Mathematical Methods Units 1 & 2

Chapter 4 — Graphs of power functions and relations

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Theory

The basic rectangular hyperbola is the graph of

$$y=\frac{1}{x}.$$

It has two branches (in the first and third quadrants), and it approaches — but never touches — the two axes. The x-axis ($y=0$) and the y-axis ($x=0$) are its **asymptotes**. The domain is $\mathbb{R}\setminus\{0\}$ and the range is $\mathbb{R}\setminus\{0\}$; there are no axis intercepts.

The general rectangular hyperbola is obtained from $y=\frac{1}{x}$ by dilation and translation:

$$y=\frac{a}{x-h}+k.$$

- **Vertical asymptote** $x=h$ (the value of x that makes the denominator zero).
- **Horizontal asymptote** $y=k$ (the value y approaches as $x\to\pm\infty$).
- **Domain** $\mathbb{R}\setminus\{h\}$; **range** $\mathbb{R}\setminus\{k\}$.
- $|a|$ is a dilation factor; if $a<0$ the graph is reflected, so the branches lie in the opposite pair of regions formed by the asymptotes.

Intercepts. The y-intercept is found by putting $x=0$ (it exists unless $h=0$); an x-intercept is found by putting $y=0$ and solving.

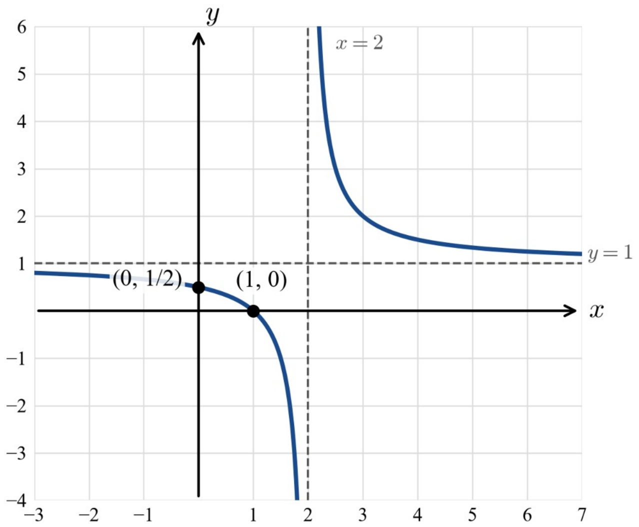
To sketch: draw the two asymptotes as dashed lines, find and mark the axis intercepts, then draw the two branches curving towards the asymptotes.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y=\frac{1}{x-2}+1$, stating the asymptotes, domain, range and axis intercepts.

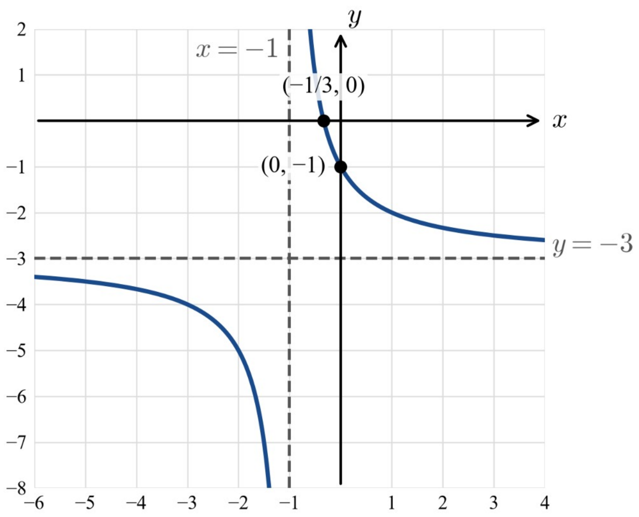
Working	Comment
Comparing with $y=\frac{a}{x-h}+k$: $a=1$, $h=2$, $k=1$.	Read h and k from the rule.
Vertical asymptote $x=2$; horizontal asymptote $y=1$.	VA where the denominator is zero; HA is k.
Domain $\mathbb{R}\setminus\{2\}$; range $\mathbb{R}\setminus\{1\}$.	The graph never reaches the asymptote values.
y-intercept: $x=0\Rightarrow y=\frac{1}{-2}+1=\frac{1}{2}$, giving $\left(0,\frac{1}{2}\right)$.	Put $x=0$.
x-intercept: $y=0\Rightarrow\frac{1}{x-2}=-1\Rightarrow x-2=-1$, so $x=1$, giving $(1,0)$.	Put $y=0$ and solve.



Example 2 — scaffolded

Sketch the graph of $y = \frac{2}{x+1} - 3$.

Working	Comment
$a = 2$, $h = -1$, $k = -3$.	Note $x + 1 = x - (-1)$, so $h = -1$.
Vertical asymptote $x = -1$; horizontal asymptote $y = -3$.	
Domain $\mathbb{R} \setminus \{-1\}$; range $\mathbb{R} \setminus \{-3\}$.	
y-intercept: $x = 0 \Rightarrow y = \frac{2}{1} - 3 = -1$, giving $(0, -1)$.	Put $x = 0$.
x-intercept: $\frac{2}{x+1} = 3 \Rightarrow x + 1 = \frac{2}{3}$, so $x = -\frac{1}{3}$, giving $\left(-\frac{1}{3}, 0\right)$.	Put $y = 0$ and solve.



Example 3 — unscaffolded

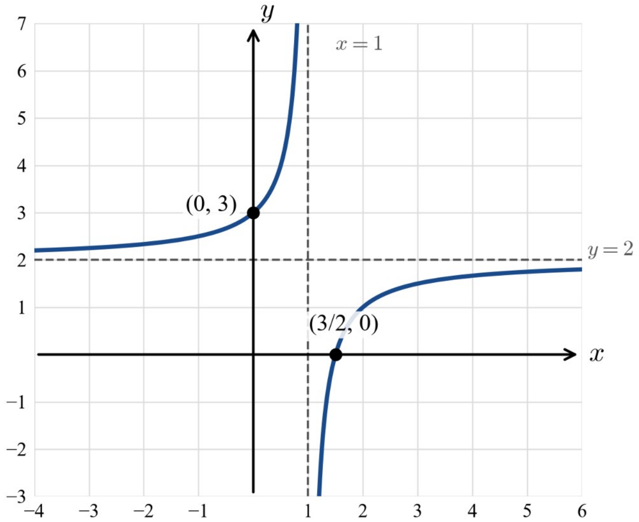
Sketch the graph of $y = -\frac{1}{x-1} + 2$.

Here $a = -1$, $h = 1$, $k = 2$. Vertical asymptote $x = 1$; horizontal asymptote $y = 2$. Domain $\mathbb{R} \setminus \{1\}$; range $\mathbb{R} \setminus \{2\}$.

y-intercept: $x=0 \Rightarrow y=-\dfrac{1}{-1}+2=1+2=3$, giving $(0,3)$.

x-intercept: $-\dfrac{1}{x-1}+2=0 \Rightarrow \dfrac{1}{x-1}=2 \Rightarrow x-1=\dfrac{1}{2}$, so $x=\dfrac{3}{2}$, giving $\left(\dfrac{3}{2},0\right)$.

Because $a<0$ the branches are reflected.



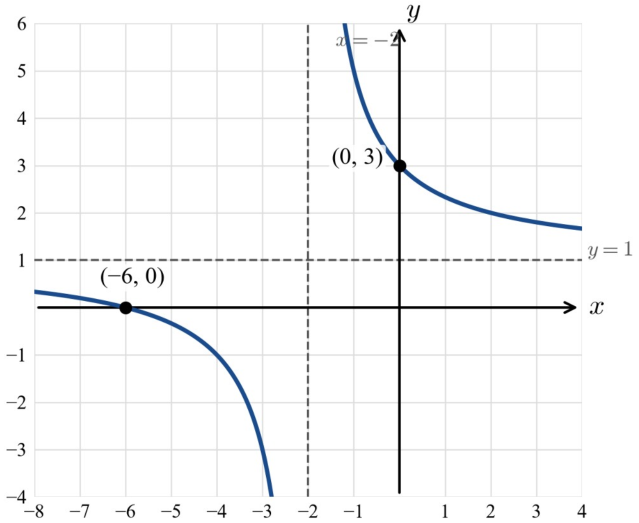
Example 4 — unscaffolded

A rectangular hyperbola has vertical asymptote $x=-2$, horizontal asymptote $y=1$, and passes through the point $(0,3)$. Find its rule and sketch the graph.

From the asymptotes the rule has the form $y=\dfrac{a}{x+2}+1$. Substituting $(0,3)$:

$$3=\frac{a}{0+2}+1 \Rightarrow \frac{a}{2}=2 \Rightarrow a=4.$$

$\therefore y=\dfrac{4}{x+2}+1$. The x-intercept is $x=-6$ and the y-intercept is $(0,3)$.



Practice questions

Scaffolded

Q1. For $y=\frac{1}{x-3}$: (a) state the asymptotes; (b) state the domain and range; (c) find the axis intercepts; (d) sketch the graph.

Q2. For $y=\frac{1}{x}+2$: (a) state the asymptotes; (b) state the domain and range; (c) find the axis intercepts; (d) sketch the graph.

Q3. For $y=\frac{3}{x-1}$: (a) state the asymptotes; (b) find the axis intercepts; (c) sketch the graph.

Q4. For $y=\frac{1}{x+2}-1$: (a) state the asymptotes; (b) state the domain and range; (c) find the axis intercepts; (d) sketch the graph.

Q5. For $y=-\frac{2}{x-1}$: (a) state the asymptotes; (b) explain how the $a=-2$ affects the graph; (c) find the axis intercepts; (d) sketch the graph.

Q6. For $y=\frac{2}{x-2}+3$: (a) state the asymptotes; (b) find the axis intercepts; (c) sketch the graph.

Unscaffolded

For each of **Q7–Q15** and **Q18**: state the equations of the asymptotes, the domain and range, and the axis intercepts, then sketch the graph.

Q7. $y=\frac{1}{x}$

Q8. $y=\frac{1}{x-4}$

Q9. $y=\frac{1}{x}+3$

Q10. $y=\frac{2}{x+1}$

Q11. $y=\frac{1}{x-2}-2$

Q12. $y=-\frac{1}{x+3}$

Q13. $y=-\frac{3}{x-1}+2$

Q14. $y=\frac{4}{x+2}-1$

Q15. $y=\frac{2}{x-3}+1$

Q16. A rectangular hyperbola has asymptotes $x=1$ and $y=-2$ and passes through the origin. Find its rule and sketch the graph.

Q17. A rectangular hyperbola has asymptotes $x = -1$ and $y = 3$ and passes through $(1, 1)$. Find its rule and sketch the graph.

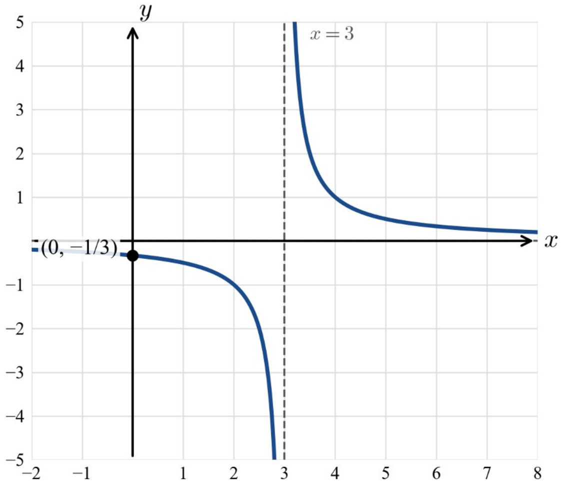
Q18. $y = \frac{5}{x-2} + 4$

Answers

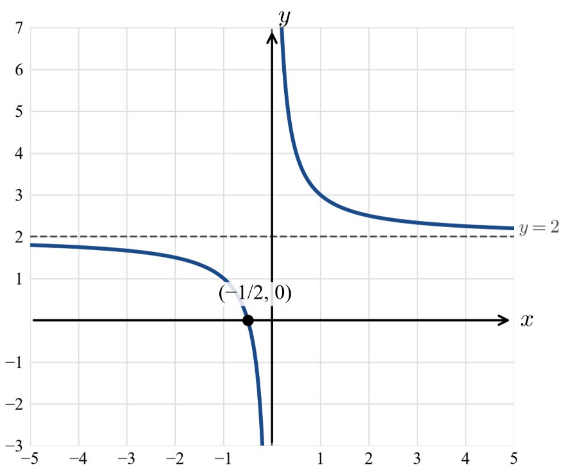
Q1. Asymptotes $x=3$, $y=0$; domain $\mathbb{R}\setminus\{3\}$, range $\mathbb{R}\setminus\{0\}$; no x-intercept, y-intercept $(0,-1/3)$. **Q2.** $x=0$, $y=2$; domain $\mathbb{R}\setminus\{0\}$, range $\mathbb{R}\setminus\{2\}$; x-intercept $(-1/2,0)$, no y-intercept. **Q3.** $x=1$, $y=0$; no x-intercept, y-intercept $(0,-3)$. **Q4.** $x=-2$, $y=-1$; domain $\mathbb{R}\setminus\{-2\}$, range $\mathbb{R}\setminus\{-1\}$; x-intercept $(-1,0)$, y-intercept $(0,-1/2)$. **Q5.** $x=1$, $y=0$; $a<0$ reflects the branches; no x-intercept, y-intercept $(0,2)$. **Q6.** $x=2$, $y=3$; x-intercept $(4/3,0)$, y-intercept $(0,2)$. **Q7.** $x=0$, $y=0$; domain & range $\mathbb{R}\setminus\{0\}$; no intercepts. **Q8.** $x=4$, $y=0$; no x-intercept, y-intercept $(0,-1/4)$. **Q9.** $x=0$, $y=3$; x-intercept $(-1/3,0)$, no y-intercept. **Q10.** $x=-1$, $y=0$; no x-intercept, y-intercept $(0,2)$. **Q11.** $x=2$, $y=-2$; x-intercept $(5/2,0)$, y-intercept $(0,-5/2)$. **Q12.** $x=-3$, $y=0$; no x-intercept, y-intercept $(0,-1/3)$. **Q13.** $x=1$, $y=2$; x-intercept $(5/2,0)$, y-intercept $(0,5)$. **Q14.** $x=-2$, $y=-1$; x-intercept $(2,0)$, y-intercept $(0,1)$. **Q15.** $x=3$, $y=1$; x-intercept $(1,0)$, y-intercept $(0,1/3)$. **Q16.** $y=-\dfrac{2}{x-1}-2$. **Q17.** $y=-\dfrac{4}{x+1}+3$. **Q18.** $x=2$, $y=4$; x-intercept $(3/4,0)$, y-intercept $(0,3/2)$.

Fully-worked solutions

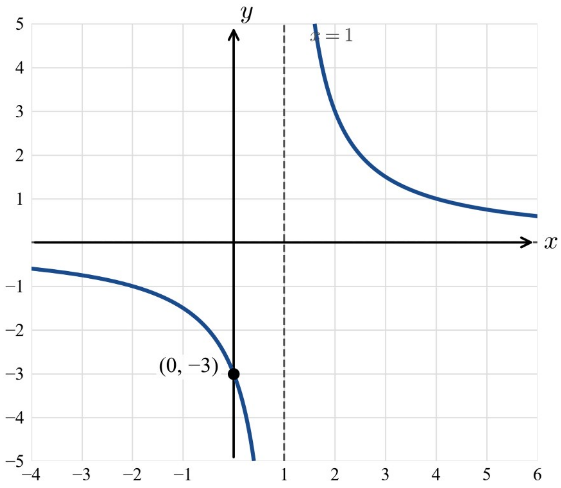
Q1. $y=\dfrac{1}{x-3}$. Asymptotes $x=3$, $y=0$; domain $\mathbb{R}\setminus\{3\}$, range $\mathbb{R}\setminus\{0\}$. $y=0$ has no solution (a non-zero constant over a linear can never be 0), so there is no x-intercept; y-intercept $x=0\Rightarrow y=\dfrac{1}{-3}=-\dfrac{1}{3}$.



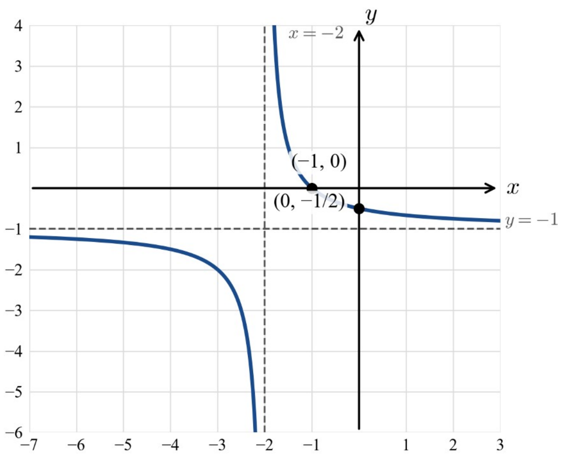
Q2. $y=\dfrac{1}{x}+2$. Asymptotes $x=0$, $y=2$; domain $\mathbb{R}\setminus\{0\}$, range $\mathbb{R}\setminus\{2\}$. x-intercept: $\dfrac{1}{x}=-2\Rightarrow x=-\dfrac{1}{2}$; no y-intercept (the y-axis is the vertical asymptote).



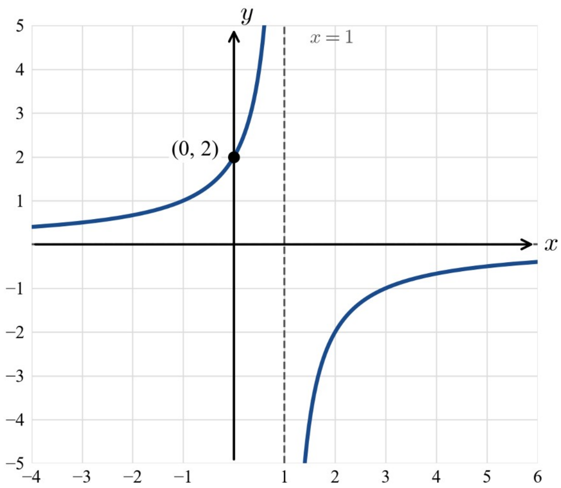
Q3. $y=\dfrac{3}{x-1}$. Asymptotes $x=1,\ y=0$. No x-intercept; y-intercept $x=0 \Rightarrow y=\dfrac{3}{-1}=-3$.



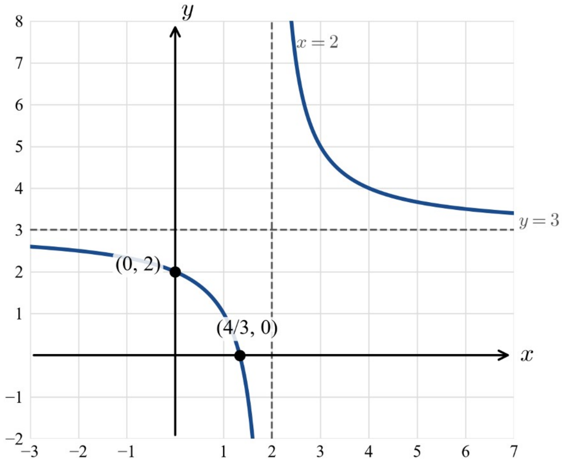
Q4. $y=\dfrac{1}{x+2}-1$. Asymptotes $x=-2,\ y=-1$; domain $\mathbb{R}\setminus\{-2\}$, range $\mathbb{R}\setminus\{-1\}$. x-intercept: $\dfrac{1}{x+2}=1 \Rightarrow x+2=1$, so $x=-1$; y-intercept $x=0 \Rightarrow \dfrac{1}{2}-1=-\dfrac{1}{2}$.



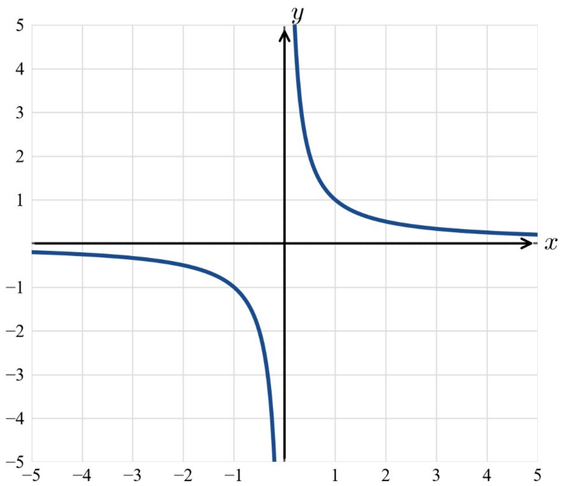
Q5. $y=-\dfrac{2}{x-1}$. Asymptotes $x=1,\ y=0$. Since $a=-2<0$, the branches are reflected (compared with $y=2/x-1$) and wider than $y=1/x-1$. No x-intercept; y-intercept $x=0 \Rightarrow y=-\dfrac{2}{-1}=2$.



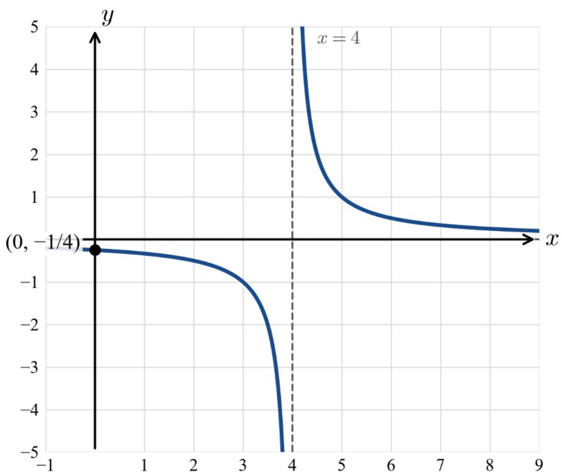
Q6. $y=\dfrac{2}{x-2}+3$. Asymptotes $x=2,\ y=3$. x-intercept: $\dfrac{2}{x-2}=-3 \Rightarrow x-2=-\dfrac{2}{3}$, so $x=\dfrac{4}{3}$; y-intercept $x=0 \Rightarrow \dfrac{2}{-2}+3=2$.



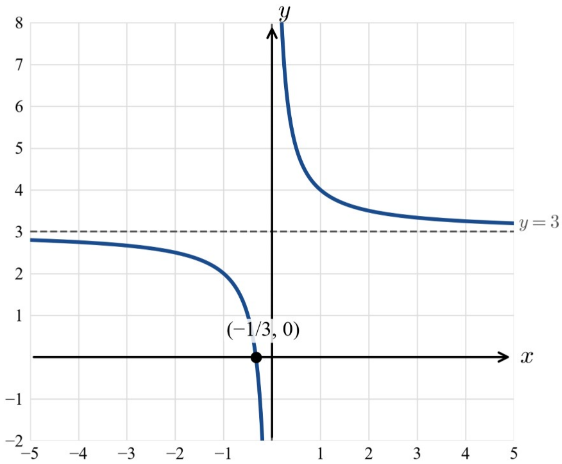
Q7. $y=\dfrac{1}{x}$. The basic rectangular hyperbola: asymptotes $x=0$ and $y=0$; domain and range $\mathbb{R}\setminus\{0\}$; no intercepts.



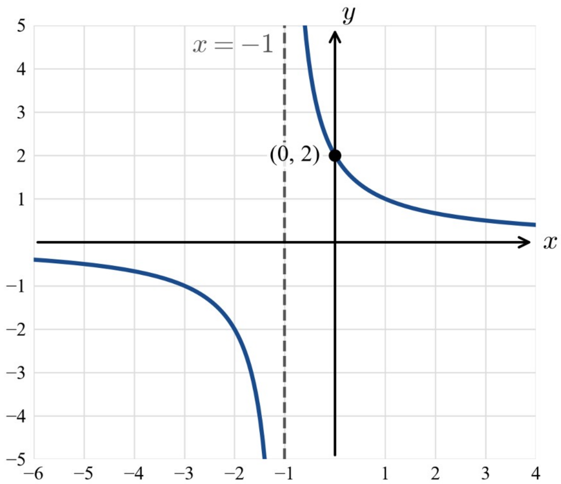
Q8. $y=\dfrac{1}{x-4}$. Asymptotes $x=4,\ y=0$. No x-intercept; y-intercept $x=0 \Rightarrow -\dfrac{1}{4}$.



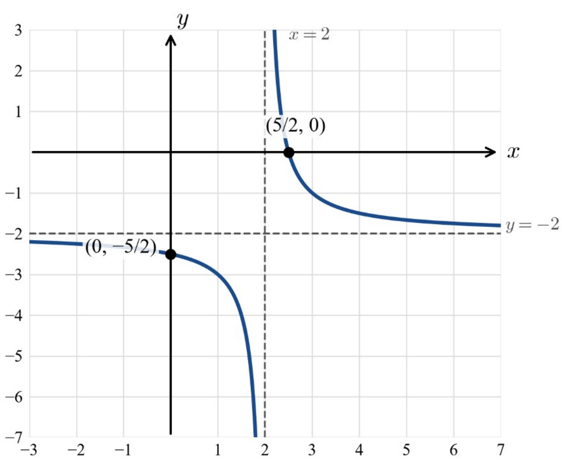
Q9. $y=\dfrac{1}{x}+3$. Asymptotes $x=0,\ y=3$. x-intercept: $\dfrac{1}{x}=-3 \Rightarrow x=-\dfrac{1}{3}$; no y-intercept.



