. Find the maximum height and the time the stone lands.

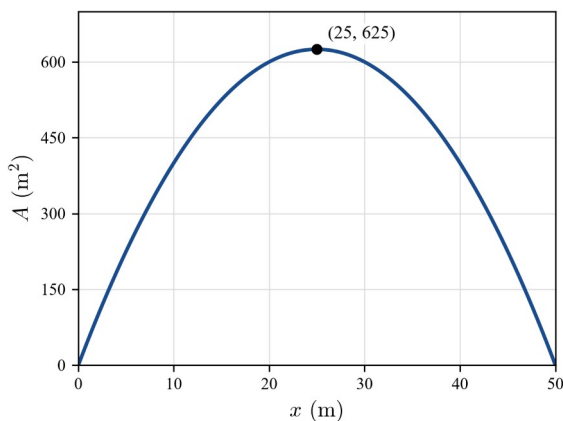
- Q10.** A taxi charges a \$5 flagfall plus \$2.50 per km. Find the cost of an 8 km trip, and the distance for which the cost is \$30.
- Q11.** A worker earns \$20 per hour for the first 40 hours and \$30 per hour beyond that. Find the weekly pay for 45 hours.
- Q12.** A reservoir holds 500 ML and is drained at 20 ML per day. Find when it holds 100 ML, and when it is empty.
- Q13.** A business has costs $C(n) = 2000 + 15n$ dollars and revenue $R(n) = 40n$ dollars for n items. Find the break-even number of items.
- Q14.** A circle has area 36π cm². Find its radius.
- Q15.** Income tax is \$0 on the first \$18,200 of income and 19% of every dollar above \$18,200. Find the tax on an income of \$25,000.
- Q16.** The volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$ cm³. Find the exact volume when $r = 3$ cm.
- Q17.** A cup of coffee cools so its temperature (°C) after t minutes is $T(t) = 80 - 6t$. Find the temperature after 5 minutes and when it reaches 20°C.
- Q18.** A rectangular pen is built against a straight wall, so only three sides need fencing. With 60 m of fencing and the two ends each x m, find the dimensions giving the greatest area.
-

Answers

Q1. (b) $0 < x < 50$; (c) $25 \text{ m} \times 25 \text{ m}$, area 625 m^2 . **Q2.** (a) $C(w) = 8 + 1.5w$, $w \geq 0$; (b) \$26; (c) \$8 fixed charge, \$1.50 per kg rate. **Q3.** (a) $[0, 5]$, lands at $t = 5 \text{ s}$; (b) 31.25 m at $t = 2.5 \text{ s}$. **Q4.** (a) $C(m) = 20 + 0.1m$; (b) \$50; (c) 150 min. **Q5.** (b) \$2, \$4; (c) it jumps at 250 g . **Q6.** (a) $A(r) = \pi r^2$; (b) $r > 0$; (c) $16\pi \text{ cm}^2$. **Q7.** $15 \text{ m} \times 15 \text{ m}$, 225 m^2 . **Q8.** $V(x) = x(30 - 2x)^2$, $V(5) = 2000 \text{ cm}^3$. **Q9.** 45 m at $t = 3 \text{ s}$; lands at $t = 6 \text{ s}$. **Q10.** \$25; 10 km . **Q11.** \$950. **Q12.** $t = 20 \text{ days}$; empty at $t = 25 \text{ days}$. **Q13.** 80 items. **Q14.** $r = 6 \text{ cm}$. **Q15.** \$1292. **Q16.** $36\pi \text{ cm}^3$. **Q17.** 50°C ; $t = 10 \text{ min}$. **Q18.** $15 \text{ m} \times 30 \text{ m}$, 450 m^2 .

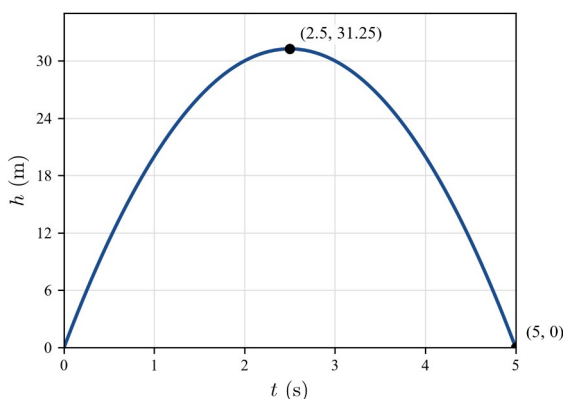
Fully-worked solutions

Q1. (a) The side opposite x is also x ; the other pair share $100 - 2x \text{ m}$, each $100 - 2x/2 = 50 - x$. So $A(x) = x(50 - x)$. (b) $0 < x < 50$. (c) Maximum at $x = 50/2 = 25$, giving $A(25) = 25(25) = 625$. The greatest area is 625 m^2 for a $25 \text{ m} \times 25 \text{ m}$ square.