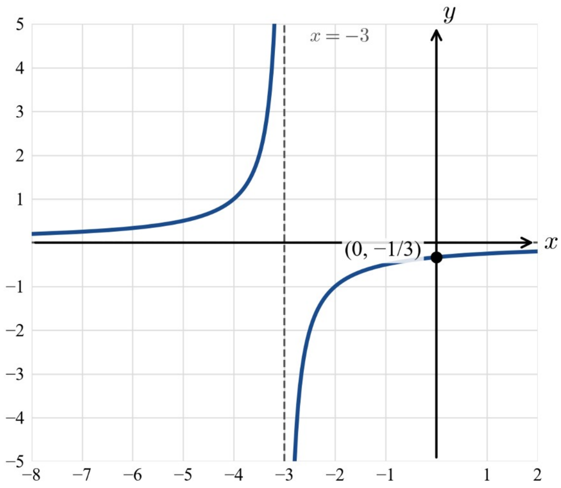
Q10. $y = \dfrac{2}{x+1}$. Asymptotes $x = -1$, $y = 0$. No x-intercept; y-intercept $x = 0 \Rightarrow \dfrac{2}{1} = 2$.



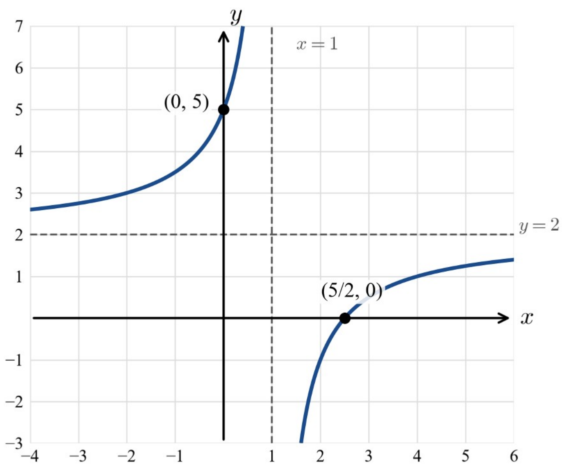
Q11. $y = \dfrac{1}{x-2} - 2$. Asymptotes $x = 2$, $y = -2$. x-intercept: $\dfrac{1}{x-2} = 2 \Rightarrow x - 2 = \dfrac{1}{2}$, so $x = \dfrac{5}{2}$; y-intercept $x = 0 \Rightarrow \dfrac{1}{-2} - 2 = -\dfrac{5}{2}$.



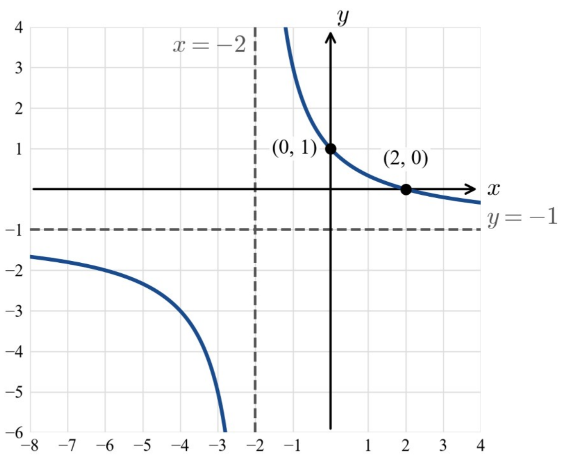
Q12. $y = -\dfrac{1}{x+3}$. Asymptotes $x = -3$, $y = 0$; $a < 0$ (reflected). No x-intercept; y-intercept $x = 0 \Rightarrow -\dfrac{1}{3}$.



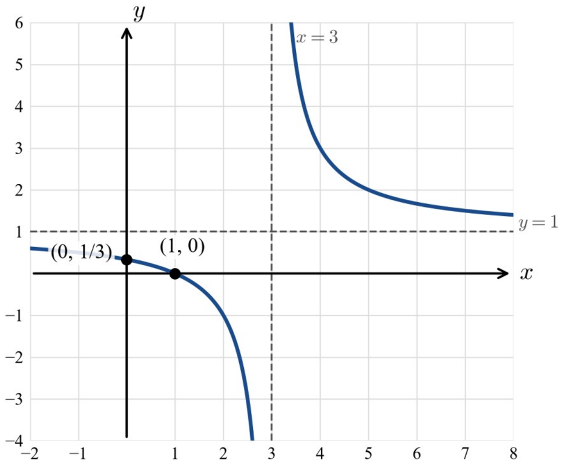
Q13. $y=-\dfrac{3}{x-1}+2$. Asymptotes $x=1$, $y=2$. x-intercept: $-\dfrac{3}{x-1}=-2 \Rightarrow \dfrac{3}{x-1}=2 \Rightarrow x-1=\dfrac{3}{2}$, so $x=\dfrac{5}{2}$; y-intercept $x=0 \Rightarrow -\dfrac{3}{-1}+2=5$.



Q14. $y=\dfrac{4}{x+2}-1$. Asymptotes $x=-2$, $y=-1$. x-intercept: $\dfrac{4}{x+2}=1 \Rightarrow x+2=4$, so $x=2$; y-intercept $x=0 \Rightarrow \dfrac{4}{2}-1=1$.

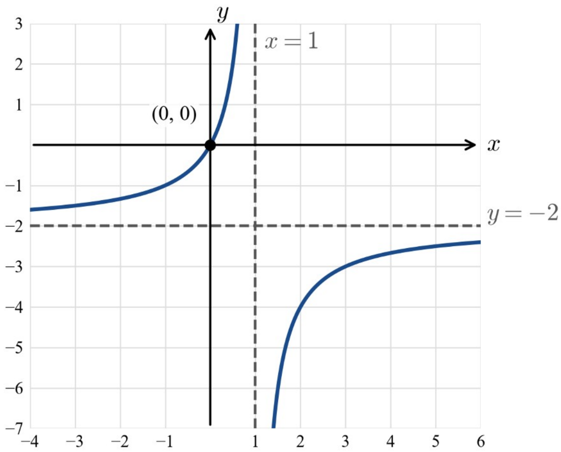


Q15. $y=\dfrac{2}{x-3}+1$. Asymptotes $x=3$, $y=1$. x-intercept: $\dfrac{2}{x-3}=-1 \Rightarrow x-3=-2$, so $x=1$; y-intercept $x=0 \Rightarrow \dfrac{2}{-3}+1=\dfrac{1}{3}$.

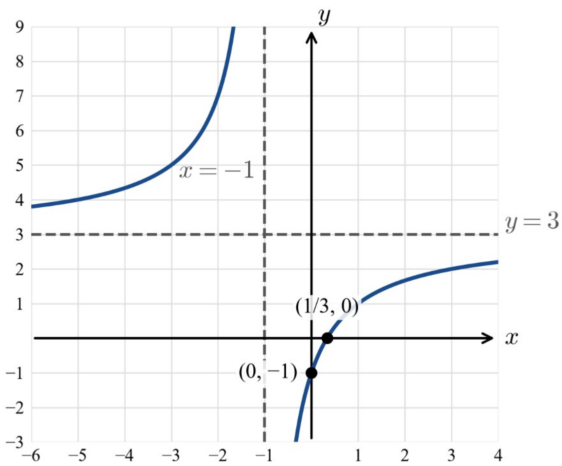


Q16. Asymptotes $x=1$, $y=-2$ give the form $y=\dfrac{a}{x-1}-2$. Through the origin $(0,0)$:

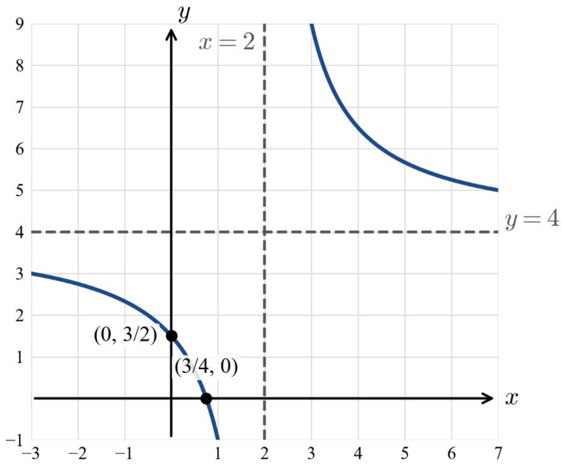
$0=\dfrac{a}{-1}-2 \Rightarrow -a-2=0 \Rightarrow a=-2$. $\therefore\ y=-\dfrac{2}{x-1}-2$.



Q17. Asymptotes $x=-1$, $y=3$ give $y=\dfrac{a}{x+1}+3$. Through $(1,1)$: $1=\dfrac{a}{2}+3 \Rightarrow \dfrac{a}{2}=-2 \Rightarrow a=-4$. $\therefore\ y=-\dfrac{4}{x+1}+3$.



Q18. $y=\dfrac{5}{x-2}+4$. Asymptotes $x=2$, $y=4$. x-intercept: $\dfrac{5}{x-2}=-4 \Rightarrow x-2=-\dfrac{5}{4}$, so $x=\dfrac{3}{4}$; y-intercept $x=0 \Rightarrow \dfrac{5}{-2}+4=\dfrac{3}{2}$.


y
x = 2
y = 4
x
(0, 3/2)
(3/4, 0)
9
8
7
6
5
4
3
2
1
−1
−3
−2
−1
1
2
3
4
5
6
7

Theory

This section covers the graph of $y=\frac{1}{x^2}$ (often called a **truncus** in VCE materials) and its transformations.

The basic graph of $y=\frac{1}{x^2}$ has two branches, both **above** the x-axis (since $x^2>0$), symmetric about the y-axis. It has a **vertical asymptote** $x=0$ (the y-axis) and a **horizontal asymptote** $y=0$ (the x-axis). Domain $\mathbb{R}\setminus\{0\}$, range $(0,\infty)$.

General form.

$$y=\frac{a}{(x-h)^2}+k.$$

- **Vertical asymptote** $x=h$; **horizontal asymptote** $y=k$.
- Both branches lie on the **same side** of the horizontal asymptote: **above** $y=k$ when $a>0$, **below** $y=k$ when $a<0$.
- Domain $\mathbb{R}\setminus\{h\}$. Range (k,∞) if $a>0$; range $(-\infty,k)$ if $a<0$.
- Obtained from $y=\frac{1}{x^2}$ by a dilation of factor a from the x-axis (with a reflection in the x-axis when $a<0$), then a translation h units horizontally and k units vertically.

Intercepts. The y-intercept (when $h\neq 0$) is found by putting $x=0$. For x-intercepts, put $y=0$ and solve $\frac{a}{(x-h)^2}=-k$; these exist only when $-\frac{a}{k}>0$.

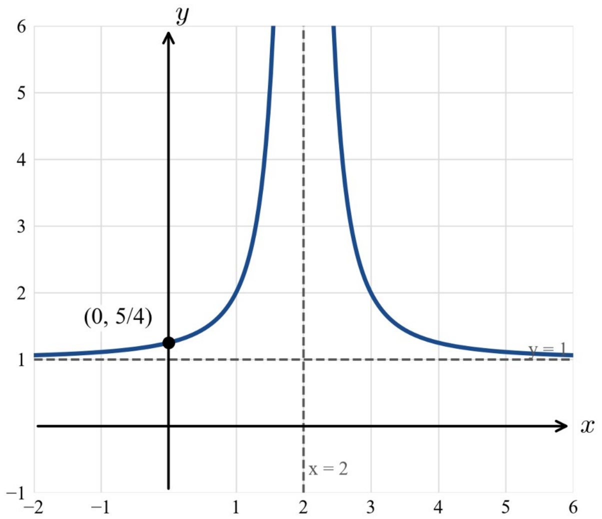
To sketch: identify a, h, k; draw the two asymptotes $x=h$ and $y=k$ (dashed); decide whether the branches are above or below $y=k$ from the sign of a; find and mark any intercepts; draw the two branches approaching the asymptotes.

Worked examples

Example 1 — scaffolded

Sketch $y=\frac{1}{(x-2)^2}+1$, stating the asymptotes, domain and range.

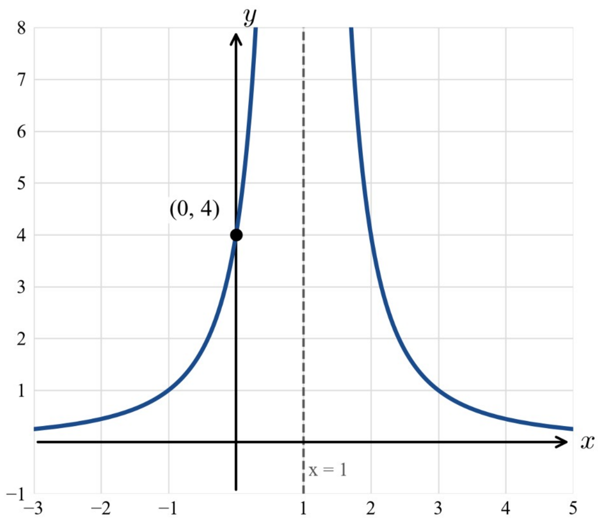
Working	Comment
$a=1$, $h=2$, $k=1$. Asymptotes $x=2$ and $y=1$.	Read h,k straight from the form.
$a=1>0$, so both branches are **above** $y=1$.	Sign of a fixes the branch position.
y-intercept: $x=0\Rightarrow y=\frac{1}{(0-2)^2}+1=\frac{1}{4}+1=\frac{5}{4}$, giving $\left(0,\frac{5}{4}\right)$.	Put $x=0$.
x-intercepts: $\frac{1}{(x-2)^2}+1=0\Rightarrow(x-2)^2=-1$, no solution — **no x-intercepts**.	The branches never cross $y=1$.
Domain $\mathbb{R}\setminus\{2\}$, range $(1,\infty)$.	From the asymptotes and branch position.



Example 2 — scaffolded

Sketch $y = \dfrac{4}{(x-1)^2}$, stating the asymptotes, domain and range.

Working	Comment
$a=4$, $h=1$, $k=0$. Asymptotes $x=1$ and $y=0$.	$k=0$: the horizontal asymptote is the x-axis.
$a=4>0$: both branches **above** the x-axis; the factor 4 makes the graph rise more steeply than $y=\dfrac{1}{(x-1)^2}$.	Dilation of factor 4 from the x-axis.
y-intercept: $x=0 \Rightarrow y=\dfrac{4}{(0-1)^2}=4$, giving $(0,4)$.	Put $x=0$.
No x-intercepts (the branches never reach $y=0$). Domain $\mathbb{R}\setminus\{1\}$, range $(0,\infty)$.	



Example 3 — unscaffolded

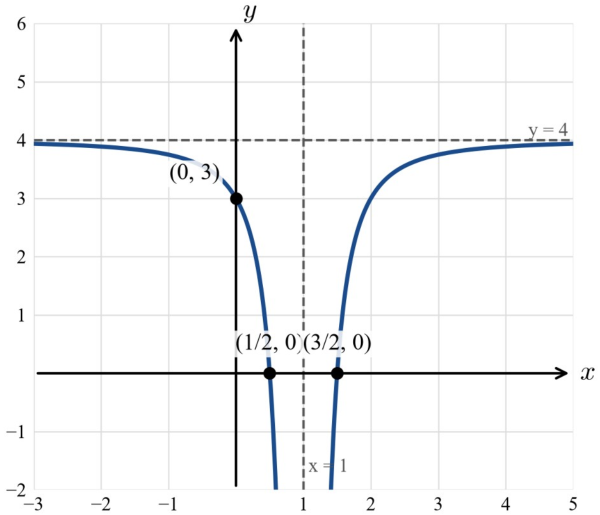
Sketch $y = -\dfrac{1}{(x-1)^2}+4$.

$a=-1$, $h=1$, $k=4$: asymptotes $x=1$ and $y=4$. Since $a=-1<0$, both branches lie **below** $y=4$.

y-intercept: $x=0 \Rightarrow y=-\dfrac{1}{(0-1)^2}+4=-1+4=3$, giving $(0,3)$.

x-intercepts: $-\dfrac{1}{(x-1)^2}+4=0 \Rightarrow (x-1)^2=\dfrac{1}{4} \Rightarrow x-1=\pm\dfrac{1}{2}$, so $x=\dfrac{1}{2}$ or $x=\dfrac{3}{2}$, giving $\left(\dfrac{1}{2},0\right)$ and $\left(\dfrac{3}{2},0\right)$.

$\therefore$ domain $\mathbb{R}\setminus\{1\}$, range $(-\infty,4)$.



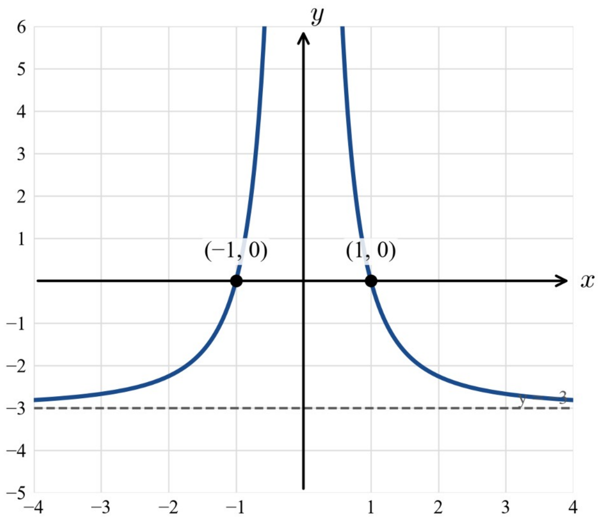
Example 4 — unscaffolded

Sketch $y=\dfrac{3}{x^2}-3$.

$a=3$, $h=0$, $k=-3$: asymptotes $x=0$ (the y-axis) and $y=-3$. Since $a=3>0$, both branches lie **above** $y=-3$. As $h=0$ there is **no y-intercept**.

x-intercepts: $\dfrac{3}{x^2}-3=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$, giving $(-1,0)$ and $(1,0)$.

$\therefore$ domain $\mathbb{R}\setminus\{0\}$, range $(-3,\infty)$.



Practice questions

Scaffolded

Q1. For $y=\dfrac{1}{(x-3)^2}$: (a) state the asymptotes; (b) find the y-intercept; (c) state the domain and range; (d) sketch the graph.

Q2. For $y=\frac{2}{(x+2)^2}$: (a) state the asymptotes; (b) find the y-intercept; (c) state the domain and range; (d) sketch the graph.

Q3. For $y=\frac{1}{x^2}+2$: (a) state the asymptotes; (b) explain why there are no axis intercepts; (c) state the domain and range; (d) sketch the graph.

Q4. For $y=\frac{1}{(x-1)^2}-1$: (a) state the asymptotes; (b) find all axis intercepts; (c) state the range; (d) sketch the graph.

Q5. For $y=-\frac{2}{(x+1)^2}$: (a) state the asymptotes and the branch position; (b) find the y-intercept; (c) state the range; (d) sketch the graph.

Q6. For $y=-\frac{1}{x^2}+1$: (a) state the asymptotes; (b) find all axis intercepts; (c) state the range; (d) sketch the graph.

Unscaffolded

For each of the following, sketch the graph, showing the asymptotes and any axis intercepts, and state the domain and range.

Q7. $y=\frac{1}{(x-4)^2}$

Q8. $y=\frac{1}{(x+3)^2}$

Q9. $y=\frac{1}{x^2}-4$

Q10. $y=\frac{3}{(x-2)^2}$

Q11. $y=\frac{1}{(x+1)^2}+1$

Q12. $y=-\frac{1}{(x-2)^2}+1$

Q13. $y=-\frac{4}{x^2}$

Q14. $y=\frac{2}{(x-1)^2}-2$

Q15. $y=\frac{1}{(x+2)^2}-1$

Q16. A curve of the form $y=\frac{a}{(x-h)^2}+k$ has asymptotes $x=1$ and $y=2$ and passes through $(0,3)$. Find its rule and sketch it.

Q17. A curve of the form $y=\frac{a}{(x-h)^2}+k$ has asymptotes $x=-2$ and $y=0$ and passes through $(0,1)$. Find its rule and sketch it.

Q18. A curve of the form $y=\frac{a}{(x-h)^2}+k$ has asymptotes $x=0$ and $y=-1$ and passes through $(1,1)$. Find its rule and sketch it.

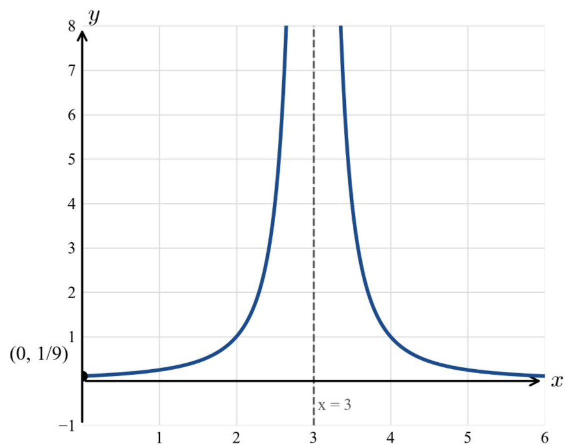
Answers

Q1. Asymptotes $x=3,\ y=0$; y-int $(0,1/9)$; domain $\mathbb{R}\setminus\{3\}$, range $(0,\infty)$. **Q2.** $x=-2,\ y=0$; y-int $(0,1/2)$; domain $\mathbb{R}\setminus\{-2\}$, range $(0,\infty)$. **Q3.** $x=0,\ y=2$; no axis intercepts; domain $\mathbb{R}\setminus\{0\}$, range $(2,\infty)$. **Q4.** $x=1$, $y=-1$; x-int $(0,0),(2,0)$; y-int $(0,0)$; range $(-1,\infty)$. **Q5.** $x=-1,\ y=0$, branches below; y-int $(0,-2)$; range $(-\infty,0)$. **Q6.** $x=0,\ y=1$; x-int $(-1,0),(1,0)$; range $(-\infty,1)$. **Q7.** $x=4,\ y=0$; y-int $(0,1/16)$; domain $\mathbb{R}\setminus\{4\}$, range $(0,\infty)$. **Q8.** $x=-3,\ y=0$; y-int $(0,1/9)$; domain $\mathbb{R}\setminus\{-3\}$, range $(0,\infty)$. **Q9.** $x=0,\ y=-4$; x-int $(-1/2,0),(1/2,0)$; domain $\mathbb{R}\setminus\{0\}$, range $(-4,\infty)$. **Q10.** $x=2,\ y=0$; y-int $(0,3/4)$; domain $\mathbb{R}\setminus\{2\}$, range $(0,\infty)$. **Q11.** $x=-1,\ y=1$; y-int $(0,2)$; domain $\mathbb{R}\setminus\{-1\}$, range $(1,\infty)$. **Q12.** $x=2,\ y=1$; y-int $(0,3/4)$; x-int $(1,0),(3,0)$; domain $\mathbb{R}\setminus\{2\}$, range $(-\infty,1)$. **Q13.** $x=0,\ y=0$, branches below; domain $\mathbb{R}\setminus\{0\}$, range $(-\infty,0)$. **Q14.** $x=1,\ y=-2$; x-int $(0,0),(2,0)$; y-int $(0,0)$; domain $\mathbb{R}\setminus\{1\}$, range $(-2,\infty)$. **Q15.** $x=-2,\ y=-1$; y-int $(0,-3/4)$; x-int $(-3,0),(-1,0)$; domain $\mathbb{R}\setminus\{-2\}$, range $(-1,\infty)$. **Q16.** $y=\dfrac{1}{(x-1)^2}+2$. **Q17.** $y=\dfrac{4}{(x+2)^2}$.

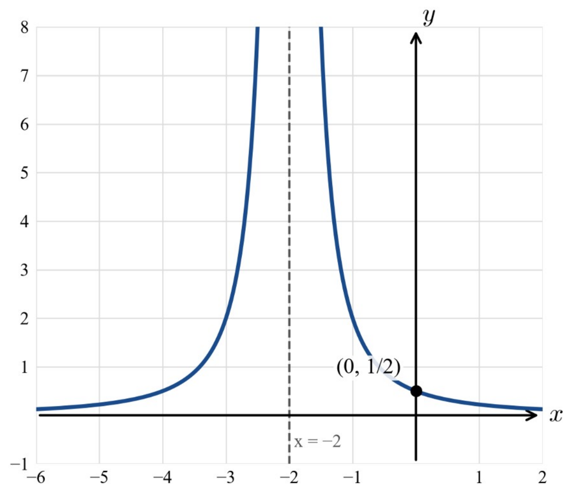
Q18. $y=\dfrac{2}{x^2}-1$ (with x-intercepts $(\pm\sqrt{2},0)$).

Fully-worked solutions

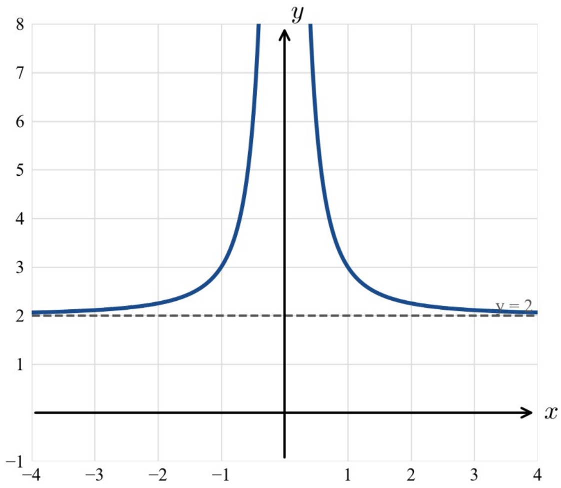
Q1. $y=\dfrac{1}{(x-3)^2}$. Asymptotes $x=3,\ y=0$; $a=1>0$ so both branches above the x-axis. y-int: $x=0\Rightarrow y=\dfrac{1}{9}$. No x-intercepts. Domain $\mathbb{R}\setminus\{3\}$, range $(0,\infty)$.



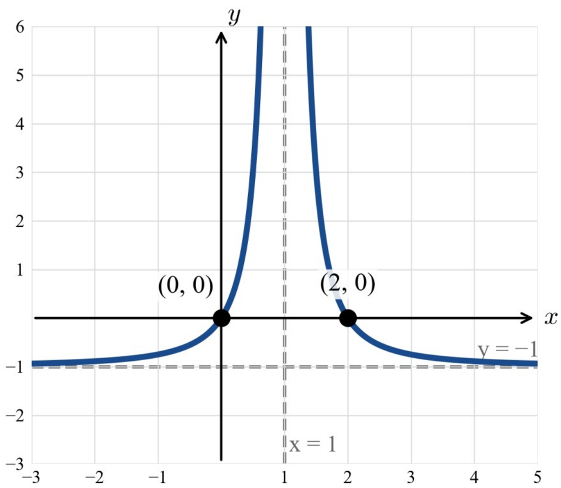
Q2. $y=\dfrac{2}{(x+2)^2}$. Asymptotes $x=-2,\ y=0$; branches above. y-int $x=0\Rightarrow y=\dfrac{2}{4}=\dfrac{1}{2}$. Domain $\mathbb{R}\setminus\{-2\}$, range $(0,\infty)$.



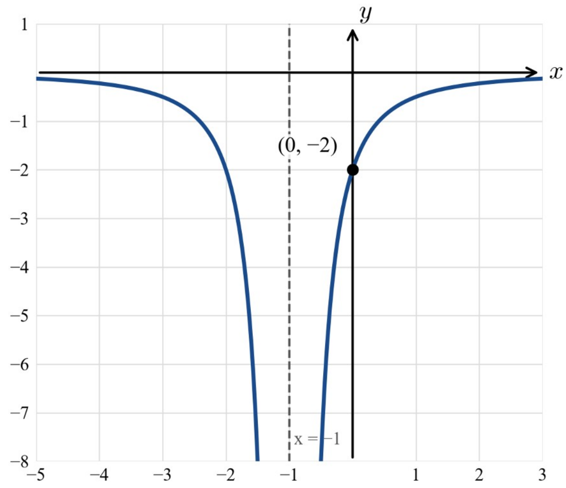
Q3. $y=\frac{1}{x^2}+2$. Asymptotes $x=0$, $y=2$; branches above $y=2$. There is no y-intercept ($x=0$ is the asymptote), and since $\frac{1}{x^2}>0$ always, $y>2$, so there are no x-intercepts. Domain $\mathbb{R}\setminus\{0\}$, range $(2,\infty)$.



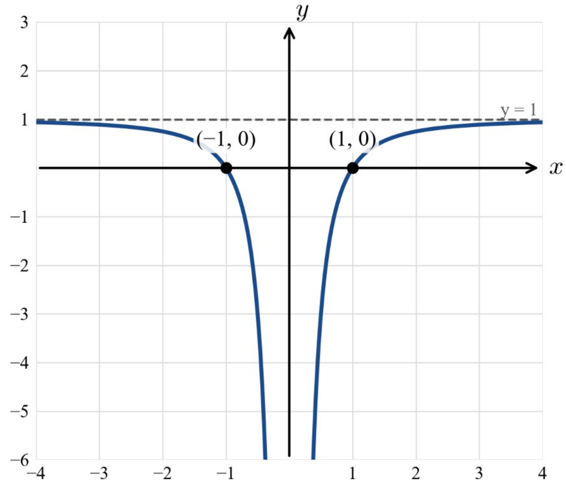
Q4. $y=\frac{1}{(x-1)^2}-1$. Asymptotes $x=1$, $y=-1$; branches above $y=-1$. y-int $x=0\Rightarrow y=1-1=0$, giving $(0,0)$. x-int: $\frac{1}{(x-1)^2}=1\Rightarrow(x-1)^2=1\Rightarrow x=0$ or 2, giving $(0,0)$ and $(2,0)$. Range $(-1,\infty)$.



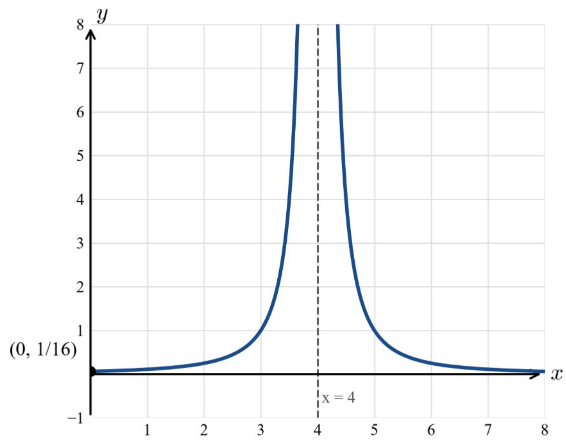
Q5. $y=-\frac{2}{(x+1)^2}$. Asymptotes $x=-1$, $y=0$; $a=-2<0$ so both branches **below** the x-axis. y-int $x=0\Rightarrow y=-2$, giving $(0,-2)$. No x-intercepts. Range $(-\infty,0)$.



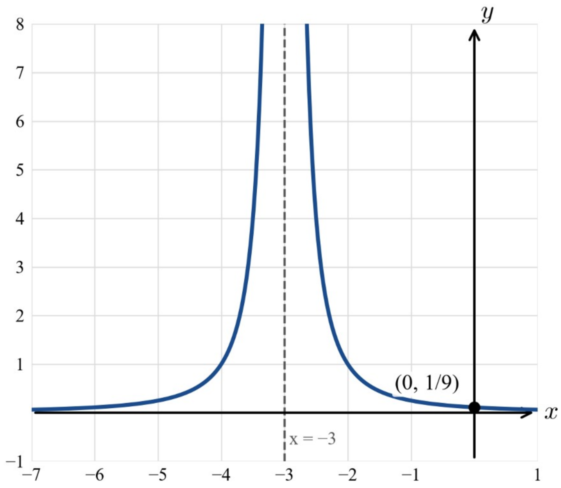
Q6. $y=-\dfrac{1}{x^2}+1$. Asymptotes $x=0$, $y=1$; branches below $y=1$. No y-intercept. x-int: $-\dfrac{1}{x^2}+1=0 \Rightarrow x^2=1 \Rightarrow x=\pm1$, giving $(-1,0)$ and $(1,0)$. Range $(-\infty,1)$.



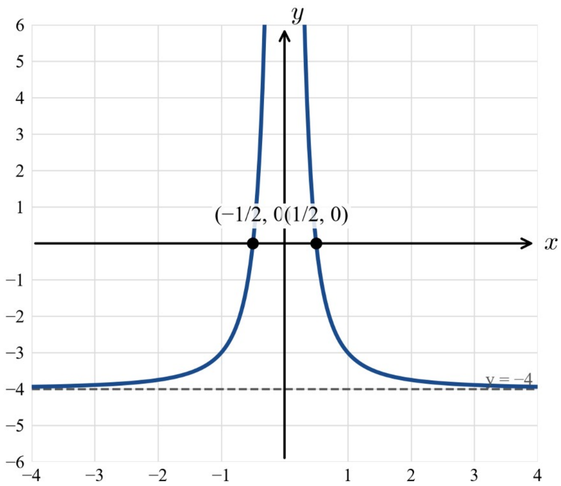
Q7. $y=\dfrac{1}{(x-4)^2}$. Asymptotes $x=4$, $y=0$; branches above. y-int $\left(0,\dfrac{1}{16}\right)$. Domain $\mathbb{R}\setminus\{4\}$, range $(0,\infty)$.



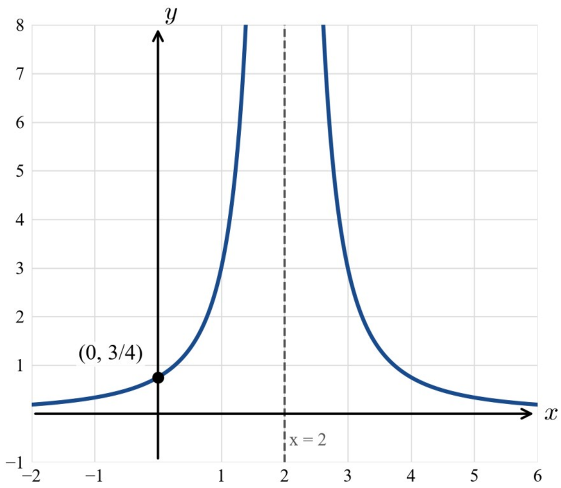
Q8. $y=\dfrac{1}{(x+3)^2}$. Asymptotes $x=-3$, $y=0$; branches above. y-int $\left(0,\dfrac{1}{9}\right)$. Domain $\mathbb{R}\setminus\{-3\}$, range $(0,\infty)$.



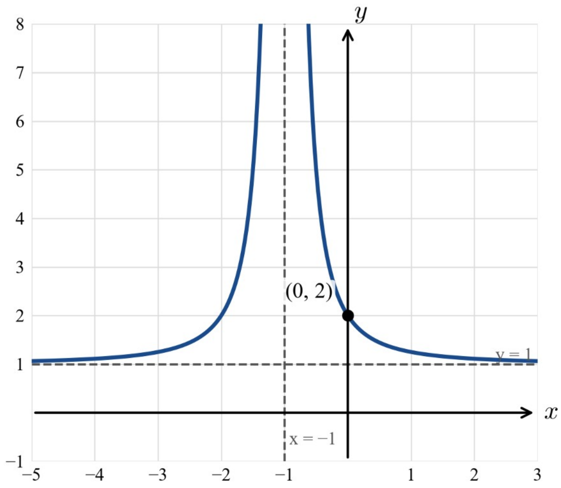
Q9. $y=\dfrac{1}{x^2}-4$. Asymptotes $x=0,\ y=-4$; branches above $y=-4$. No y-intercept. x-int: $\dfrac{1}{x^2}=4 \Rightarrow x^2=\dfrac{1}{4} \Rightarrow x=\pm\dfrac{1}{2}$, giving $\left(-\dfrac{1}{2},0\right)$ and $\left(\dfrac{1}{2},0\right)$. Domain $\mathbb{R}\setminus\{0\}$, range $(-4,\infty)$.



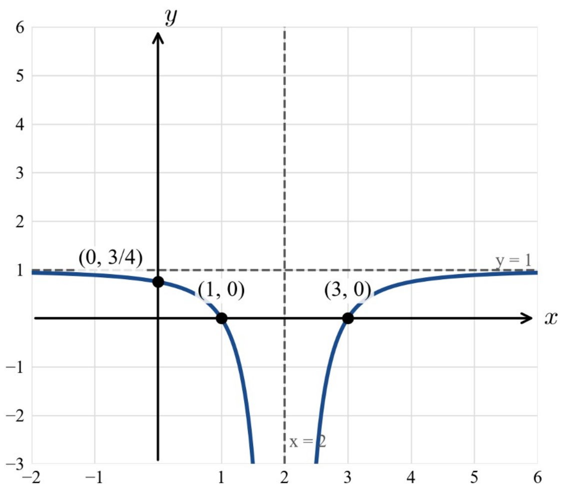
Q10. $y=\dfrac{3}{(x-2)^2}$. Asymptotes $x=2,\ y=0$; branches above. y-int $x=0 \Rightarrow y=\dfrac{3}{4}$. Domain $\mathbb{R}\setminus\{2\}$, range $(0,\infty)$.



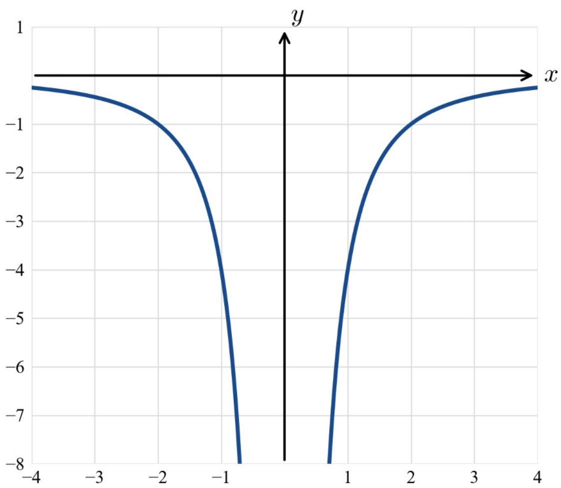
Q11. $y=\dfrac{1}{(x+1)^2}+1$. Asymptotes $x=-1,\ y=1$; branches above $y=1$. y-int $x=0 \Rightarrow y=1+1=2$, giving $(0,2)$. No x-intercepts. Domain $\mathbb{R}\setminus\{-1\}$, range $(1,\infty)$.



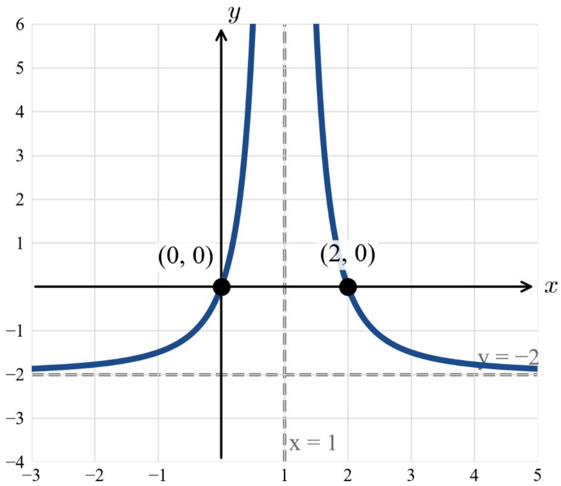
Q12. $y=-\frac{1}{(x-2)^2}+1$. Asymptotes $x=2,\ y=1$; branches below $y=1$. y-int $x=0 \Rightarrow y=-\frac{1}{4}+1=\frac{3}{4}$, giving $\left(0,\frac{3}{4}\right)$. x-int: $(x-2)^2=1 \Rightarrow x=1$ or 3, giving $(1,0)$ and $(3,0)$. Domain $\mathbb{R}\setminus\{2\}$, range $(-\infty,1)$.



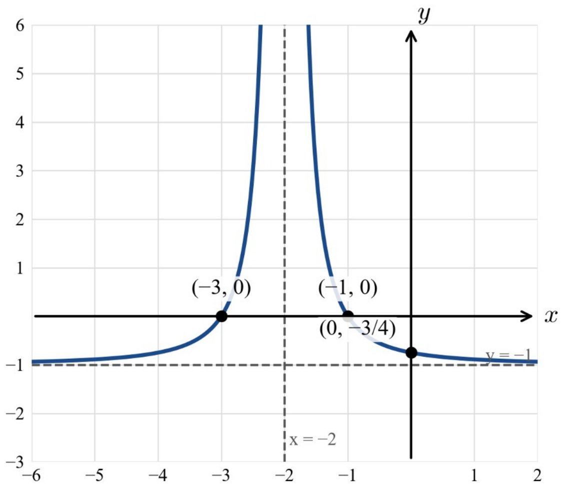
Q13. $y=-\frac{4}{x^2}$. Asymptotes $x=0,\ y=0$; $a=-4<0$ so both branches below the x-axis. No axis intercepts. Domain $\mathbb{R}\setminus\{0\}$, range $(-\infty,0)$.



Q14. $y=\dfrac{2}{(x-1)^2}-2$. Asymptotes $x=1$, $y=-2$; branches above $y=-2$. y-int $x=0 \Rightarrow y=2-2=0$, giving $(0,0)$. x-int: $\dfrac{2}{(x-1)^2}=2 \Rightarrow (x-1)^2=1 \Rightarrow x=0$ or 2, giving $(0,0)$ and $(2,0)$. Domain $\mathbb{R}\setminus\{1\}$, range $(-2,\infty)$.

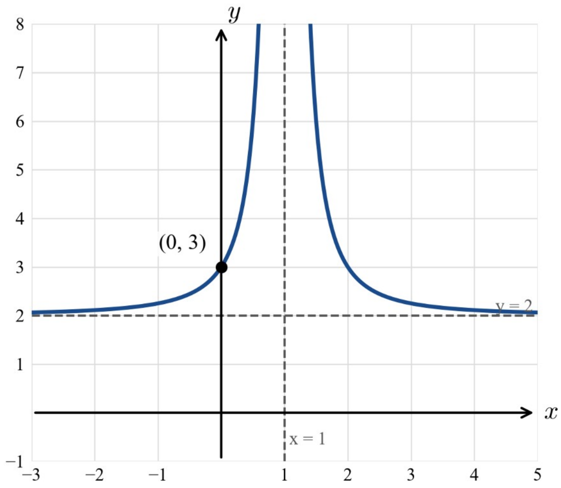


Q15. $y=\dfrac{1}{(x+2)^2}-1$. Asymptotes $x=-2$, $y=-1$; branches above $y=-1$. y-int $x=0 \Rightarrow y=\dfrac{1}{4}-1=-\dfrac{3}{4}$. x-int: $(x+2)^2=1 \Rightarrow x=-1$ or -3, giving $(-1,0)$ and $(-3,0)$. Domain $\mathbb{R}\setminus\{-2\}$, range $(-1,\infty)$.

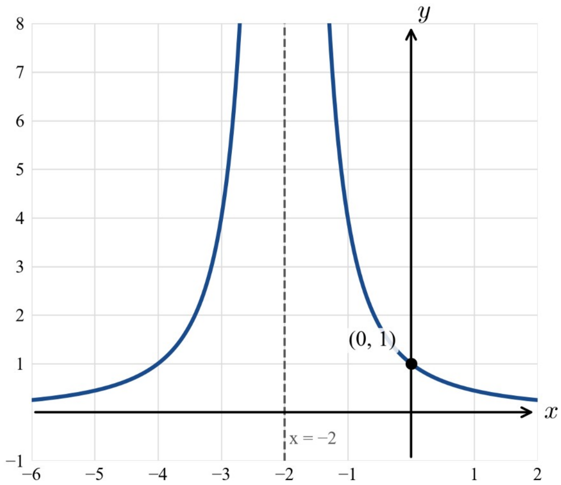


Q16. The asymptotes give $h=1$, $k=2$, so $y=\dfrac{a}{(x-1)^2}+2$. Substituting $(0,3)$: $3=\dfrac{a}{(0-1)^2}+2=a+2 \Rightarrow a=1$.

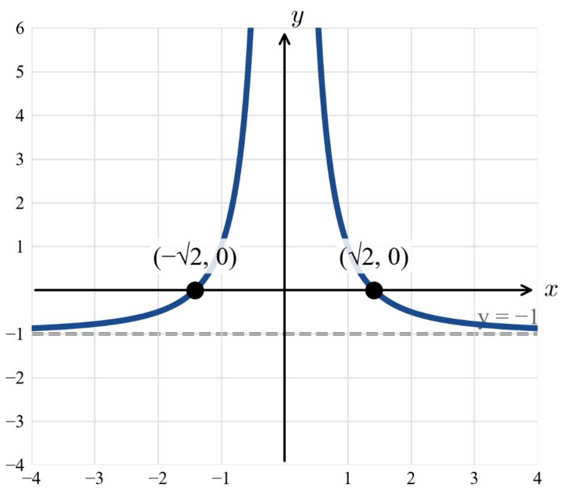
$\therefore y=\dfrac{1}{(x-1)^2}+2$.



Q17. Here $h=-2$, $k=0$, so $y=\dfrac{a}{(x+2)^2}$. Substituting $(0,1)$: $1=\dfrac{a}{(0+2)^2}=\dfrac{a}{4}\Rightarrow a=4$. $\therefore y=\dfrac{4}{(x+2)^2}$.



Q18. Here $h=0$, $k=-1$, so $y=\dfrac{a}{x^2}-1$. Substituting $(1,1)$: $1=\dfrac{a}{1}-1\Rightarrow a=2$. $\therefore y=\dfrac{2}{x^2}-1$ (its x-intercepts are $(\pm\sqrt{2},0)$).



Theory

The relation $y^2 = x$ has a graph that is a **sideways parabola**. Because $y = \pm\sqrt{x}$, each value of $x > 0$ gives **two** values of y, so the curve opens to the **right**, with its vertex at the origin and symmetric about the x-axis.

It is not a function. A vertical line $x = c$ (for $c > 0$) meets the curve **twice**, so $y^2 = x$ fails the vertical line test. Useful guide points are $(0, 0)$, $(1, \pm 1)$, $(4, \pm 2)$ and $(9, \pm 3)$.

General form. Every such relation can be written

$$(y - k)^2 = a(x - h).$$

- The **vertex** is (h, k) and the axis of symmetry is the horizontal line $y = k$.
- If $a > 0$ the parabola opens to the **right**; if $a < 0$ it opens to the **left**. $|a|$ controls the width.
- The two halves are $y = k \pm \sqrt{a(x - h)}$, defined where $a(x - h) \geq 0$.
- **Domain:** $[h, \infty)$ if $a > 0$, or $(-\infty, h]$ if $a < 0$. **Range:** $\mathbb{R}$.

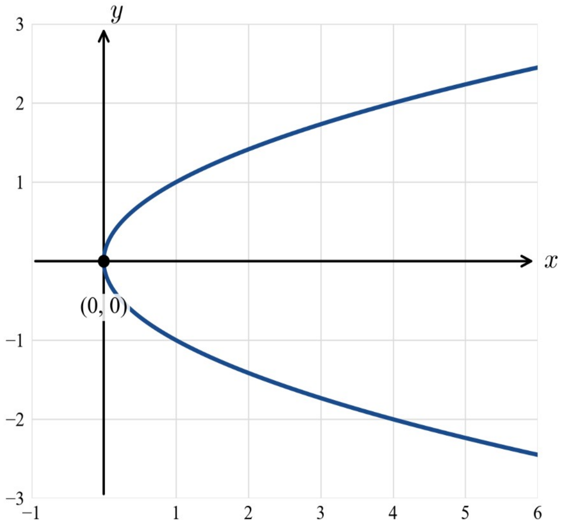
Intercepts. For the x-intercept, set $y = 0$. For the y-intercept(s), set $x = 0$: there may be none, one or two, depending on whether $-ah$ is negative, zero or positive.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y^2 = x$, stating the vertex, the domain and range, and whether it is a function.

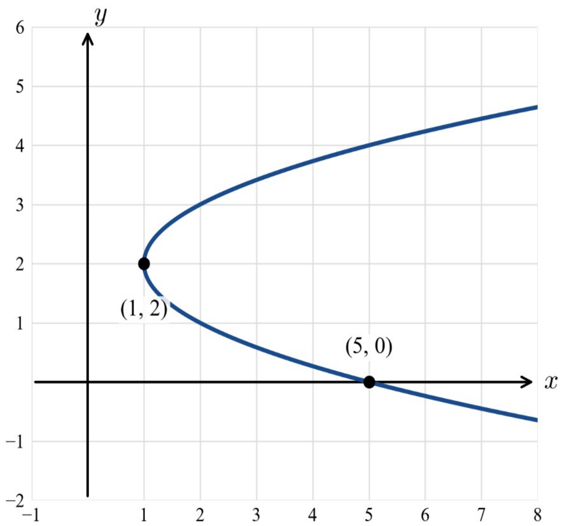
Working	Comment
$y = \pm\sqrt{x}$, so the curve opens to the **right** with vertex $(0, 0)$.	Compare with $(y - k)^2 = a(x - h)$: here $a = 1$, $h = 0$, $k = 0$.
Guide points: $(1, \pm 1)$ and $(4, \pm 2)$.	Choose x-values that give whole-number square roots.
Domain $[0, \infty)$, range $\mathbb{R}$.	$x = y^2 \geq 0$, and y takes every real value.
A vertical line such as $x = 4$ meets the curve at $(4, 2)$ and $(4, -2)$, so it is **not a function**.	Vertical line test.



Example 2 — scaffolded

Sketch the graph of $(y-2)^2 = x-1$, labelling the vertex and all axis intercepts.

Working	Comment
$a=1>0$, $h=1$, $k=2$: opens **right**, vertex $(1,2)$.	Read h and k from the general form.
x-intercept ($y=0$): $(0-2)^2 = x-1 \Rightarrow 4 = x-1 \Rightarrow x=5$, giving $(5,0)$.	Put $y=0$.
y-intercept ($x=0$): $(y-2)^2 = -1$, which has **no real solution**, so there is no y-intercept.	The domain is $[1,\infty)$, so the curve never reaches $x=0$.
Domain $[1,\infty)$, range $\mathbb{R}$.	Vertex x-value gives the left endpoint.