Q2. (a) $C(w) = 8 + 1.5w$, $w \geq 0$. (b) $C(12) = 8 + 1.5(12) = 8 + 18 = 26$, i.e. \$26. (c) The \$8 is the fixed charge (the cost for 0 kg); the \$1.50 is the rate — the extra cost per kilogram (the gradient).

Q3. (a) $h \geq 0$ requires $25t - 5t^2 \geq 0 \Rightarrow 5t(5 - t) \geq 0$, so $0 \leq t \leq 5$; the ball lands when $h = 0$ at $t = 5 \text{ s}$. (b) Maximum at $t = 25/2 \times 5 = 2.5$, $h(2.5) = 25(2.5) - 5(2.5)^2 = 62.5 - 31.25 = 31.25 \text{ m}$.

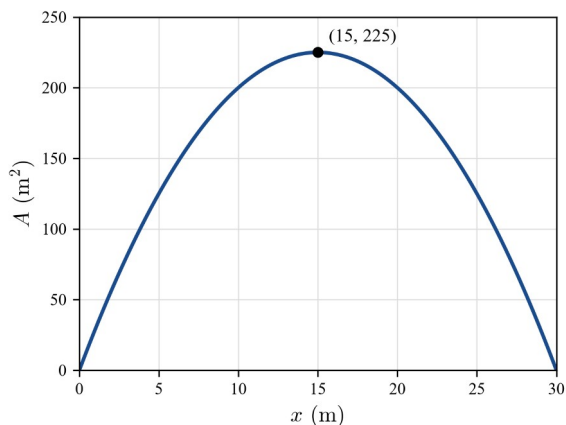


Q4. (a) $C(m) = 20 + 0.1m$. (b) $C(300) = 20 + 0.1(300) = 20 + 30 = 50$, i.e. \$50. (c) $20 + 0.1m = 35 \Rightarrow 0.1m = 15 \Rightarrow m = 150$ minutes.

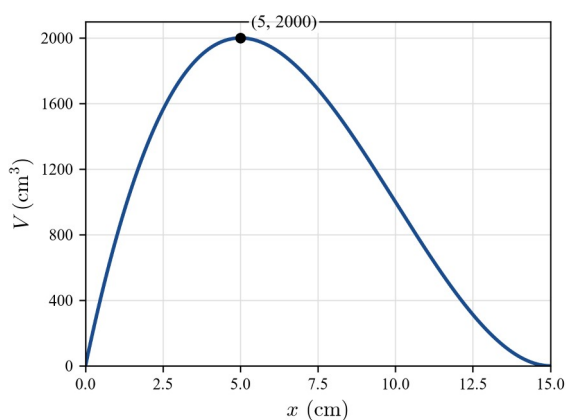
Q5. (a) $P(w) = \begin{cases} 2 & 0 < w \leq 250 \\ 4 & 250 < w \leq 500 \end{cases}$ (w in grams). (b) $P(200) = 2$ and $P(400) = 4$, i.e. \$2 and \$4. (c) The cost jumps suddenly from \$2 to \$4 at $w = 250 \text{ g}$, so the graph has a break — it is a step (piecewise) function, not a smooth curve.

Q6. (a) $A(r) = \pi r^2$. (b) $r > 0$. (c) $A(4) = \pi(4)^2 = 16\pi \text{ cm}^2$.

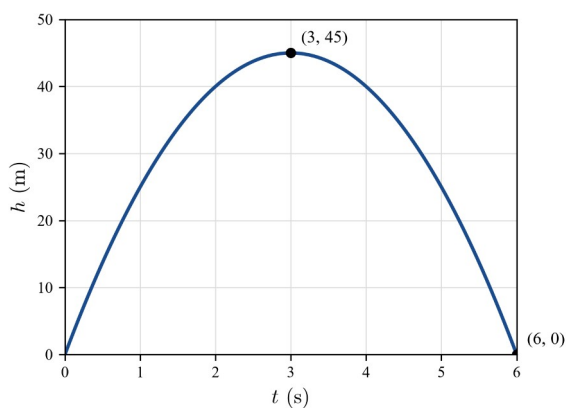
Q7. With x and $30 - x$ as the sides, $A(x) = x(30 - x)$, domain $0 < x < 30$. Maximum at $x = 15$:
 $A(15) = 15(15) = 225$. $\therefore 15 \text{ m} \times 15 \text{ m}$ gives the greatest area, 225 m^2 .



Q8. $V(x) = x(30 - 2x)^2$, domain $0 < x < 15$. $V(5) = 5(30 - 10)^2 = 5(20)^2 = 5 \times 400 = 2000$. $\therefore V(5) = 2000 \text{ cm}^3$.