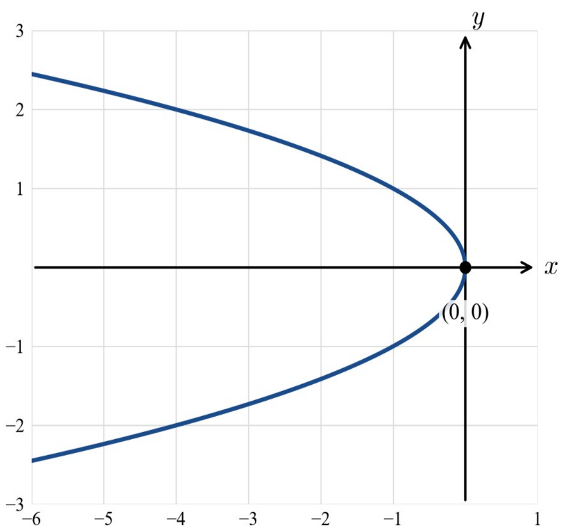


Example 3 — unscaffolded

Sketch the graph of $y^2 = -x$.

$a=-1<0$, so the parabola opens to the **left**, with vertex $(0,0)$ and axis of symmetry the x-axis. Guide points: $(-1,\pm 1)$ and $(-4,\pm 2)$.

Domain $(-\infty,0]$, range $\mathbb{R}$. $\therefore$ it is the graph of $y^2 = x$ reflected in the y-axis (not a function).



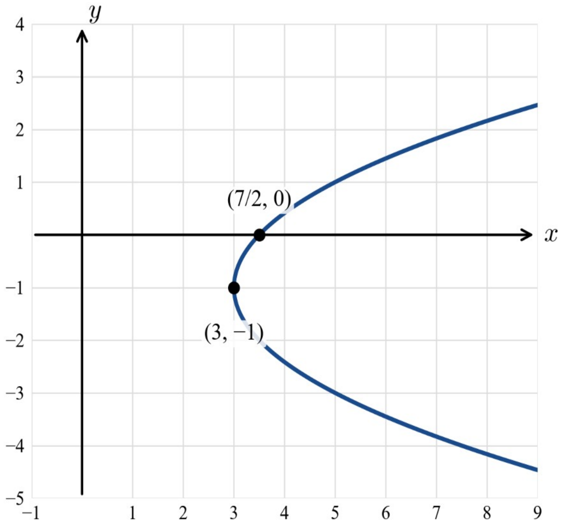
Example 4 — unscaffolded

Sketch the graph of $(y+1)^2 = 2(x-3)$.

$a = 2 > 0$, $h = 3$, $k = -1$: opens **right**, vertex $(3, -1)$.

x-intercept: $y = 0 \Rightarrow (1)^2 = 2(x-3) \Rightarrow 1 = 2x - 6 \Rightarrow x = 7/2$, giving $(7/2, 0)$. y-intercept: $x = 0 \Rightarrow (y+1)^2 = -6$, no real solution, so none.

Domain $[3, \infty)$, range $\mathbb{R}$.



Practice questions

Scaffolded

For each relation: state the vertex and the direction it opens; find the axis intercepts; state the domain and range; and sketch the graph.

Q1. $y^2 = x$

Q2. $y^2 = 2x$

Q3. $(y-3)^2 = x$

Q4. $y^2 = x - 4$

Q5. $y^2 = -2x$

Q6. $(y+1)^2 = x - 2$

Unscaffolded

Sketch each graph, labelling the vertex and all axis intercepts, and state the domain.

Q7. $y^2 = 4x$

Q8. $y^2 = -3x$

Q9. $(y-1)^2 = x$

Q10. $(y+2)^2 = x$

Q11. $y^2 = x - 1$

Q12. $y^2 = x + 2$

Q13. $(y-2)^2 = x - 3$

Q14. $(y+1)^2 = x+2$

Q15. $y^2 = -(x-2)$

Q16. $(y-1)^2 = -(x-3)$

Q17. $(y+2)^2 = 2(x-1)$

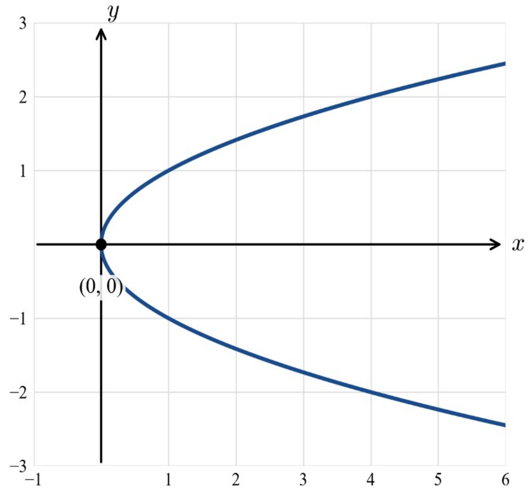
Q18. $(y-3)^2 = -2(x+1)$

Answers

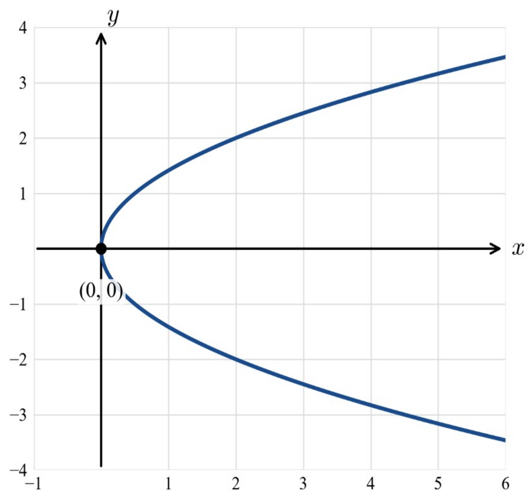
Q1. V$(0,0)$, opens right; through $(1,\pm1),(4,\pm2)$; domain $[0,\infty)$, range $\mathbb{R}$. **Q2.** V$(0,0)$, opens right; through $(2,\pm2),(8,\pm4)$; domain $[0,\infty)$, range $\mathbb{R}$. **Q3.** V$(0,3)$, opens right; x-int $(9,0)$, y-int $(0,3)$; domain $[0,\infty)$, range $\mathbb{R}$. **Q4.** V$(4,0)$, opens right; x-int $(4,0)$, no y-int; domain $[4,\infty)$, range $\mathbb{R}$. **Q5.** V$(0,0)$, opens left; through $(-2,\pm2)$; domain $(-\infty,0]$, range $\mathbb{R}$. **Q6.** V$(2,-1)$, opens right; x-int $(3,0)$, no y-int; domain $[2,\infty)$, range $\mathbb{R}$. **Q7.** V$(0,0)$, opens right; through $(1,\pm2),(4,\pm4)$; domain $[0,\infty)$. **Q8.** V$(0,0)$, opens left; through $(-3,\pm3)$; domain $(-\infty,0]$. **Q9.** V$(0,1)$, opens right; x-int $(1,0)$, y-int $(0,1)$; domain $[0,\infty)$. **Q10.** V$(0,-2)$, opens right; x-int $(4,0)$, y-int $(0,-2)$; domain $[0,\infty)$. **Q11.** V$(1,0)$, opens right; x-int $(1,0)$, no y-int; domain $[1,\infty)$. **Q12.** V$(-2,0)$, opens right; x-int $(-2,0)$, y-int $(0,\pm\sqrt{2})$; domain $[-2,\infty)$. **Q13.** V$(3,2)$, opens right; x-int $(7,0)$, no y-int; domain $[3,\infty)$. **Q14.** V$(-2,-1)$, opens right; x-int $(-1,0)$, y-int $(0,-1\pm\sqrt{2})$; domain $[-2,\infty)$. **Q15.** V$(2,0)$, opens left; x-int $(2,0)$, y-int $(0,\pm\sqrt{2})$; domain $(-\infty,2]$. **Q16.** V$(3,1)$, opens left; x-int $(2,0)$, y-int $(0,1\pm\sqrt{3})$; domain $(-\infty,3]$. **Q17.** V$(1,-2)$, opens right; x-int $(3,0)$, no y-int; domain $[1,\infty)$. **Q18.** V$(-1,3)$, opens left; x-int $(-11/2,0)$, no y-int; domain $(-\infty,-1]$.

Fully-worked solutions

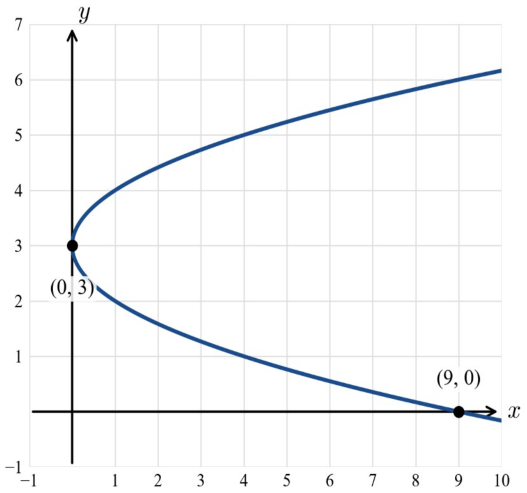
Q1. $y^2=x$: $a=1$, vertex $(0,0)$, opens right. Through $(1,\pm1)$ and $(4,\pm2)$; domain $[0,\infty)$, range $\mathbb{R}$.



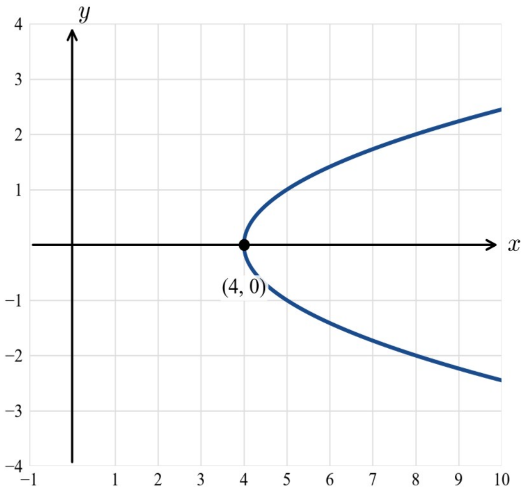
Q2. $y^2=2x$: vertex $(0,0)$, opens right. Since $y=\pm\sqrt{2x}$, it passes through $(2,\pm2)$ and $(8,\pm4)$; domain $[0,\infty)$, range $\mathbb{R}$.



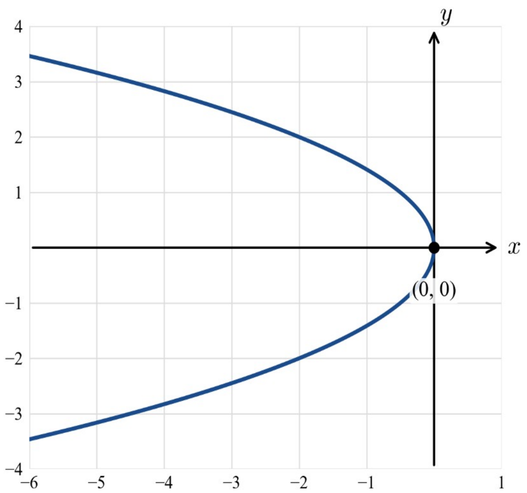
Q3. $(y-3)^2=x$: vertex $(0,3)$, opens right. x-int ($y=0$): $x=9$, giving $(9,0)$; y-int ($x=0$): $(y-3)^2=0\Rightarrow y=3$, giving $(0,3)$. Domain $[0,\infty)$, range $\mathbb{R}$.



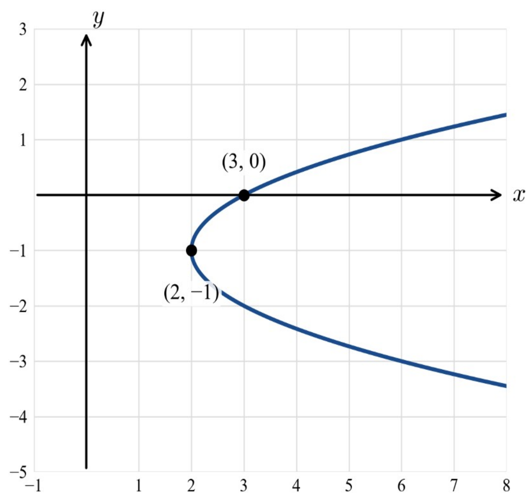
Q4. $y^2 = x - 4$: vertex $(4, 0)$, opens right. x-int $(4, 0)$; no y-int (the domain does not reach $x = 0$). Domain $[4, \infty)$, range $\mathbb{R}$.



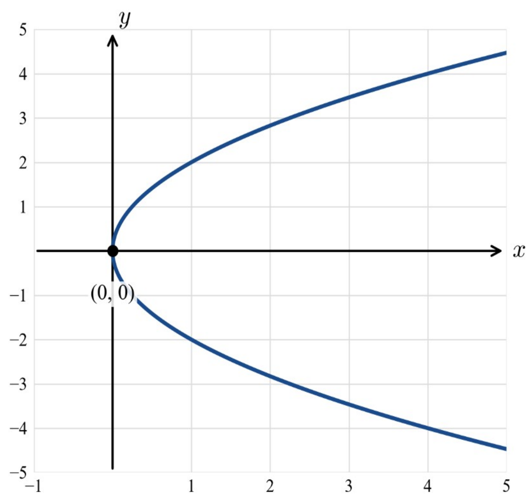
Q5. $y^2 = -2x$: $a = -2 < 0$, vertex $(0, 0)$, opens left. Through $(-2, \pm 2)$; domain $(-\infty, 0]$, range $\mathbb{R}$.



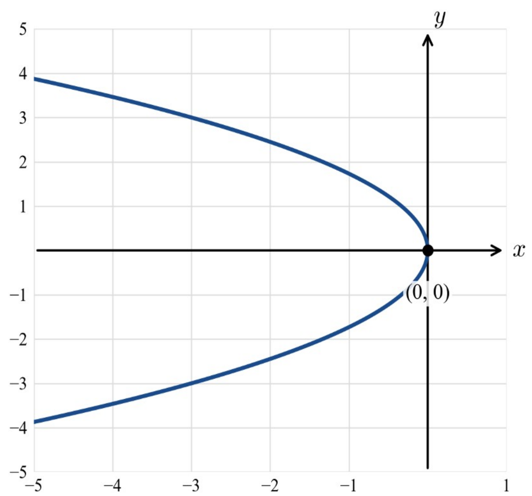
Q6. $(y+1)^2 = x - 2$: vertex $(2, -1)$, opens right. x-int: $y = 0 \Rightarrow 1 = x - 2 \Rightarrow x = 3$, giving $(3, 0)$; no y-int. Domain $[2, \infty)$, range $\mathbb{R}$.



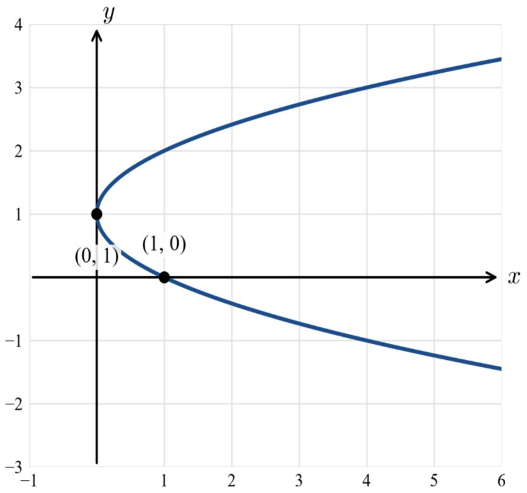
Q7. $y^2 = 4x$: vertex $(0, 0)$, opens right. $y = \pm 2\sqrt{x}$, through $(1, \pm 2)$ and $(4, \pm 4)$; domain $[0, \infty)$.



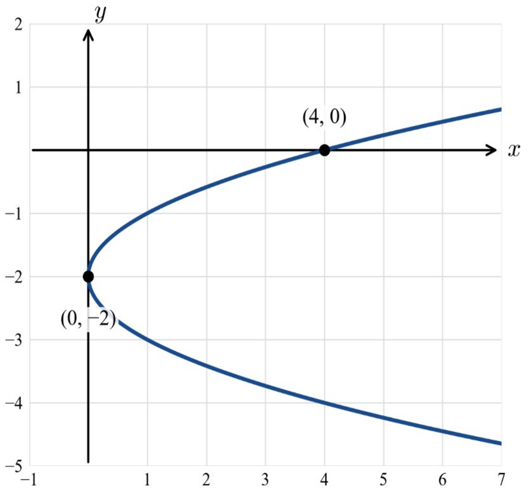
Q8. $y^2 = -3x$: vertex $(0, 0)$, opens left; through $(-3, \pm 3)$; domain $(-\infty, 0]$.



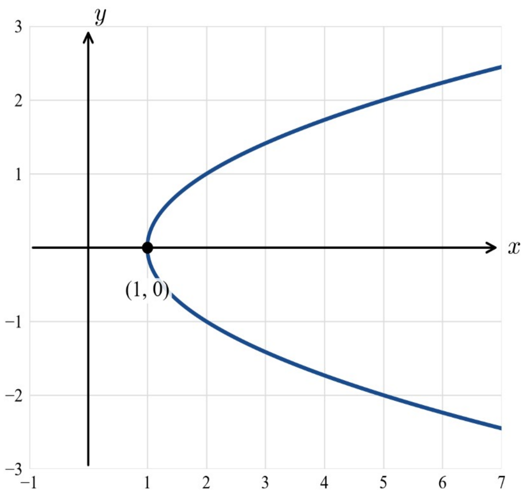
Q9. $(y-1)^2 = x$: vertex $(0, 1)$, opens right. x-int $(1, 0)$; y-int $(0, 1)$; domain $[0, \infty)$.



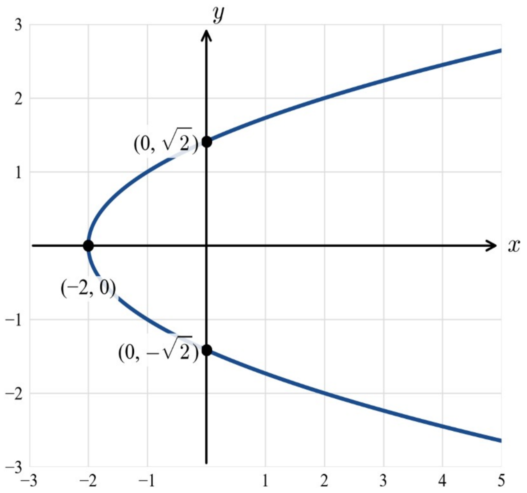
Q10. $(y+2)^2 = x$: vertex $(0, -2)$, opens right. x-int: $y=0 \Rightarrow x=4$, giving $(4, 0)$; y-int $(0, -2)$; domain $[0, \infty)$.



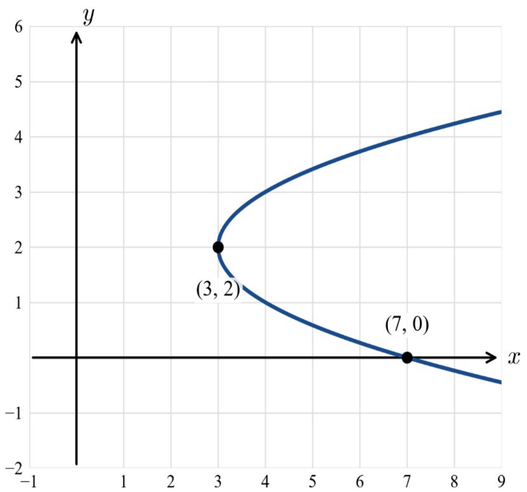
Q11. $y^2 = x - 1$: vertex $(1, 0)$, opens right. x-int $(1, 0)$; no y-int; domain $[1, \infty)$.



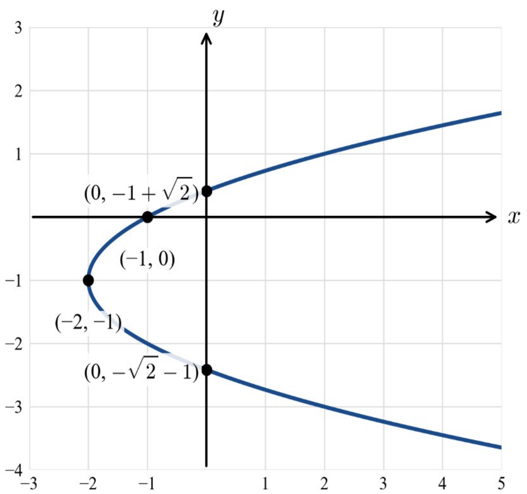
Q12. $y^2 = x + 2$: vertex $(-2, 0)$, opens right. x-int $(-2, 0)$; y-int: $x=0 \Rightarrow y^2=2 \Rightarrow y=\pm\sqrt{2}$, giving $(0, \pm\sqrt{2})$; domain $[-2, \infty)$.



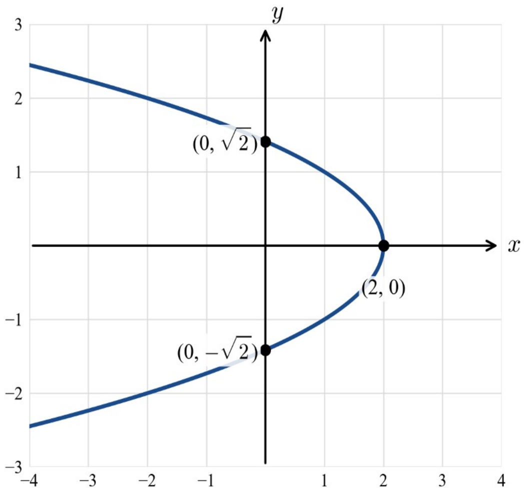
Q13. $(y-2)^2=x-3$: vertex $(3,2)$, opens right. x-int: $y=0\Rightarrow 4=x-3\Rightarrow x=7$, giving $(7,0)$; no y-int; domain $[3,\infty)$.



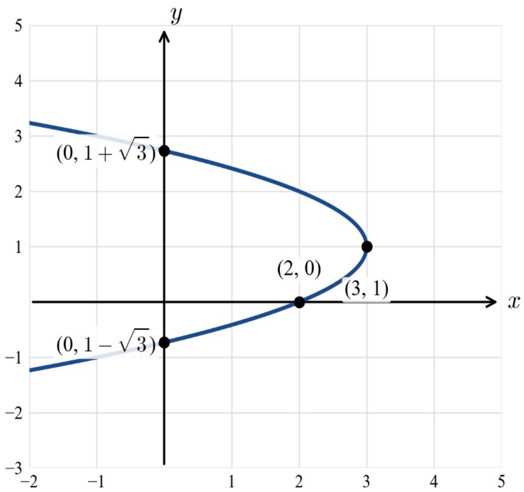
Q14. $(y+1)^2=x+2$: vertex $(-2,-1)$, opens right. x-int: $y=0\Rightarrow 1=x+2\Rightarrow x=-1$, giving $(-1,0)$; y-int: $x=0\Rightarrow(y+1)^2=2\Rightarrow y=-1\pm\sqrt{2}$, giving $(0,-1\pm\sqrt{2})$; domain $[-2,\infty)$.



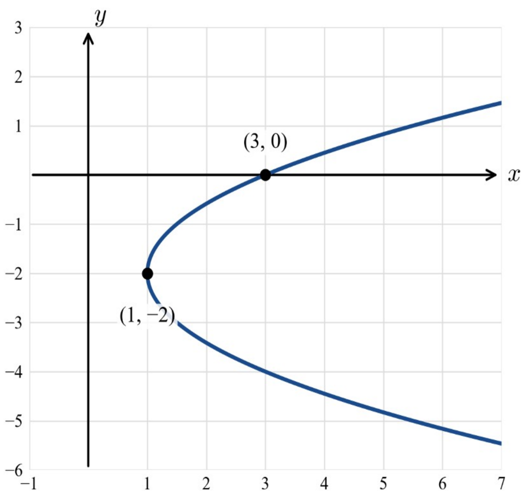
Q15. $y^2=-(x-2)$: $a=-1$, vertex $(2,0)$, opens left. x-int $(2,0)$; y-int: $x=0\Rightarrow y^2=2\Rightarrow y=\pm\sqrt{2}$, giving $(0,\pm\sqrt{2})$; domain $(-\infty,2]$.



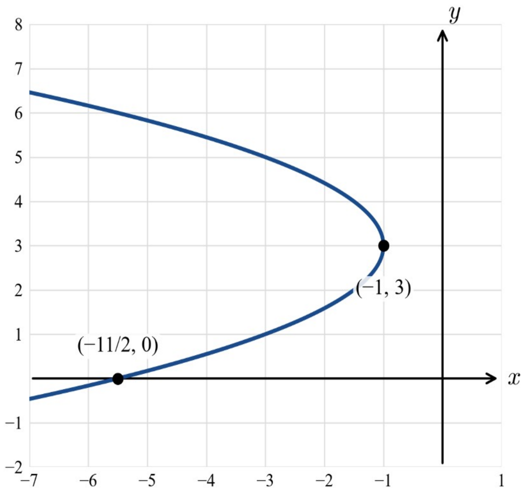
Q16. $(y-1)^2=-(x-3)$: $a=-1$, vertex $(3,1)$, opens left. x-int: $y=0 \Rightarrow 1=-(x-3) \Rightarrow x=2$, giving $(2,0)$; y-int: $x=0 \Rightarrow (y-1)^2=3 \Rightarrow y=1\pm\sqrt{3}$, giving $(0,1\pm\sqrt{3})$; domain $(-\infty,3]$.