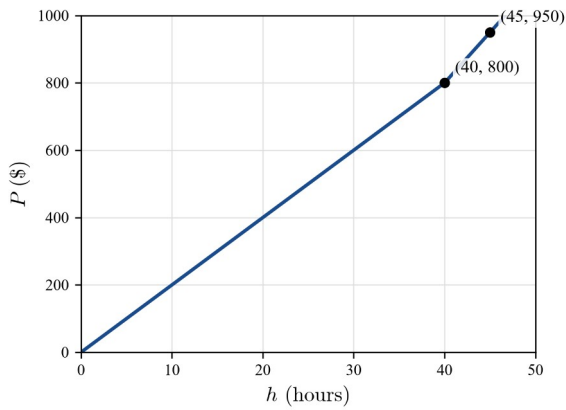


Q9. $h(t) = 30t - 5t^2$, domain $[0, 6]$ (since $5t(6 - t) \geq 0$). Maximum at $t = 30/2 \times 5 = 3$, $h(3) = 90 - 45 = 45 \text{ m}$.
 Lands when $h = 0$ at $t = 6 \text{ s}$.

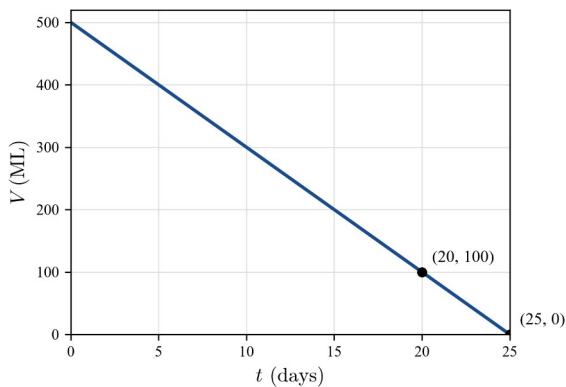


Q10. $C(d) = 5 + 2.5d$, $d \geq 0$. $C(8) = 5 + 2.5(8) = 5 + 20 = 25$, i.e. \$25. $5 + 2.5d = 30 \Rightarrow 2.5d = 25 \Rightarrow d = 10 \text{ km}$.

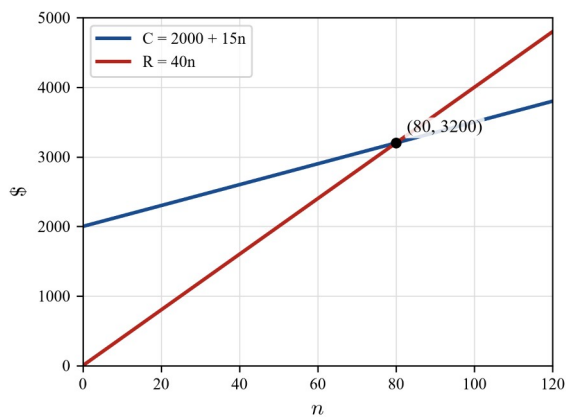
Q11. $P(h) = \begin{cases} 20h & 0 \leq h \leq 40 \\ 800 + 30(h - 40) & h > 40 \end{cases}$ Since $45 > 40$: $P(45) = 800 + 30(5) = 800 + 150 = 950$, i.e. \$950.



Q12. $V(t) = 500 - 20t$, domain $0 \leq t \leq 25$. $V = 100$: $500 - 20t = 100 \Rightarrow 20t = 400 \Rightarrow t = 20$ days. Empty when $V = 0$: $t = 25$ days.



Q13. Break-even is where $R(n) = C(n)$: $40n = 2000 + 15n \Rightarrow 25n = 2000 \Rightarrow n = 80$. $\therefore$ the business breaks even at 80 items (both cost and revenue = \$3200).



Q14. $\pi r^2 = 36\pi \Rightarrow r^2 = 36 \Rightarrow r = 6$ (taking the positive root, since $r > 0$). $\therefore r = 6$ cm.

Q15. Tax = 19% of the amount above \$18,200: $0.19 \times (25\,000 - 18\,200) = 0.19 \times 6800 = 1292$. $\therefore$ the tax is \$1292.

Q16. $V(3) = \frac{4}{3}\pi(3)^3 = \frac{4}{3}\pi \times 27 = 36\pi$ cm³.

Q17. $T(5) = 80 - 6(5) = 80 - 30 = 50^\circ\text{C}$. $T = 20$: $80 - 6t = 20 \Rightarrow 6t = 60 \Rightarrow t = 10$ min.

Q18. The wall replaces one long side, so the fencing covers two ends of x m and one side of $60 - 2x$ m:
 $A(x) = x(60 - 2x)$, domain $0 < x < 30$. Maximum at $x = 15$: $A(15) = 15(60 - 30) = 15 \times 30 = 450$. $\therefore$ the ends are 15 m and the long side 30 m, area 450 m².