Q17. $(y+2)^2=2(x-1)$: vertex $(1,-2)$, opens right. x-int: $y=0 \Rightarrow 4=2(x-1) \Rightarrow x=3$, giving $(3,0)$; no y-int; domain $[1,\infty)$.



Q18. $(y-3)^2=-2(x+1)$: $a=-2$, vertex $(-1,3)$, opens left. x-int: $y=0 \Rightarrow 9=-2(x+1) \Rightarrow x=-11/2$, giving $(-11/2,0)$; no y-int; domain $(-\infty,-1]$.


y
x
8
7
6
5
4
3
2
1
−1
−2
−7
−6
−5
−4
−3
−2
−1
1
(−1, 3)
(−11/2, 0)

Chapter 4 Graphs of power functions and relations — 4D The square-root and cube-root functions

Theory

The **square-root function** is $y = \sqrt{x}$. Because a square root of a negative number is not real, the domain is $[0, \infty)$ and the range is $[0, \infty)$. The graph starts at the origin $(0, 0)$ and increases, becoming less steep as x increases.

General form. Every graph in this family can be written as

$$y = a\sqrt{x - h} + k.$$

- The graph starts at the **endpoint** (h, k).
- **Domain** $[h, \infty)$; **range** $[k, \infty)$ if $a > 0$, or $(-\infty, k]$ if $a < 0$.
- a is a dilation from the x-axis; if $a < 0$ the graph is reflected in the x-axis (it goes down from the endpoint instead of up).

Reflection in the y-axis. Replacing x with $-x$ gives $y = \sqrt{-x}$, with domain $(-\infty, 0]$. More generally $y = a\sqrt{h - x} + k$ has endpoint (h, k) and domain $(-\infty, h]$ — it opens to the **left**.

To sketch $y = a\sqrt{\pm(x - h)} + k$: (1) mark the endpoint (h, k); (2) decide the direction — right if the expression under the root is $x - h$, left if it is $h - x$; up if $a > 0$, down if $a < 0$; (3) find any axis intercepts; (4) plot one extra point (a perfect square under the root is easiest) and draw a smooth curve.

The cube-root function. The **cube-root function** is $y = \sqrt[3]{x} = x^{1/3}$. Every real number has exactly one real cube root, so — unlike $y = \sqrt{x}$ — the **domain is $\mathbb{R}$ and the range is $\mathbb{R}$**. The graph passes through the origin $(0, 0)$, is **increasing everywhere**, and has **odd symmetry** (since $\sqrt[3]{-x} = -\sqrt[3]{x}$): it has the same shape below the origin as above, turned through half a revolution. At the origin the curve has a **vertical tangent** — it is steepest there and flattens as $|x|$ grows — and $(0, 0)$ is a **point of inflection**. Because cubing and taking a cube root are inverse operations, the graph of $y = \sqrt[3]{x}$ is the reflection of $y = x^3$ in the line $y = x$.

General form. Every graph in this family can be written as

$$y = a\sqrt[3]{x - h} + k, \text{where} \sqrt[3]{x - h} = (x - h)^{1/3}.$$

- The graph has its **point of inflection** at (h, k) — the analogue of the square-root endpoint.
- **Domain $\mathbb{R}$** and **range $\mathbb{R}$** for every $a \neq 0$; the curve runs off to infinity in both directions, so there is always **exactly one** x-intercept and **exactly one** y-intercept.
- a is a dilation of factor $|a|$ from the x-axis; if $a < 0$ the graph is **reflected in the x-axis**, so it **decreases** everywhere instead of increasing.

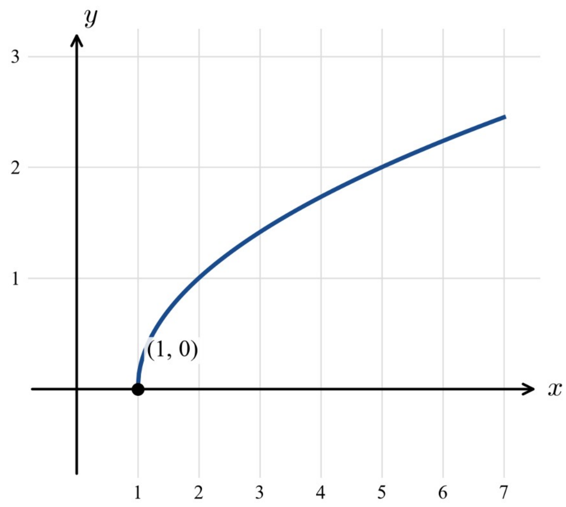
To sketch $y = a\sqrt[3]{x - h} + k$: (1) mark the point of inflection (h, k); (2) decide the direction — increasing if $a > 0$, decreasing if $a < 0$; (3) find the x- and y-intercepts exactly (choosing the number under the root to be a perfect cube keeps them exact); (4) draw a smooth curve through (h, k) with a vertical tangent there.

Worked examples

Example 1 — scaffolded

Sketch $y = \sqrt{x - 1}$, stating the domain and range.

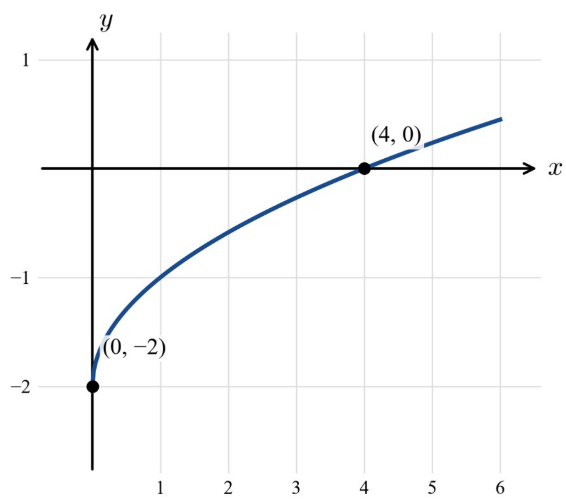
Working	Comment
Compare with $y=a\sqrt{x-h}+k$: $a=1$, $h=1$, $k=0$. Endpoint $(1,0)$.	The endpoint is (h,k).
The root is $x-1$, so the graph opens to the **right**; $a=1>0$, so it goes **up**. Domain $[1,\infty)$, range $[0,\infty)$.	Direction from the sign inside the root and the sign of a.
Extra point: $x=5\Rightarrow y=\sqrt{4}=2$, giving $(5,2)$.	Choose x so the root is a perfect square.



Example 2 — scaffolded

Sketch $y=\sqrt{x}-2$, showing all axis intercepts.

Working	Comment
Here $h=0$, $k=-2$: endpoint $(0,-2)$. Domain $[0,\infty)$, range $[-2,\infty)$.	Read the endpoint and domain/range.
y-intercept: $x=0\Rightarrow y=-2$, giving $(0,-2)$ (the endpoint).	The endpoint is on the y-axis here.
x-intercept: $y=0\Rightarrow\sqrt{x}=2\Rightarrow x=4$, giving $(4,0)$.	Solve $\sqrt{x}=2$ by squaring.



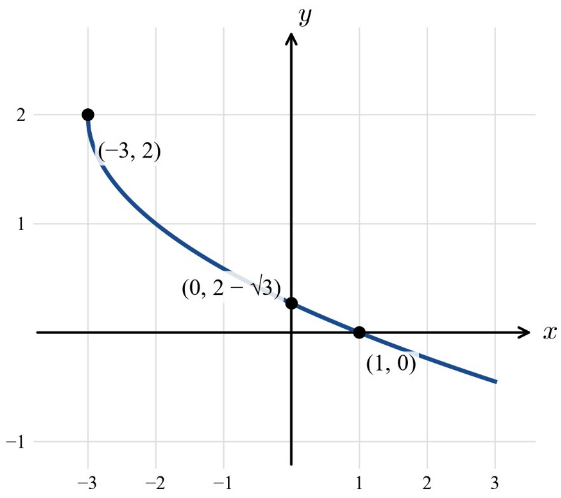
Example 3 — unscaffolded

Sketch $y=-\sqrt{x+3}+2$, stating the domain, range and all intercepts.

$a=-1$, $h=-3$, $k=2$: endpoint $(-3,2)$; the root is $x+3$ (opens right) and $a<0$ (goes down). Domain $[-3,\infty)$, range $(-\infty,2]$.

y-intercept: $x=0 \Rightarrow y=-\sqrt{3}+2=2-\sqrt{3}$, giving $(0,2-\sqrt{3})$.

x-intercept: $-\sqrt{x+3}+2=0 \Rightarrow \sqrt{x+3}=2 \Rightarrow x+3=4 \Rightarrow x=1$, giving $(1,0)$.

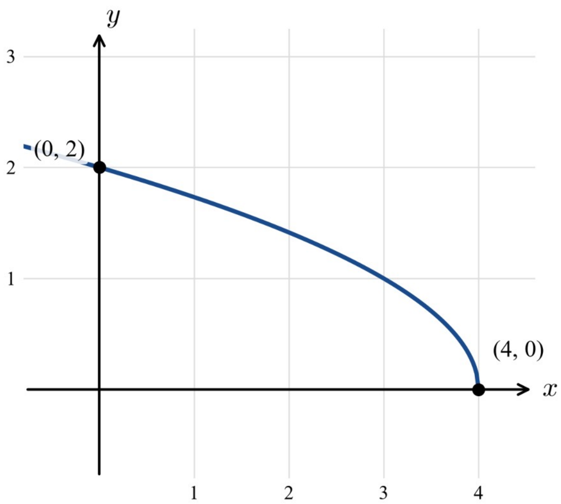


Example 4 — unscaffolded

Sketch $y=\sqrt{4-x}$, stating the domain and range.

Write $4-x=-(x-4)$, so $h=4$, $k=0$, and the graph opens to the **left**. Endpoint $(4,0)$; domain $(-\infty,4]$, range $[0,\infty)$.

y-intercept: $x=0 \Rightarrow y=\sqrt{4}=2$, giving $(0,2)$. x-intercept $(4,0)$ (the endpoint).

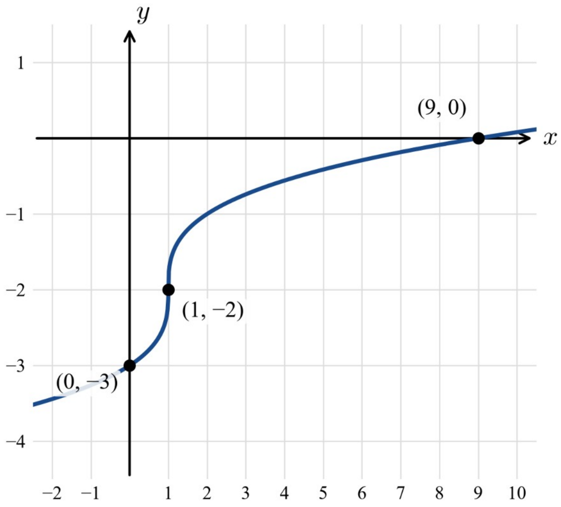


Example 5 — scaffolded

Sketch $y=\sqrt[3]{x-1}-2$, stating the domain, range and all axis intercepts.

Working	Comment
Compare with $y=a\sqrt[3]{x-h}+k$: $a=1$, $h=1$, $k=-2$. Point of inflection $(1,-2)$.	The point of inflection is (h,k).
$a=1>0$, so the graph is **increasing**. Domain $\mathbb{R}$, range $\mathbb{R}$.	A cube root is defined for every real number.
x-intercept: $y=0 \Rightarrow \sqrt[3]{x-1}=2 \Rightarrow x-1=8 \Rightarrow x=9$, giving $(9,0)$.	Cube both sides ($2^3=8$).

Working	Comment
y-intercept: $x=0 \Rightarrow y=\sqrt[3]{-1}-2=-1-2=-3$, giving $(0,-3)$.	$\sqrt[3]{-1}=-1$.



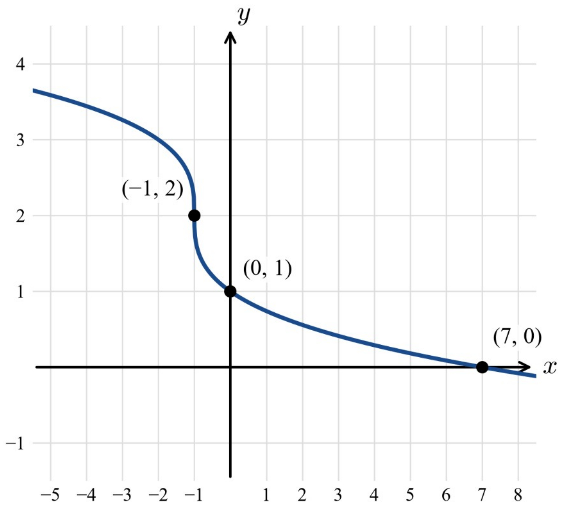
Example 6 — unscaffolded

Sketch $y=-\sqrt[3]{x+1}+2$, stating the domain, range and all intercepts.

$a=-1$, $h=-1$, $k=2$: point of inflection $(-1,2)$. Since $a<0$ the graph is **decreasing** (the increasing shape reflected in the x-axis). Domain $\mathbb{R}$, range $\mathbb{R}$.

x-intercept: $-\sqrt[3]{x+1}+2=0 \Rightarrow \sqrt[3]{x+1}=2 \Rightarrow x+1=8 \Rightarrow x=7$, giving $(7,0)$.

y-intercept: $x=0 \Rightarrow y=-\sqrt[3]{1}+2=-1+2=1$, giving $(0,1)$.



Practice questions

Scaffolded

Q1. For $y=\sqrt{x-4}$: (a) state the endpoint; (b) state the domain and range; (c) sketch the graph.

Q2. For $y=\sqrt{x}+1$: (a) state the endpoint, domain and range; (b) find the axis intercepts; (c) sketch the graph.

Q3. For $y=\sqrt{x+2}-1$: (a) state the endpoint; (b) find the x- and y-intercepts; (c) sketch the graph.

Q4. For $y=2\sqrt{x}$: (a) state the endpoint, domain and range; (b) describe the effect of the 2; (c) sketch the graph.

Q5. For $y = -\sqrt{x} + 3$: (a) state the endpoint and range; (b) find the axis intercepts; (c) sketch the graph.

Q6. For $y = \sqrt{2 - x}$: (a) state the endpoint and domain; (b) find the axis intercepts; (c) sketch the graph.

Unscaffolded

For each of the following: sketch the graph, and state the endpoint, domain, range and all axis intercepts.

Q7. $y = \sqrt{x - 5}$

Q8. $y = \sqrt{x} - 3$

Q9. $y = \sqrt{x + 1} + 2$

Q10. $y = -\sqrt{x - 1}$

Q11. $y = -\sqrt{x} + 4$

Q12. $y = 3\sqrt{x}$

Q13. $y = \sqrt{-x}$

Q14. $y = \sqrt{1 - x}$

Q15. $y = \sqrt{x + 4} - 2$

Q16. $y = 2\sqrt{x - 1} + 1$

Q17. $y = -2\sqrt{x + 2}$

Q18. A square-root graph of the form $y = a\sqrt{x - h} + k$ has endpoint $(2, -1)$ and passes through the point $(6, 1)$. Find its rule, then sketch it.

Cube-root functions

For **Q19–Q22**: sketch the graph, and state the point of inflection, domain, range and all axis intercepts.

Q19. $y = \sqrt[3]{x - 1}$

Q20. $y = \sqrt[3]{x} + 2$

Q21. $y = \sqrt[3]{x + 1} - 2$

Q22. $y = -\sqrt[3]{x - 1} + 1$

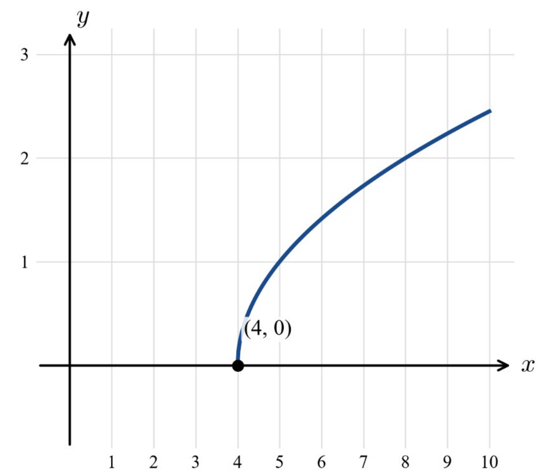
Q23. A cube-root graph of the form $y = a\sqrt[3]{x - h} + k$ has point of inflection $(1, 2)$ and passes through the point $(9, 4)$. Find its rule, then sketch it.

Answers

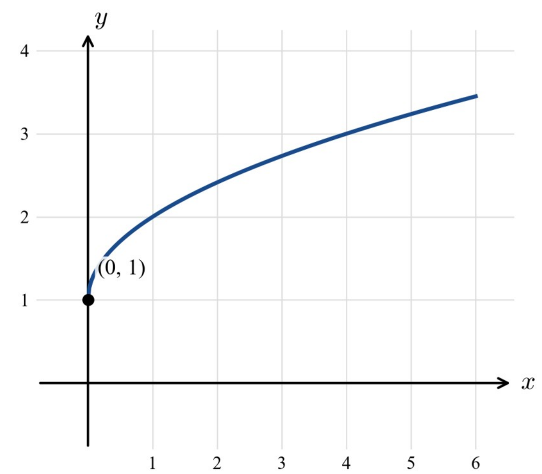
Q1. Endpoint $(4,0)$; domain $[4,\infty)$, range $[0,\infty)$. **Q2.** Endpoint $(0,1)$; domain $[0,\infty)$, range $[1,\infty)$; y-int $(0,1)$; no x-intercept. **Q3.** Endpoint $(-2,-1)$; x-int $(-1,0)$; y-int $(0,\sqrt{2}-1)$. **Q4.** Endpoint $(0,0)$; domain $[0,\infty)$, range $[0,\infty)$; dilation of factor 2 from the x-axis (steeper). **Q5.** Endpoint $(0,3)$; range $(-\infty,3]$; y-int $(0,3)$; x-int $(9,0)$. **Q6.** Endpoint $(2,0)$; domain $(-\infty,2]$; x-int $(2,0)$; y-int $(0,\sqrt{2})$. **Q7.** Endpoint $(5,0)$; domain $[5,\infty)$, range $[0,\infty)$. **Q8.** Endpoint $(0,-3)$; domain $[0,\infty)$, range $[-3,\infty)$; x-int $(9,0)$; y-int $(0,-3)$. **Q9.** Endpoint $(-1,2)$; domain $[-1,\infty)$, range $[2,\infty)$; y-int $(0,3)$; no x-intercept. **Q10.** Endpoint $(1,0)$; domain $[1,\infty)$, range $(-\infty,0]$. **Q11.** Endpoint $(0,4)$; domain $[0,\infty)$, range $(-\infty,4]$; x-int $(16,0)$; y-int $(0,4)$. **Q12.** Endpoint $(0,0)$; domain $[0,\infty)$, range $[0,\infty)$; dilation of factor 3. **Q13.** Endpoint $(0,0)$; domain $(-\infty,0]$, range $[0,\infty)$; reflection of $y=\sqrt{x}$ in the y-axis. **Q14.** Endpoint $(1,0)$; domain $(-\infty,1]$, range $[0,\infty)$; y-int $(0,1)$. **Q15.** Endpoint $(-4,-2)$; domain $[-4,\infty)$, range $[-2,\infty)$; passes through the origin $(0,0)$. **Q16.** Endpoint $(1,1)$; domain $[1,\infty)$, range $[1,\infty)$; no axis intercepts. **Q17.** Endpoint $(-2,0)$; domain $[-2,\infty)$, range $(-\infty,0]$; x-int $(-2,0)$; y-int $(0,-2\sqrt{2})$. **Q18.** $y=\sqrt{x-2}-1$; x-int $(3,0)$. **Q19.** Inflection $(1,0)$; domain $\mathbb{R}$, range $\mathbb{R}$; x-int $(1,0)$; y-int $(0,-1)$. **Q20.** Inflection $(0,2)$; domain $\mathbb{R}$, range $\mathbb{R}$; x-int $(-8,0)$; y-int $(0,2)$. **Q21.** Inflection $(-1,-2)$; domain $\mathbb{R}$, range $\mathbb{R}$; x-int $(7,0)$; y-int $(0,-1)$. **Q22.** Inflection $(1,1)$; decreasing; domain $\mathbb{R}$, range $\mathbb{R}$; x-int $(2,0)$; y-int $(0,2)$. **Q23.** $y=\sqrt[3]{x-1}+2$; inflection $(1,2)$; x-int $(-7,0)$; y-int $(0,1)$.

Fully-worked solutions

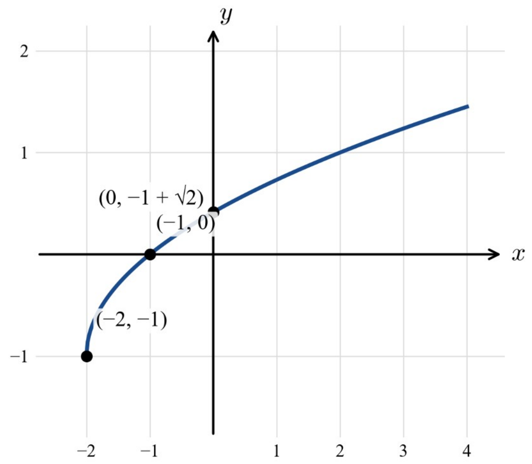
Q1. $y=\sqrt{x-4}$: $a=1$, $h=4$, $k=0$. Endpoint $(4,0)$; opens right and up. Domain $[4,\infty)$, range $[0,\infty)$.



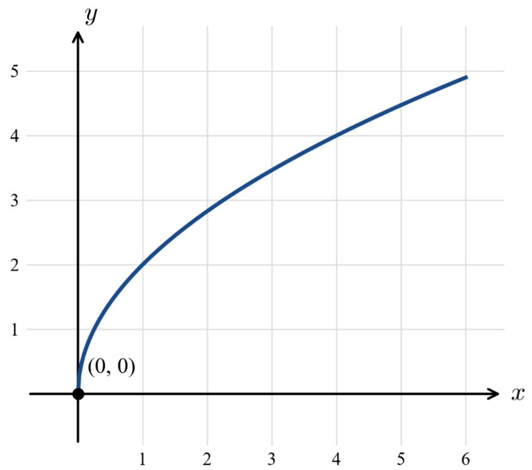
Q2. $y=\sqrt{x}+1$: endpoint $(0,1)$; domain $[0,\infty)$, range $[1,\infty)$. y-intercept $(0,1)$; $y=0$ gives $\sqrt{x}=-1$, impossible, so there is no x-intercept.



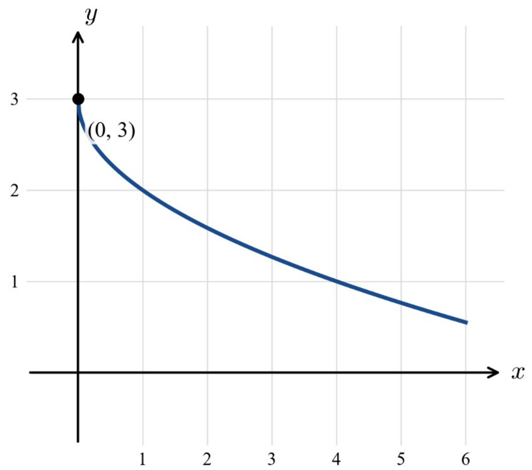
Q3. $y=\sqrt{x+2}-1$: endpoint $(-2,-1)$. x-intercept: $\sqrt{x+2}=1 \Rightarrow x+2=1 \Rightarrow x=-1$, giving $(-1,0)$. y-intercept: $x=0 \Rightarrow y=\sqrt{2}-1$, giving $(0,\sqrt{2}-1)$.



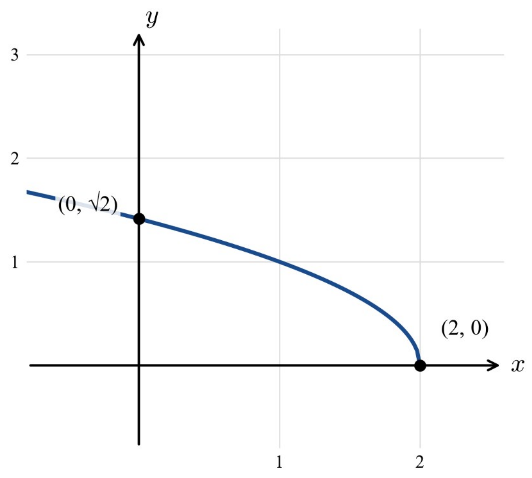
Q4. $y=2\sqrt{x}$: endpoint $(0,0)$; domain $[0,\infty)$, range $[0,\infty)$. The factor 2 is a dilation of factor 2 from the x-axis, so the graph rises twice as fast as $y=\sqrt{x}$ (e.g. at $x=4$, $y=4$).



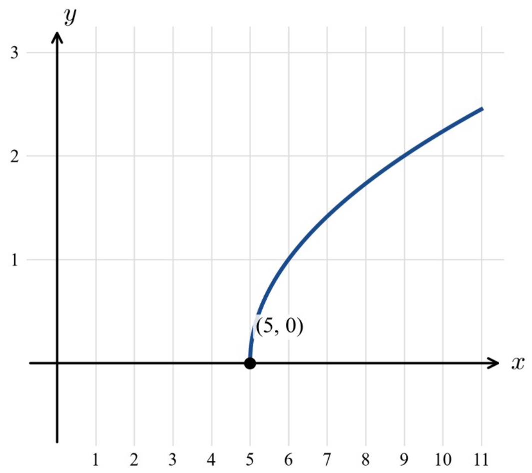
Q5. $y=-\sqrt{x}+3$: endpoint $(0,3)$; $a=-1$ so it goes down; range $(-\infty,3]$. y-intercept $(0,3)$; x-intercept: $\sqrt{x}=3 \Rightarrow x=9$, giving $(9,0)$.



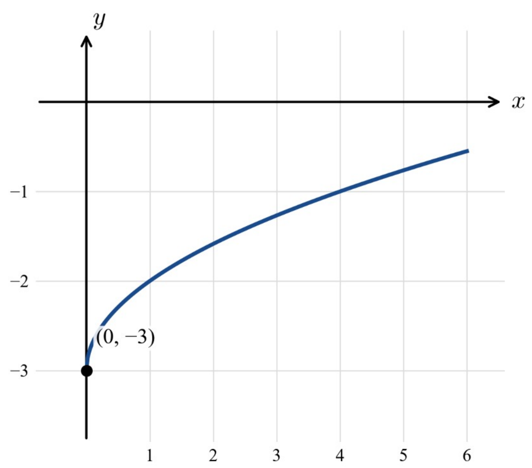
Q6. $y=\sqrt{2-x}$: $2-x=-(x-2)$, so endpoint $(2,0)$, opens left. Domain $(-\infty,2]$. x-intercept $(2,0)$; y-intercept: $x=0 \Rightarrow y=\sqrt{2}$, giving $(0,\sqrt{2})$.



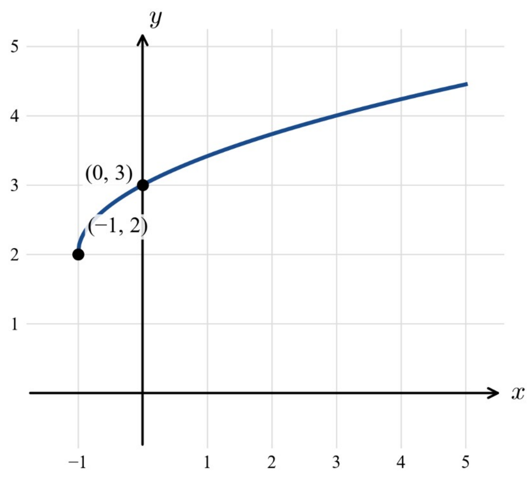
Q7. $y=\sqrt{x-5}$: endpoint $(5,0)$; opens right and up. Domain $[5,\infty)$, range $[0,\infty)$.



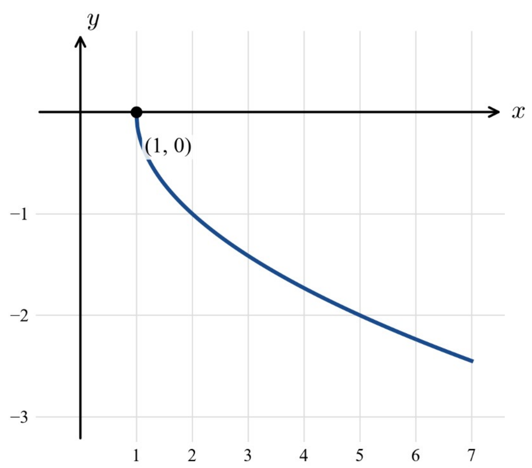
Q8. $y=\sqrt{x}-3$: endpoint $(0,-3)$; domain $[0,\infty)$, range $[-3,\infty)$. $\sqrt{x}=3 \Rightarrow x=9$: x-intercept $(9,0)$; y-intercept $(0,-3)$.



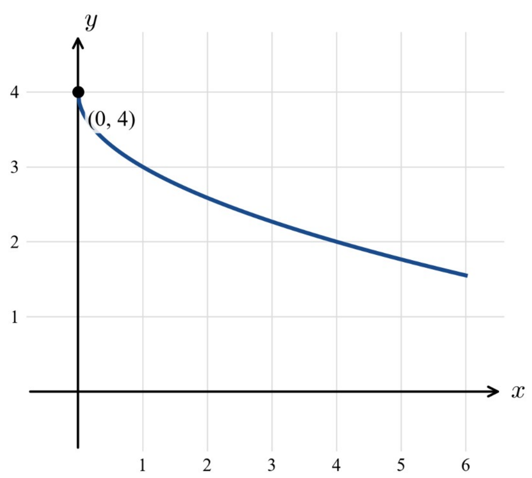
Q9. $y=\sqrt{x+1}+2$: endpoint $(-1,2)$; domain $[-1,\infty)$, range $[2,\infty)$. y-intercept: $x=0 \Rightarrow y=3$, giving $(0,3)$. Since $y \geq 2$ there is no x-intercept.



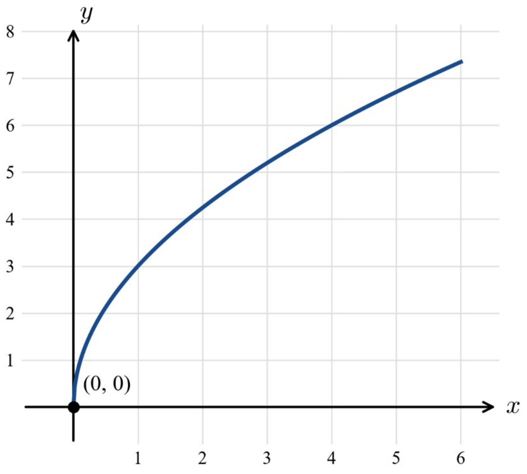
Q10. $y=-\sqrt{x-1}$: endpoint $(1,0)$; $a=-1$ so it goes down. Domain $[1,\infty)$, range $(-\infty,0]$.



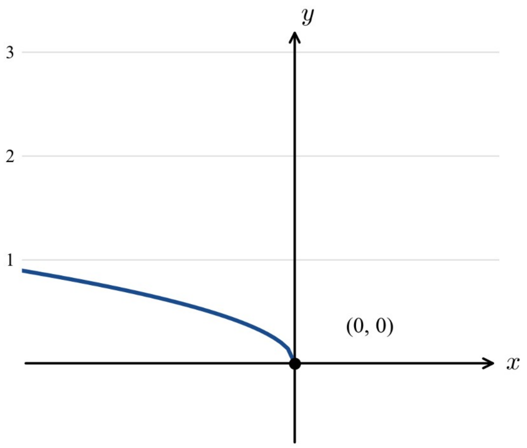
Q11. $y=-\sqrt{x}+4$: endpoint $(0,4)$; goes down; range $(-\infty,4]$. $\sqrt{x}=4 \Rightarrow x=16$: x-intercept $(16,0)$; y-intercept $(0,4)$.



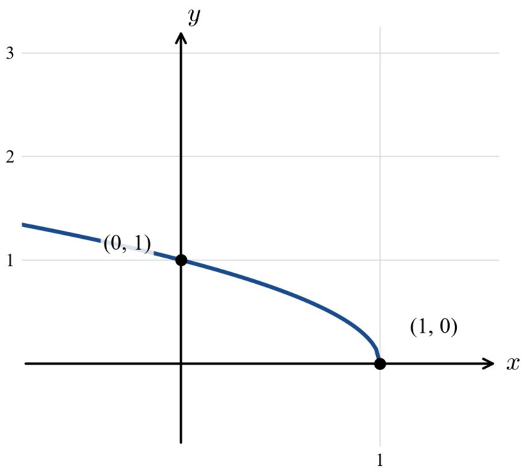
Q12. $y=3\sqrt{x}$: endpoint $(0,0)$; domain $[0,\infty)$, range $[0,\infty)$; a dilation of factor 3 from the x-axis (e.g. at $x=4$, $y=6$).



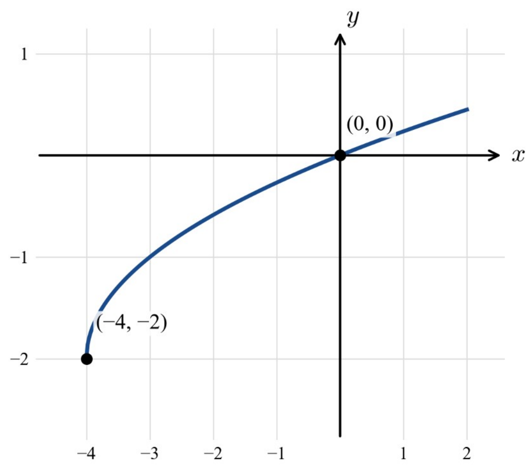
Q13. $y=\sqrt{-x}$: endpoint $(0,0)$; opens left. Domain $(-\infty,0]$, range $[0,\infty)$ — the reflection of $y=\sqrt{x}$ in the y-axis.



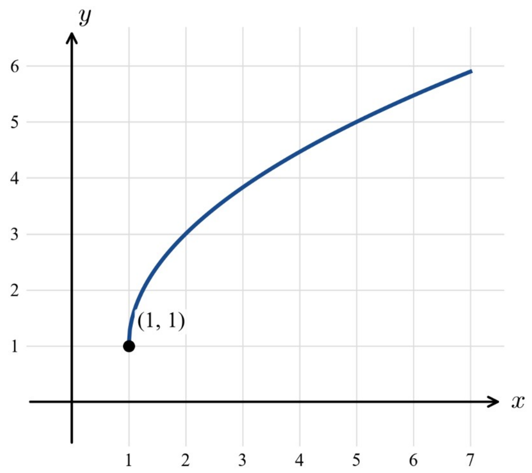
Q14. $y=\sqrt{1-x}$: $1-x=-(x-1)$, endpoint $(1,0)$, opens left. Domain $(-\infty,1]$. y-intercept: $x=0 \Rightarrow y=1$, giving $(0,1)$.



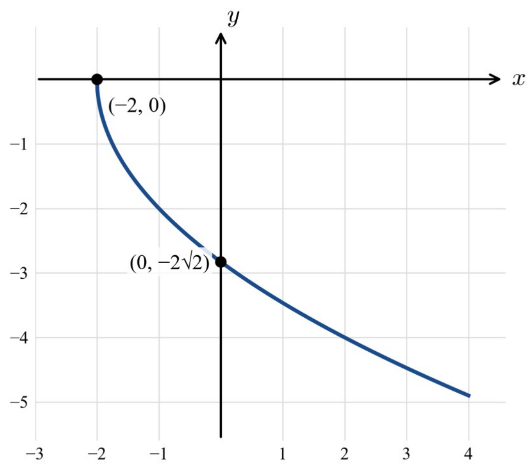
Q15. $y=\sqrt{x+4}-2$: endpoint $(-4,-2)$; domain $[-4,\infty)$, range $[-2,\infty)$. x-intercept: $\sqrt{x+4}=2 \Rightarrow x+4=4 \Rightarrow x=0$, so the graph passes through the origin $(0,0)$ (both intercepts).



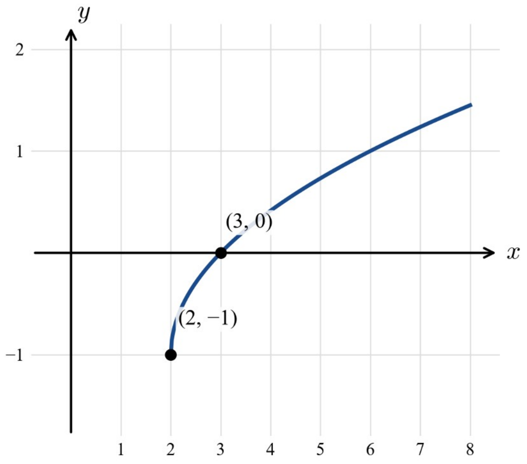
Q16. $y=2\sqrt{x-1}+1$: endpoint $(1,1)$; domain $[1,\infty)$, range $[1,\infty)$. Since $y\geq 1$ there are no axis intercepts.



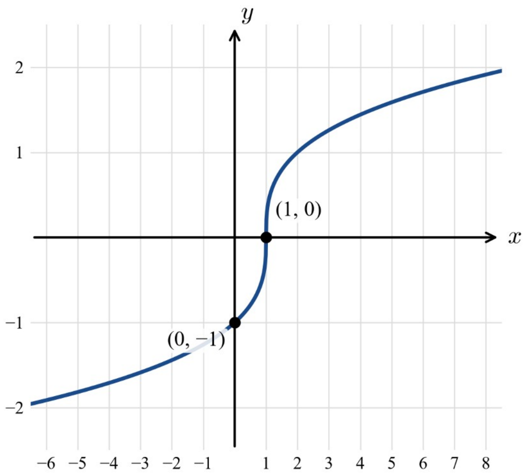
Q17. $y=-2\sqrt{x+2}$: endpoint $(-2,0)$; $a=-2$ (down, steeper). Domain $[-2,\infty)$, range $(-\infty,0]$. x-intercept $(-2,0)$; y-intercept: $x=0\Rightarrow y=-2\sqrt{2}$, giving $(0,-2\sqrt{2})$.



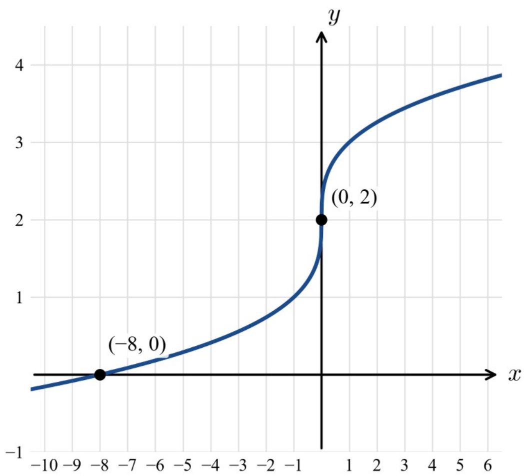
Q18. Endpoint $(2,-1)$ gives $h=2$, $k=-1$, so $y=a\sqrt{x-2}-1$. Substituting the point $(6,1)$: $1=a\sqrt{6-2}-1=2a-1\Rightarrow a=1$. $\therefore\ y=\sqrt{x-2}-1$. x-intercept: $\sqrt{x-2}=1\Rightarrow x=3$, giving $(3,0)$.



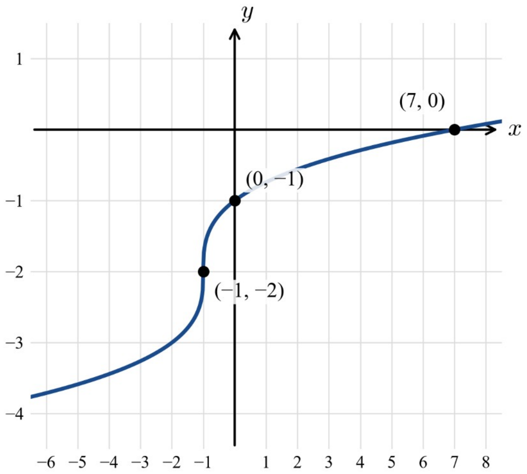
Q19. $y=\sqrt[3]{x-1}$: $a=1$, $h=1$, $k=0$. Point of inflection $(1,0)$; increasing. Domain $\mathbb{R}$, range $\mathbb{R}$. x-intercept: $\sqrt[3]{x-1}=0 \Rightarrow x=1$, giving $(1,0)$ (the point of inflection). y-intercept: $x=0 \Rightarrow y=\sqrt[3]{-1}=-1$, giving $(0,-1)$.



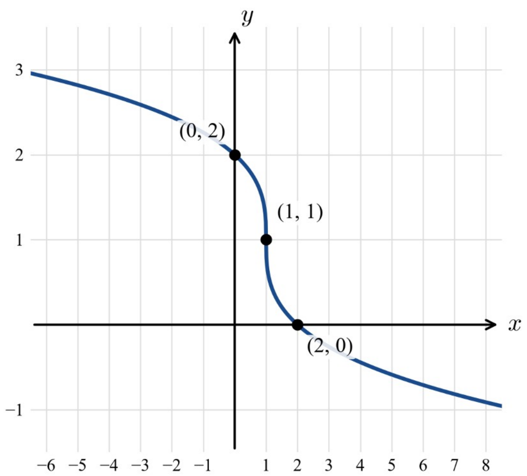
Q20. $y=\sqrt[3]{x}+2$: point of inflection $(0,2)$; domain $\mathbb{R}$, range $\mathbb{R}$. y-intercept $(0,2)$. x-intercept: $\sqrt[3]{x}=-2 \Rightarrow x=(-2)^3=-8$, giving $(-8,0)$.



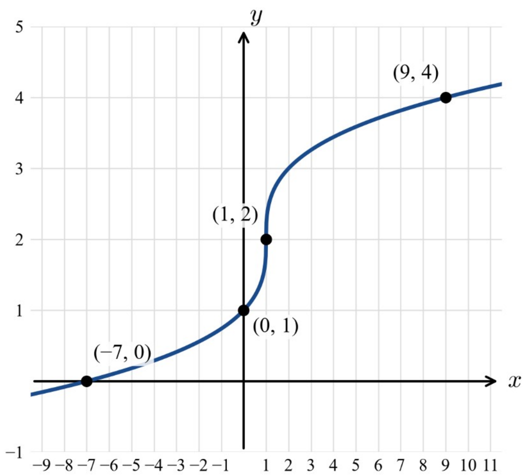
Q21. $y=\sqrt[3]{x+1}-2$: point of inflection $(-1,-2)$; domain $\mathbb{R}$, range $\mathbb{R}$. x-intercept: $\sqrt[3]{x+1}=2 \Rightarrow x+1=8 \Rightarrow x=7$, giving $(7,0)$. y-intercept: $x=0 \Rightarrow y=\sqrt[3]{1}-2=-1$, giving $(0,-1)$.



Q22. $y=-\sqrt[3]{x-1}+1$: $a=-1$, so the graph is **decreasing**; point of inflection $(1,1)$; domain $\mathbb{R}$, range $\mathbb{R}$. x-intercept: $-\sqrt[3]{x-1}+1=0 \Rightarrow \sqrt[3]{x-1}=1 \Rightarrow x-1=1 \Rightarrow x=2$, giving $(2,0)$. y-intercept: $x=0 \Rightarrow y=-\sqrt[3]{-1}+1=-(-1)+1=2$, giving $(0,2)$.



Q23. Point of inflection $(1,2)$ gives $h=1$, $k=2$, so $y=a\sqrt[3]{x-1}+2$. Substituting the point $(9,4)$: $4=a\sqrt[3]{9-1}+2=a\sqrt[3]{8}+2=2a+2 \Rightarrow a=1$. $\therefore\ y=\sqrt[3]{x-1}+2$. x-intercept: $\sqrt[3]{x-1}=-2 \Rightarrow x-1=-8 \Rightarrow x=-7$, giving $(-7,0)$; y-intercept: $x=0 \Rightarrow y=\sqrt[3]{-1}+2=1$, giving $(0,1)$.



Theory

The circle with centre (h, k) and radius r is the set of all points at distance r from the centre. By the distance formula, its equation is

$$(x-h)^2+(y-k)^2=r^2.$$

- The circle $x^2+y^2=r^2$ has centre the origin $(0, 0)$ and radius r.
- The **domain** is $[h-r, h+r]$ and the **range** is $[k-r, k+r]$.

General form. Expanding gives an equation of the form $x^2+y^2+Dx+Ey+F=0$. To find the centre and radius, **complete the square** in x and in y.

Axis intercepts. Put $x=0$ for the y-intercepts and $y=0$ for the x-intercepts. A circle may cross an axis twice, touch it once, or not at all.

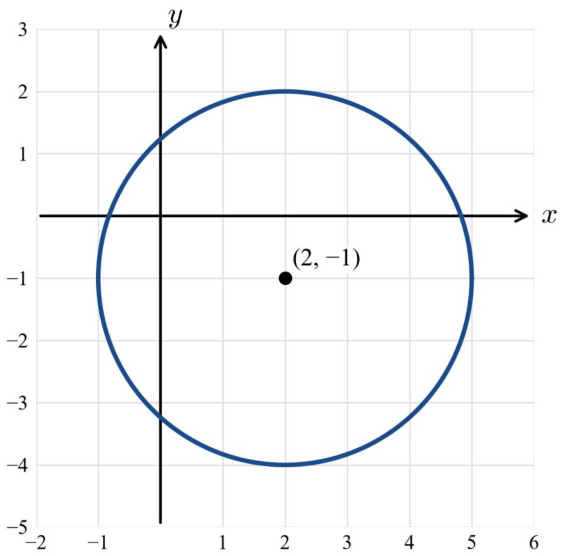
Semicircles as functions. The full circle is not a function (it fails the vertical-line test), but each half is. Solving $x^2+y^2=r^2$ for y gives the **upper** semicircle $y=\sqrt{r^2-x^2}$ (range $[0, r]$) and the **lower** semicircle $y=-\sqrt{r^2-x^2}$ (range $[-r, 0]$); both have domain $[-r, r]$.

Worked examples

Example 1 — scaffolded

State the centre and radius of $(x-2)^2+(y+1)^2=9$, and sketch its graph.

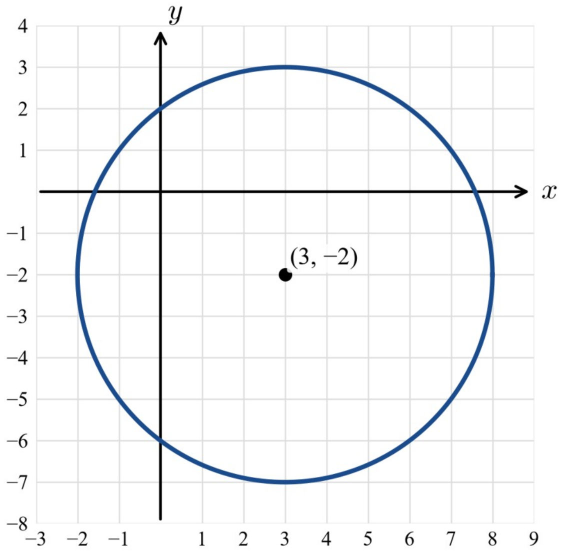
Working	Comment
Compare with $(x-h)^2+(y-k)^2=r^2$: $h=2$, $k=-1$.	Note $(y+1)^2=(y-(-1))^2$, so $k=-1$.
Centre $(2, -1)$.	The centre is (h, k).
$r^2=9 \Rightarrow r=3$.	Take the positive square root.
Plot the centre and draw a circle of radius 3.	The circle reaches 3 units in every direction from the centre.



Example 2 — scaffolded

Find the centre and radius of the circle $x^2+y^2-6x+4y-12=0$.

Working	Comment
Group: $(x^2-6x)+(y^2+4y)=12$.	Move the constant to the right.
Complete the square: $(x^2-6x+9)+(y^2+4y+4)=12+9+4$.	Add $(6/2)^2=9$ and $(4/2)^2=4$ to **both** sides.
$(x-3)^2+(y+2)^2=25$.	Write each bracket as a perfect square.
Centre $(3,-2)$, radius $r=\sqrt{25}=5$.	Read off (h,k) and r.



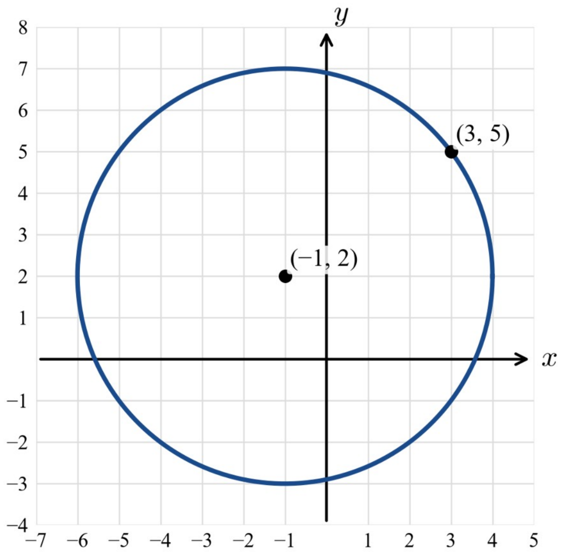
Example 3 — unscaffolded

Find the equation of the circle with centre $(-1,2)$ that passes through the point $(3,5)$.

The radius is the distance from the centre to $(3,5)$:

$$r^2=(3-(-1))^2+(5-2)^2=4^2+3^2=16+9=25.$$

$\therefore$ the equation is $(x+1)^2+(y-2)^2=25$ (radius 5).

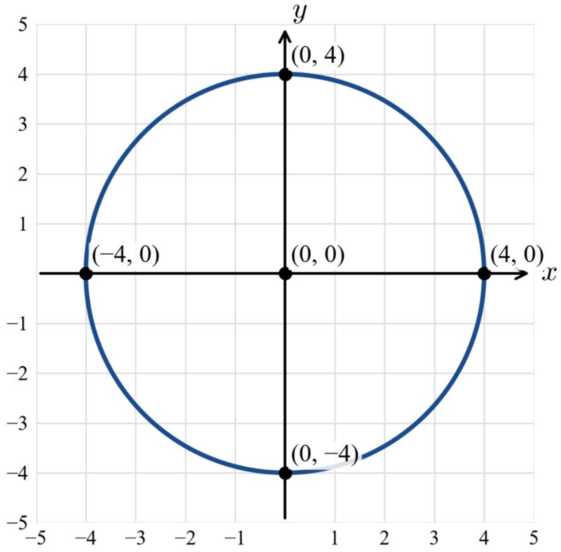


Example 4 — unscaffolded

For the circle $x^2 + y^2 = 16$, find the axis intercepts, then write the rule of the **upper** semicircle as a function, stating its domain and range.

Centre $(0, 0)$, radius 4. x-intercepts: $y = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$, giving $(4, 0)$ and $(-4, 0)$. y-intercepts: $(0, 4)$ and $(0, -4)$.

Upper semicircle: $y = \sqrt{16 - x^2}$, with domain $[-4, 4]$ and range $[0, 4]$.



Practice questions

Scaffolded

Q1. For $(x-1)^2 + (y-3)^2 = 4$: (a) state the centre; (b) state the radius; (c) sketch.

Q2. For $(x+2)^2 + (y-1)^2 = 16$: (a) state the centre and radius; (b) sketch.

Q3. For $x^2 + y^2 - 4x - 2y - 4 = 0$: (a) complete the square; (b) hence state the centre and radius; (c) sketch.

Q4. Write the equation of the circle with centre the origin and radius $\sqrt{7}$.

Q5. Write the equation of the circle with centre $(2, -3)$ and radius 4.

Q6. For $x^2 + y^2 = 9$: (a) write the rule of the upper semicircle as a function; (b) state its domain and range; (c) sketch it.

Unscaffolded

Q7. For $(x-4)^2 + (y+2)^2 = 25$, state the centre and radius, find the y-intercepts, and sketch.

Q8. State the centre and radius of $x^2 + (y-5)^2 = 1$ and sketch. Does the circle cross the x-axis? Explain.

Q9. Find the centre and radius of $x^2 + y^2 - 8x + 6y + 9 = 0$ and sketch.

Q10. Find the centre and radius of $x^2 + y^2 + 2x - 10y + 17 = 0$ and sketch.

Q11. Write the equation of the circle with centre $(-3, 4)$ and radius 6.

Q12. Find the equation of the circle with centre $(1, 2)$ passing through $(4, 6)$.

Q13. Find all axis intercepts of $(x-3)^2 + (y-4)^2 = 25$ and sketch.

Q14. Find the axis intercepts of $x^2 + y^2 = 10$, giving exact values, and sketch.

Q15. Write the rule of the **lower** semicircle of $x^2 + y^2 = 25$ as a function, state its domain and range, and sketch it.

Q16. Sketch $y = \sqrt{4 - x^2}$, stating its domain and range and describing the graph.

Q17. The points $(1, 2)$ and $(7, 10)$ are the endpoints of a diameter of a circle. Find the centre, the radius, and the equation of the circle.

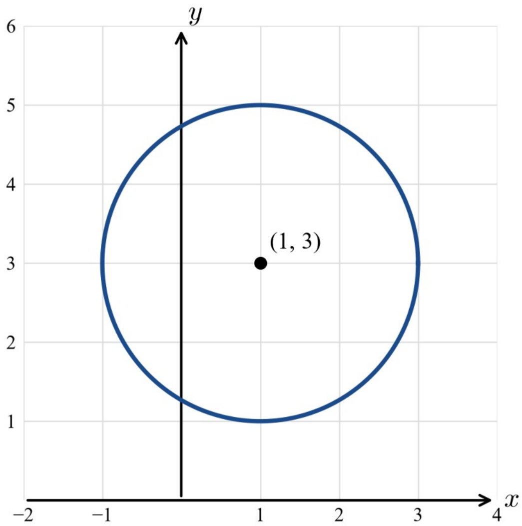
Q18. Determine whether the point $(2, 3)$ lies inside, on, or outside the circle $x^2 + y^2 = 16$.

Answers

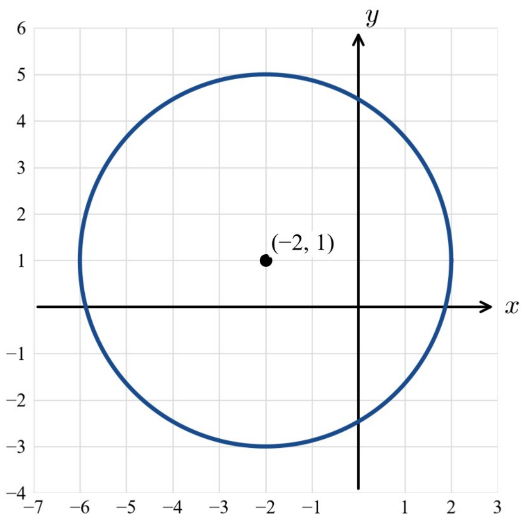
Q1. Centre $(1,3)$; radius 2. **Q2.** Centre $(-2,1)$; radius 4. **Q3.** $(x-2)^2+(y-1)^2=9$; centre $(2,1)$; radius 3. **Q4.** $x^2+y^2=7$. **Q5.** $(x-2)^2+(y+3)^2=16$. **Q6.** $y=\sqrt{9-x^2}$; domain $[-3,3]$, range $[0,3]$. **Q7.** Centre $(4,-2)$, radius 5; y-intercepts $(0,1)$ and $(0,-5)$. **Q8.** Centre $(0,5)$, radius 1; no — the lowest point is $(0,4)$, so it does not reach the x-axis. **Q9.** Centre $(4,-3)$, radius 4. **Q10.** Centre $(-1,5)$, radius 3. **Q11.** $(x+3)^2+(y-4)^2=36$. **Q12.** $(x-1)^2+(y-2)^2=25$. **Q13.** $(0,0)$, $(6,0)$ and $(0,8)$. **Q14.** $(\sqrt{10},0)$, $(-\sqrt{10},0)$, $(0,\sqrt{10})$, $(0,-\sqrt{10})$. **Q15.** $y=-\sqrt{25-x^2}$; domain $[-5,5]$, range $[-5,0]$. **Q16.** Upper semicircle, radius 2; domain $[-2,2]$, range $[0,2]$. **Q17.** Centre $(4,6)$, radius 5; $(x-4)^2+(y-6)^2=25$. **Q18.** Inside.

Fully-worked solutions

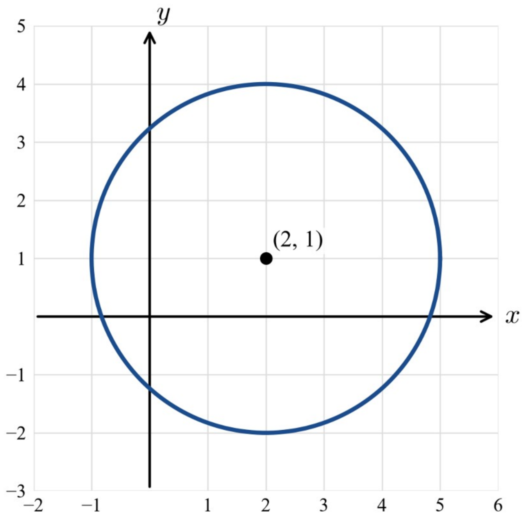
Q1. Compare with $(x-h)^2+(y-k)^2=r^2$: (a) centre $(1,3)$; (b) $r=\sqrt{4}=2$.



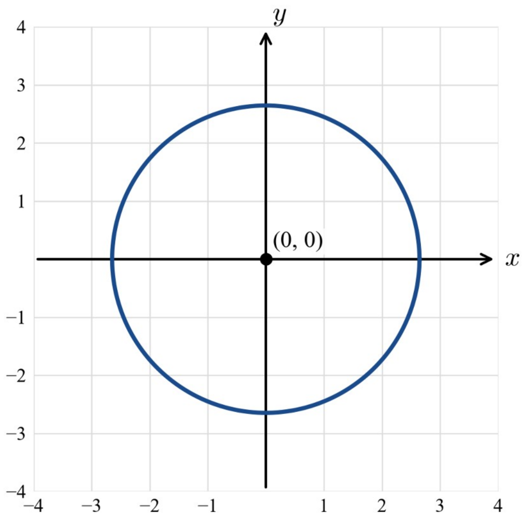
Q2. (a) Centre $(-2,1)$, radius $\sqrt{16}=4$.



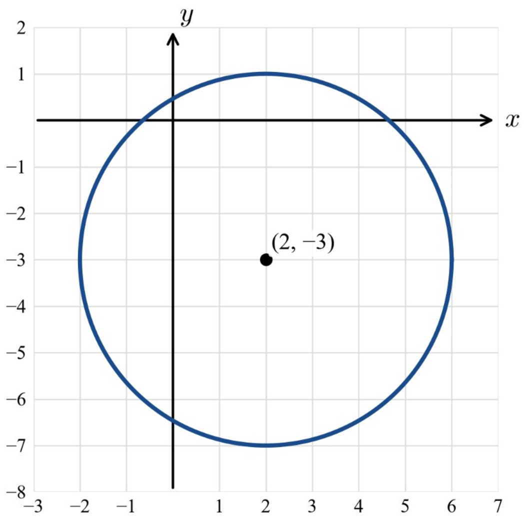
Q3. (a) $(x^2-4x+4)+(y^2-2y+1)=4+4+1 \Rightarrow (x-2)^2+(y-1)^2=9$. (b) Centre $(2,1)$, radius 3.



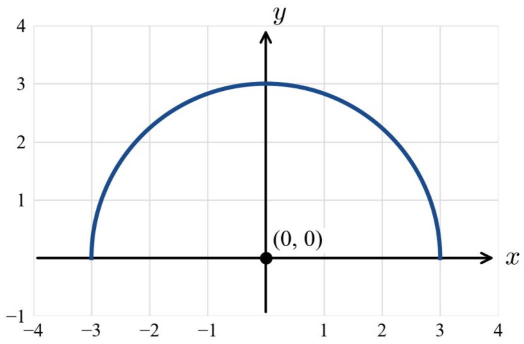
Q4. $x^2 + y^2 = (\sqrt{7})^2 = 7$.



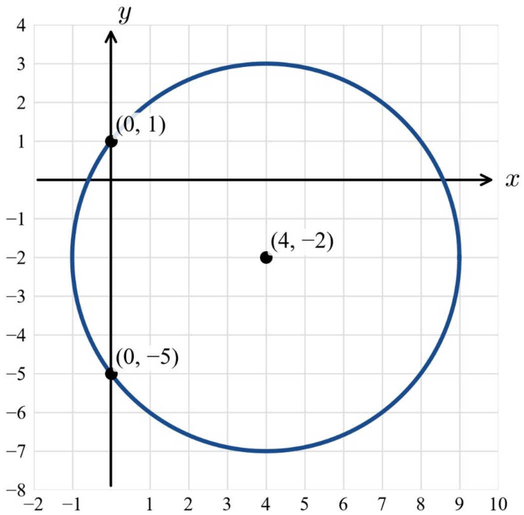
Q5. $(x-2)^2 + (y-(-3))^2 = 4^2$, i.e. $(x-2)^2 + (y+3)^2 = 16$.



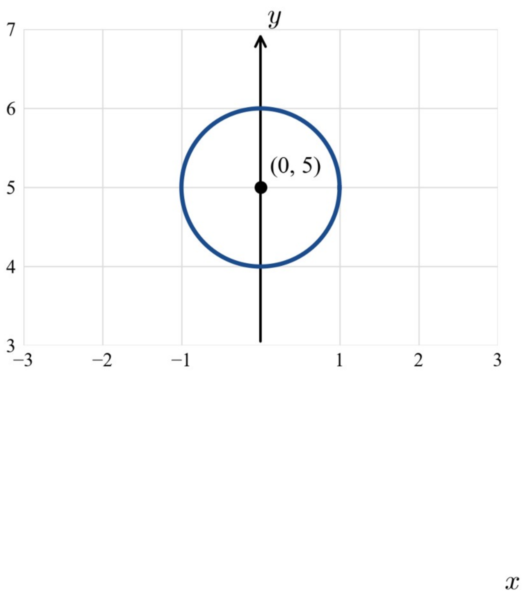
Q6. (a) $y^2 = 9 - x^2 \Rightarrow y = \sqrt{9 - x^2}$ (positive root for the upper half). (b) Domain $[-3, 3]$, range $[0, 3]$.



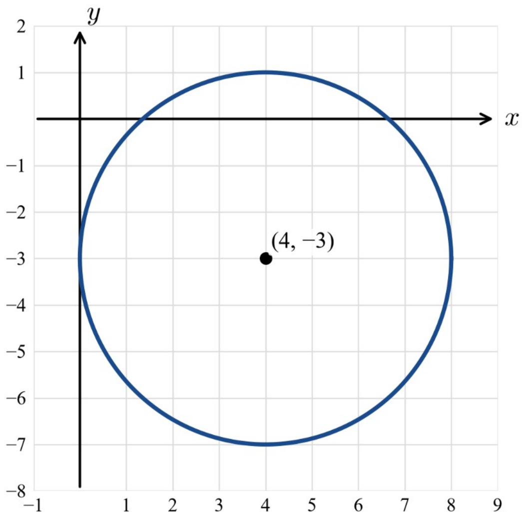
Q7. Centre $(4, -2)$, radius 5. y-intercepts: $x = 0 \Rightarrow 16 + (y+2)^2 = 25 \Rightarrow (y+2)^2 = 9 \Rightarrow y + 2 = \pm 3$, so $y = 1$ or $y = -5$, giving $(0, 1)$ and $(0, -5)$.



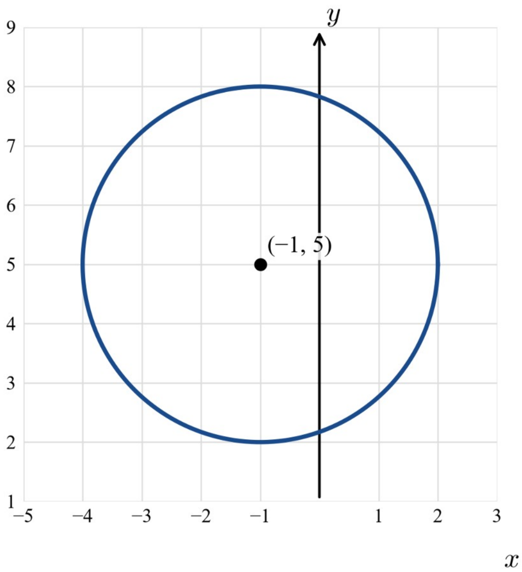
Q8. Centre $(0, 5)$, radius 1. The circle spans $y \in [4, 6]$, so it does **not** cross the x-axis (its lowest point $(0, 4)$ is above it).



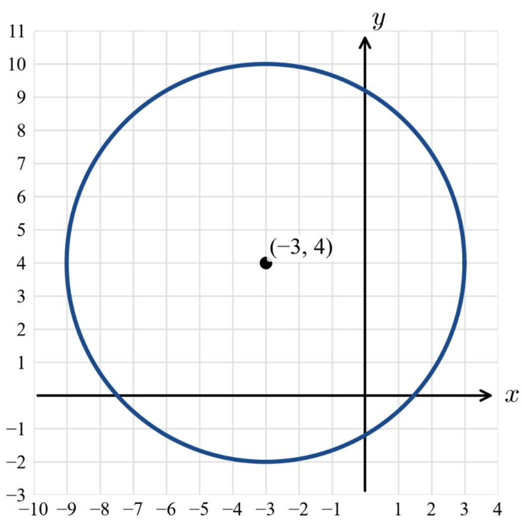
Q9. $(x^2 - 8x + 16) + (y^2 + 6y + 9) = -9 + 16 + 9 \Rightarrow (x-4)^2 + (y+3)^2 = 16$. Centre $(4, -3)$, radius 4.



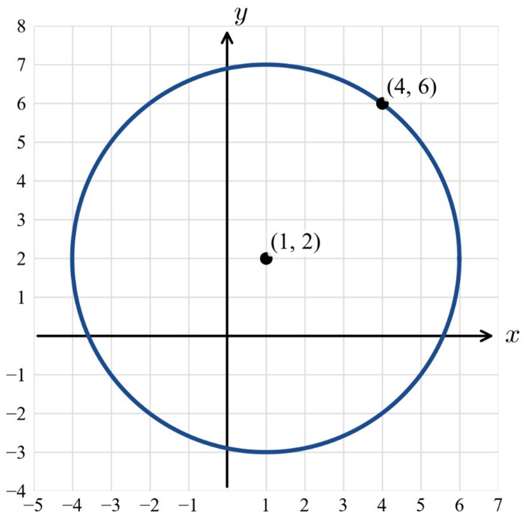
Q10. $(x^2+2x+1)+(y^2-10y+25)=-17+1+25 \Rightarrow (x+1)^2+(y-5)^2=9$. Centre $(-1,5)$, radius 3.



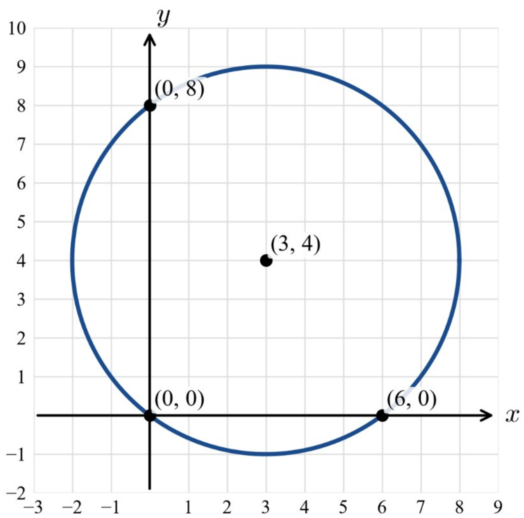
Q11. $(x-(-3))^2+(y-4)^2=6^2$, i.e. $(x+3)^2+(y-4)^2=36$.



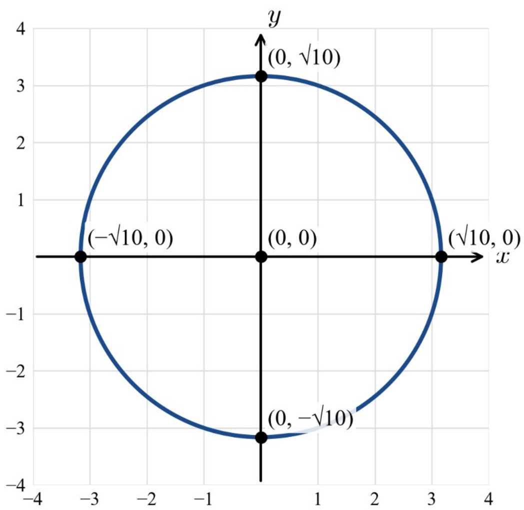
Q12. $r^2=(4-1)^2+(6-2)^2=9+16=25$. $\therefore (x-1)^2+(y-2)^2=25$.



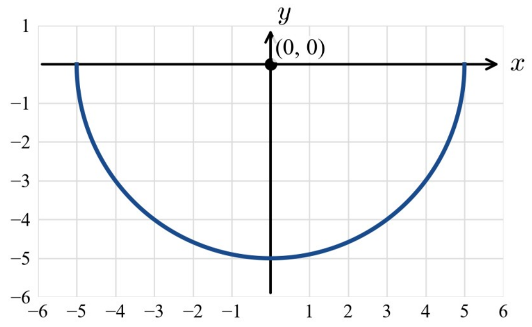
Q13. x-intercepts: $y=0 \Rightarrow (x-3)^2+16=25 \Rightarrow (x-3)^2=9 \Rightarrow x=0$ or 6, giving $(0,0)$ and $(6,0)$. y-intercepts: $x=0 \Rightarrow 9+(y-4)^2=25 \Rightarrow (y-4)^2=16 \Rightarrow y=0$ or 8, giving $(0,0)$ and $(0,8)$.



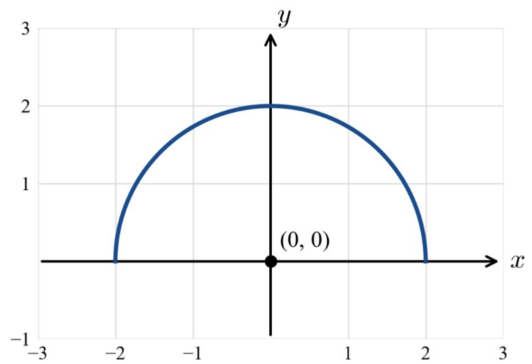
Q14. x-intercepts: $y=0 \Rightarrow x^2=10 \Rightarrow x=\pm\sqrt{10}$. y-intercepts: $x=0 \Rightarrow y=\pm\sqrt{10}$. Points $(\pm\sqrt{10},0)$ and $(0,\pm\sqrt{10})$.



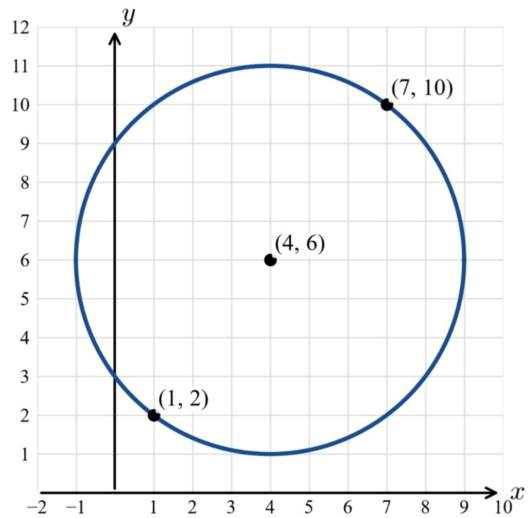
Q15. $y^2=25-x^2 \Rightarrow y=-\sqrt{25-x^2}$ (negative root for the lower half). Domain $[-5,5]$, range $[-5,0]$.



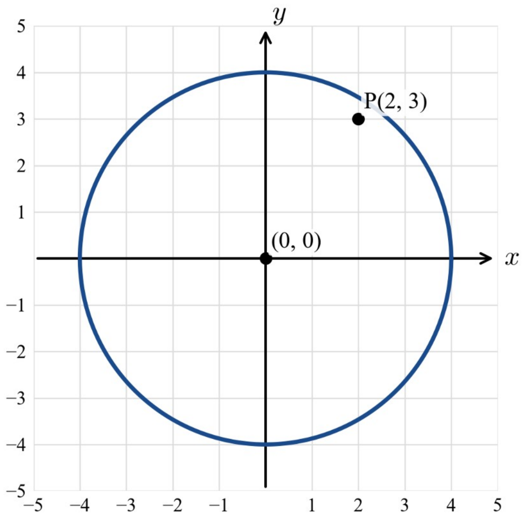
Q16. $y=\sqrt{4-x^2}$ is the **upper** semicircle of $x^2+y^2=4$ (centre origin, radius 2). Domain $[-2,2]$, range $[0,2]$.



Q17. The centre is the midpoint of the diameter: $\left(\frac{1+7}{2},\frac{2+10}{2}\right)=(4,6)$. The radius is half the diameter's length: $r=1/2\sqrt{(7-1)^2+(10-2)^2}=1/2\sqrt{36+64}=1/2\sqrt{100}=5$. $\therefore (x-4)^2+(y-6)^2=25$.



Q18. Compare the distance from the origin with the radius: $2^2+3^2=4+9=13<16=r^2$. Since $13<16$, the point $(2,3)$ lies **inside** the circle.


y
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P(2, 3)
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1
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5

Theory

Each graph in this chapter belongs to a **family** with a standard form. To find the rule of a particular graph, first **identify the family** from its key features, then use a **known point** to find the remaining constant.

Rectangular hyperbola $y=\dfrac{a}{x-h}+k$:

- vertical asymptote $x=h$, horizontal asymptote $y=k$;
- substitute a point on the curve to find a (the sign of a decides which pair of regions the branches occupy).

Inverse-square curve $y=\dfrac{a}{(x-h)^2}+k$:

- vertical asymptote $x=h$, horizontal asymptote $y=k$;
- both branches lie **above** the horizontal asymptote when $a>0$ and **below** it when $a<0$.

Square root $y=a\sqrt{x-h}+k$:

- the graph has an **endpoint** at (h,k) and is defined for $x\geq h$;
- the sign of a decides whether the curve rises or falls from the endpoint.

Circle $(x-h)^2+(y-k)^2=r^2$:

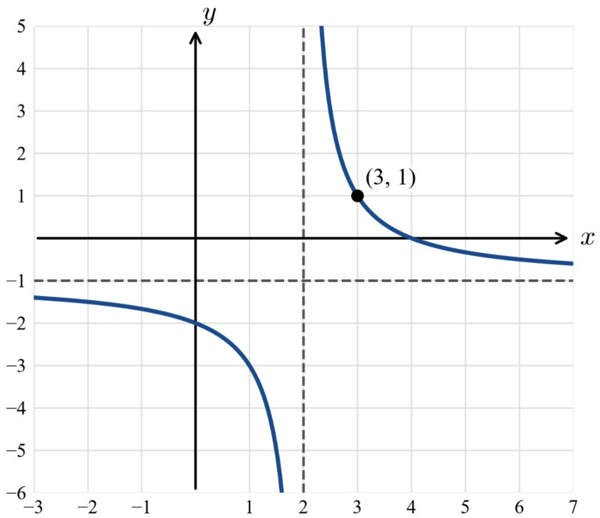
- centre (h,k), radius r;
- given the centre and a point on the circle, r is the distance between them; given the two ends of a **diameter**, the centre is their midpoint and r is half the diameter's length.

Worked examples

Example 1 — scaffolded

A rectangular hyperbola has vertical asymptote $x=2$, horizontal asymptote $y=-1$, and passes through $(3,1)$. Find its rule.

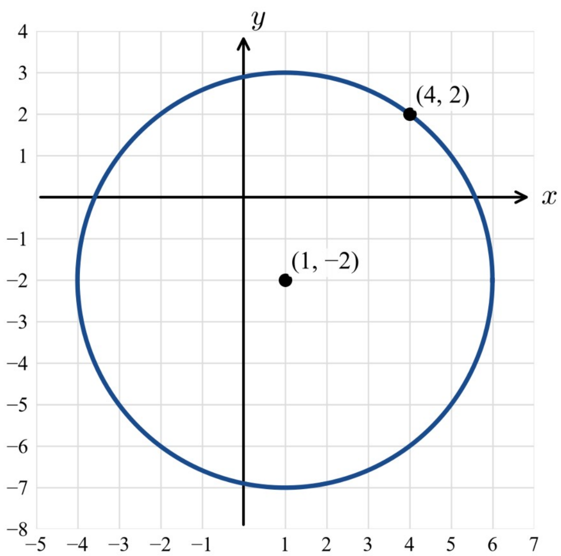
Working	Comment
The asymptotes give the form $y=\dfrac{a}{x-2}-1$.	Vertical asymptote $x=h$ and horizontal asymptote $y=k$.
Substitute $(3,1)$: $1=\dfrac{a}{3-2}-1=a-1$.	Use the known point to find a.
$a=2$.	Solve for a.
$\therefore y=\dfrac{2}{x-2}-1$.	State the rule.



Example 2 — scaffolded

A circle has centre $(1,-2)$ and passes through $(4,2)$. Find its equation.

Working	Comment
$(x-1)^2+(y+2)^2=r^2$.	Centre $(h,k)=(1,-2)$.
$r^2=(4-1)^2+(2-(-2))^2=9+16=25$.	r is the distance from the centre to the point.
$\therefore (x-1)^2+(y+2)^2=25$.	State the equation (radius 5).

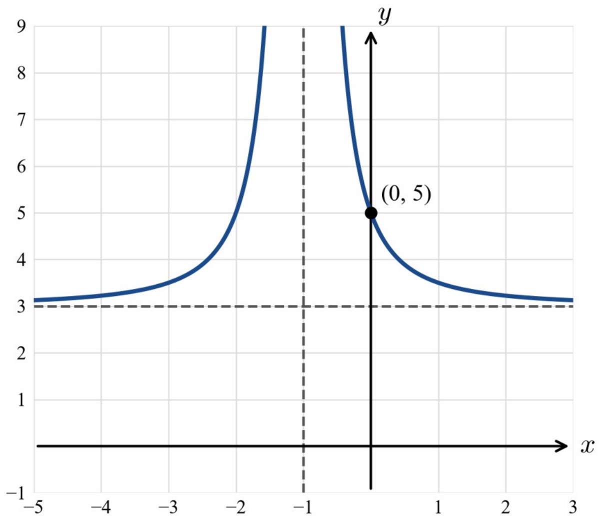


Example 3 — unscaffolded

A curve of the form $y=\dfrac{a}{(x-h)^2}+k$ has asymptotes $x=-1$ and $y=3$ and passes through $(0,5)$. Find its rule.

Form: $y=\dfrac{a}{(x+1)^2}+3$. Substitute $(0,5)$: $5=\dfrac{a}{(0+1)^2}+3=a+3$, so $a=2$.

$\therefore y=\dfrac{2}{(x+1)^2}+3$.

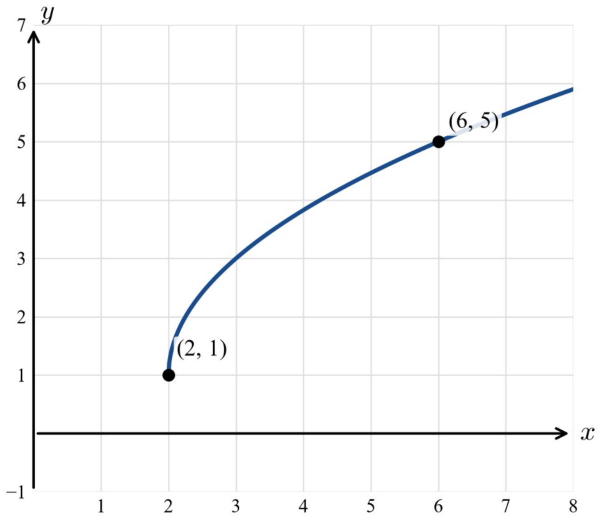


Example 4 — unscaffolded

A square root graph $y = a\sqrt{x-h}+k$ has endpoint $(2,1)$ and passes through $(6,5)$. Find its rule.

Endpoint gives $h=2$, $k=1$: $y=a\sqrt{x-2}+1$. Substitute $(6,5)$: $5=a\sqrt{6-2}+1=2a+1$, so $a=2$.

$\therefore y=2\sqrt{x-2}+1$.



Practice questions

Scaffolded

Q1. A hyperbola has asymptotes $x=-3$ and $y=2$ and passes through $(-1,4)$. (a) Write the form using the asymptotes. (b) Substitute the point to find a. (c) State the rule.

Q2. A circle has centre $(-2,3)$ and passes through $(1,3)$. (a) Find r^2. (b) State the equation.

Q3. A curve of the form $y=\dfrac{a}{(x-h)^2}+k$ has asymptotes $x=1$ and $y=0$ and passes through $(2,5)$. (a) Write the form. (b) Find a. (c) State the rule.

Q4. A square root graph $y=a\sqrt{x}$ passes through $(4,6)$. (a) Find a. (b) State the rule.

Q5. A hyperbola has asymptotes $x=0$ and $y=1$ and passes through $(2,3)$. Find its rule.

Q6. A circle has a diameter with endpoints $(1,2)$ and $(5,8)$. (a) Find the centre. (b) Find r^2. (c) State the equation.

Unscaffolded

Find the rule (or equation) of each graph from the information given.

Q7. A hyperbola with asymptotes $x=4$, $y=-2$, through $(5,1)$.

Q8. A hyperbola with asymptotes $x=-1$, $y=3$, through $(0,1)$.

Q9. A curve $y=\dfrac{a}{(x-h)^2}+k$ with asymptotes $x=2$, $y=1$, through $(3,4)$.

Q10. A curve $y=\dfrac{a}{(x-h)^2}+k$ with asymptotes $x=0$, $y=-2$, through $(1,1)$.

Q11. A square root graph with endpoint $(0,0)$, through $(9,6)$.

Q12. A square root graph with endpoint $(1,2)$, through $(5,6)$.

Q13. A square root graph with endpoint $(0,3)$, through $(4,-1)$.

Q14. A circle with centre $(2,-1)$ and radius 5.

Q15. A circle with centre $(-3,0)$ passing through $(0,4)$.

Q16. A circle with a diameter from $(-1,2)$ to $(3,-2)$.

Q17. A hyperbola with both axes as asymptotes ($x=0$ and $y=0$) passing through $(1,3)$.

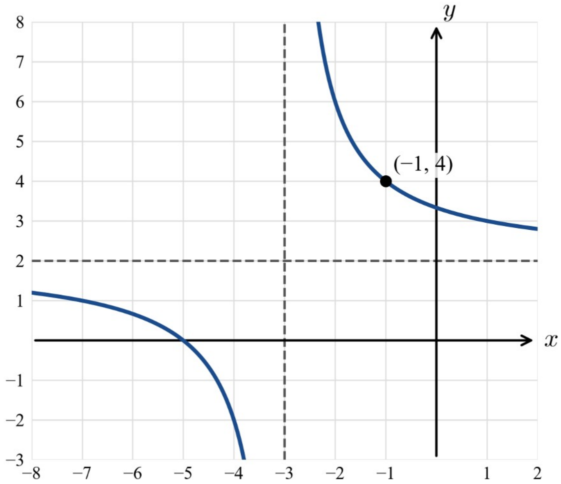
Q18. A square root graph with endpoint $(-2,0)$ passing through $(2,4)$.

Answers

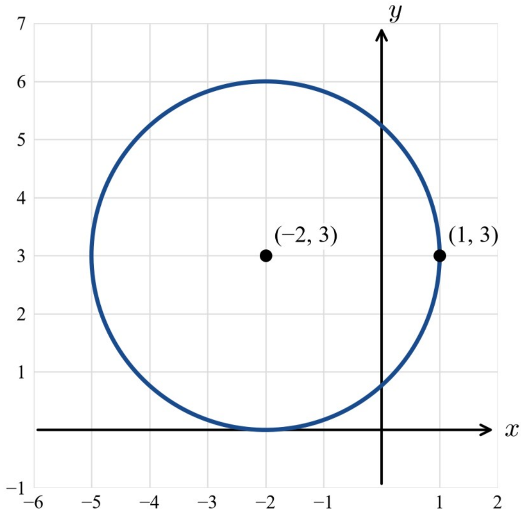
Q1. $y=\frac{4}{x+3}+2$. **Q2.** $(x+2)^2+(y-3)^2=9$. **Q3.** $y=\frac{5}{(x-1)^2}$. **Q4.** $y=3\sqrt{x}$. **Q5.** $y=\frac{4}{x}+1$. **Q6.** centre $(3,5)$, $(x-3)^2+(y-5)^2=13$. **Q7.** $y=\frac{3}{x-4}-2$. **Q8.** $y=\frac{-2}{x+1}+3$. **Q9.** $y=\frac{3}{(x-2)^2}+1$. **Q10.** $y=\frac{3}{x^2}-2$. **Q11.** $y=2\sqrt{x}$. **Q12.** $y=2\sqrt{x-1}+2$. **Q13.** $y=-2\sqrt{x}+3$. **Q14.** $(x-2)^2+(y+1)^2=25$. **Q15.** $(x+3)^2+y^2=25$. **Q16.** $(x-1)^2+y^2=8$. **Q17.** $y=\frac{3}{x}$. **Q18.** $y=2\sqrt{x+2}$.

Fully-worked solutions

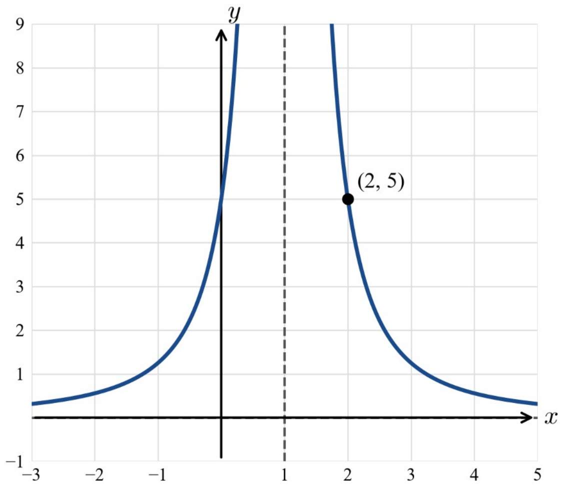
Q1. Asymptotes give $y=\frac{a}{x+3}+2$. Substitute $(-1,4)$: $4=\frac{a}{-1+3}+2=\frac{a}{2}+2$, so $\frac{a}{2}=2$ and $a=4$. $\therefore y=\frac{4}{x+3}+2$.



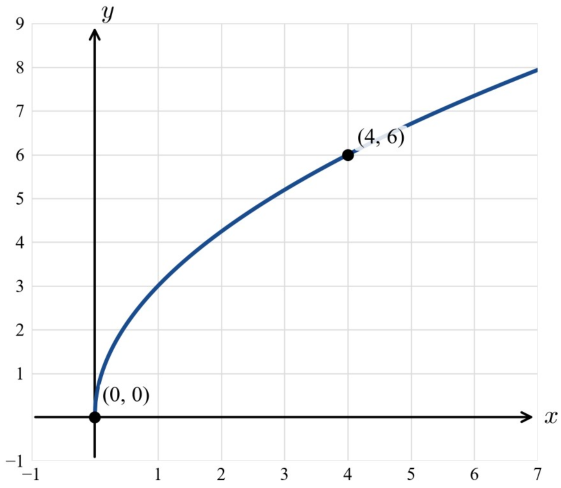
Q2. $(x+2)^2+(y-3)^2=r^2$; $r^2=(1-(-2))^2+(3-3)^2=9$. $\therefore (x+2)^2+(y-3)^2=9$.



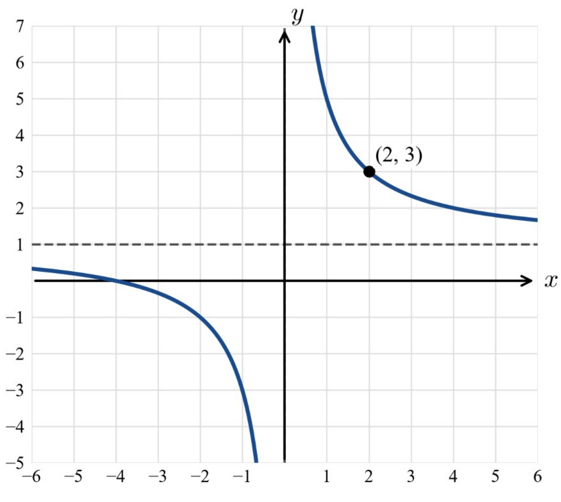
Q3. $y=\frac{a}{(x-1)^2}+0$. Substitute $(2,5)$: $5=\frac{a}{(2-1)^2}=a$, so $a=5$. $\therefore y=\frac{5}{(x-1)^2}$.



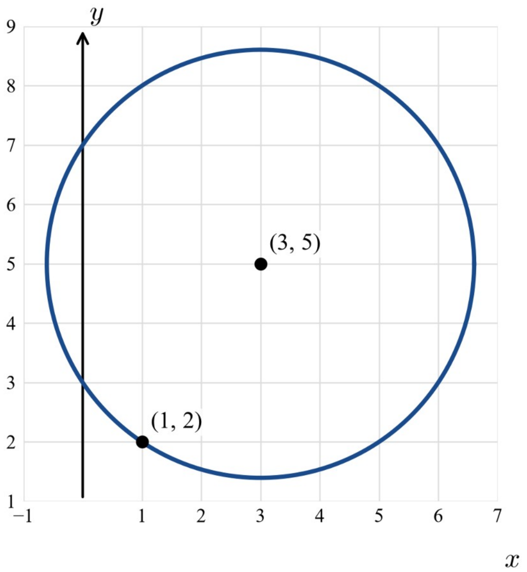
Q4. $6 = a\sqrt{4} = 2a$, so $a = 3$. $\therefore y = 3\sqrt{x}$.



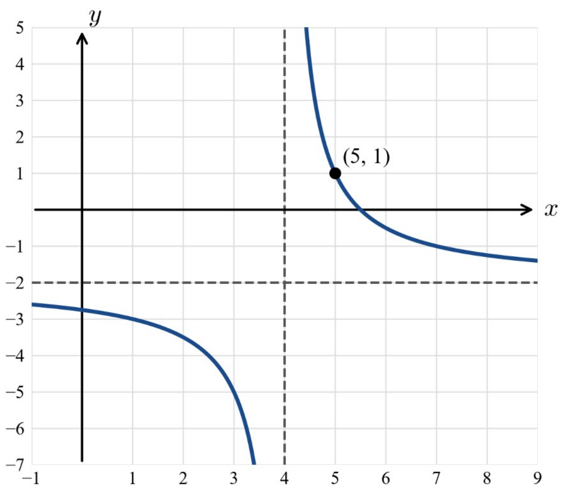
Q5. $y = \frac{a}{x} + 1$. Substitute $(2, 3)$: $3 = \frac{a}{2} + 1$, so $a = 4$. $\therefore y = \frac{4}{x} + 1$.



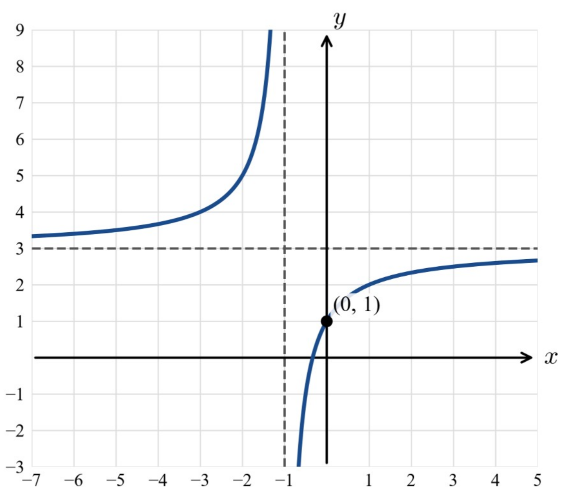
Q6. Centre $= \left(\frac{1+5}{2}, \frac{2+8}{2}\right) = (3, 5)$; $r^2 = (3-1)^2 + (5-2)^2 = 4 + 9 = 13$. $\therefore (x-3)^2 + (y-5)^2 = 13$.



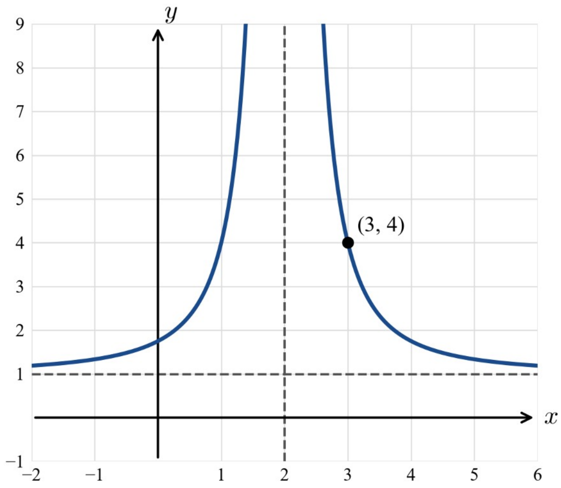
Q7. $y=\dfrac{a}{x-4}-2$; $1=\dfrac{a}{5-4}-2=a-2$, so $a=3$. $\therefore\ y=\dfrac{3}{x-4}-2$.



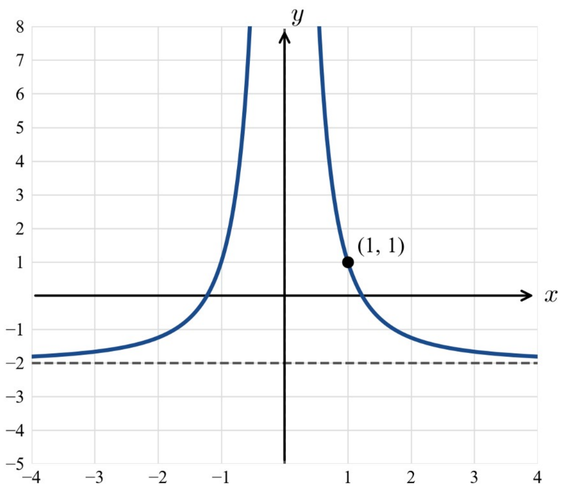
Q8. $y=\dfrac{a}{x+1}+3$; $1=\dfrac{a}{0+1}+3=a+3$, so $a=-2$. $\therefore\ y=\dfrac{-2}{x+1}+3$.



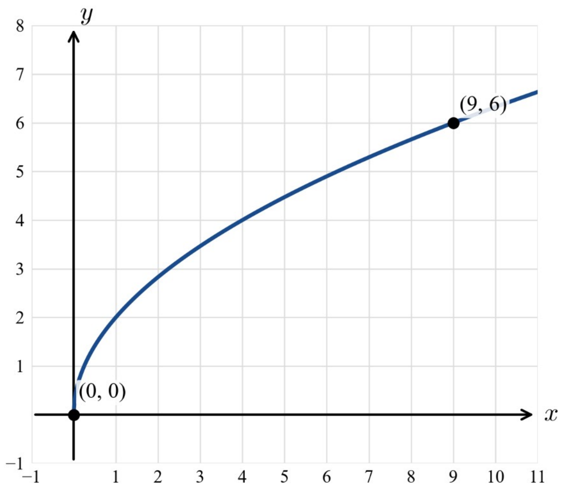
Q9. $y=\dfrac{a}{(x-2)^2}+1$; $4=\dfrac{a}{(3-2)^2}+1=a+1$, so $a=3$. $\therefore\ y=\dfrac{3}{(x-2)^2}+1$.



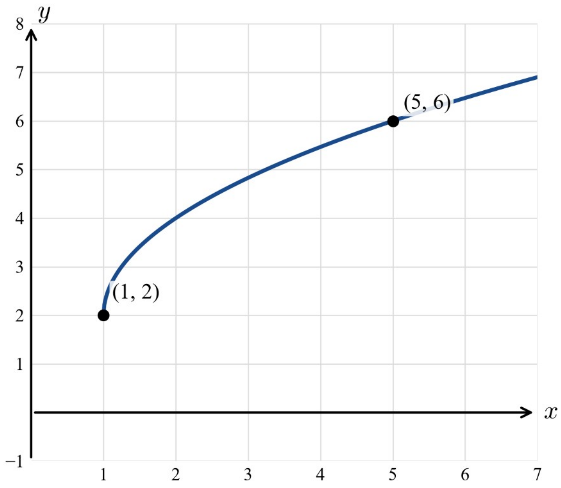
Q10. $y=\dfrac{a}{x^2}-2$; $1=\dfrac{a}{1^2}-2=a-2$, so $a=3$. $\therefore y=\dfrac{3}{x^2}-2$.



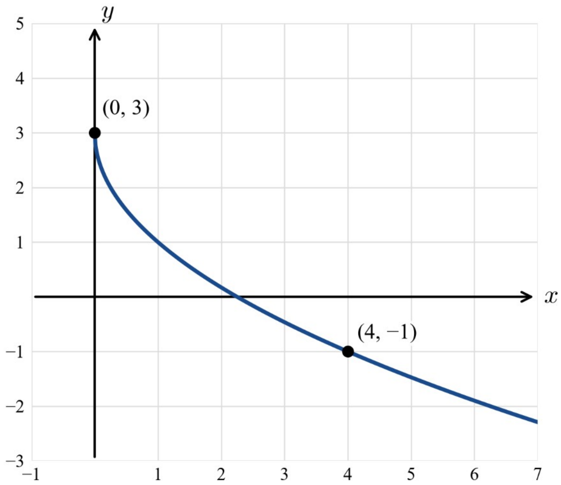
Q11. $6=a\sqrt{9}=3a$, so $a=2$. $\therefore y=2\sqrt{x}$.



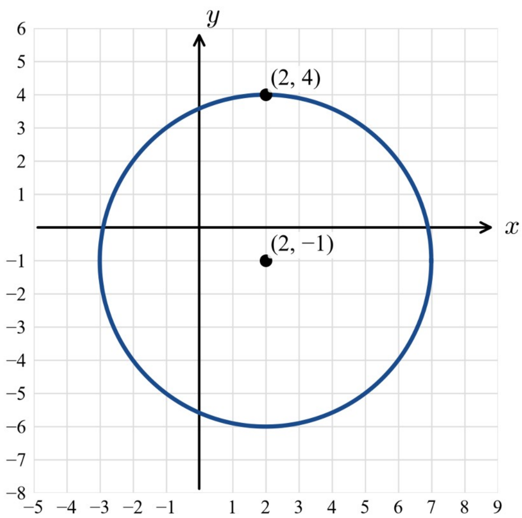
Q12. $y=a\sqrt{x-1}+2$; $6=a\sqrt{5-1}+2=2a+2$, so $a=2$. $\therefore y=2\sqrt{x-1}+2$.



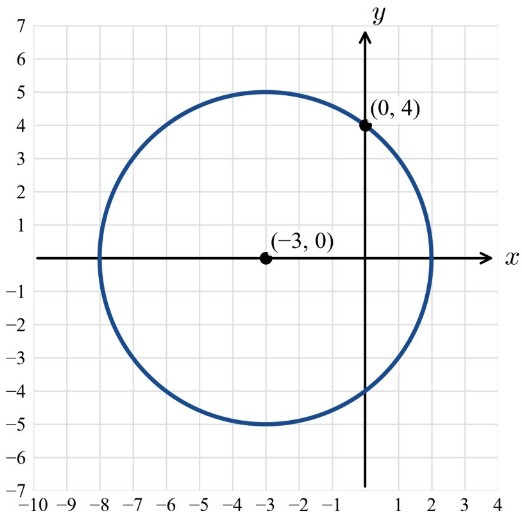
Q13. $y = a\sqrt{x} + 3$; $-1 = a\sqrt{4} + 3 = 2a + 3$, so $a = -2$. $\therefore y = -2\sqrt{x} + 3$.



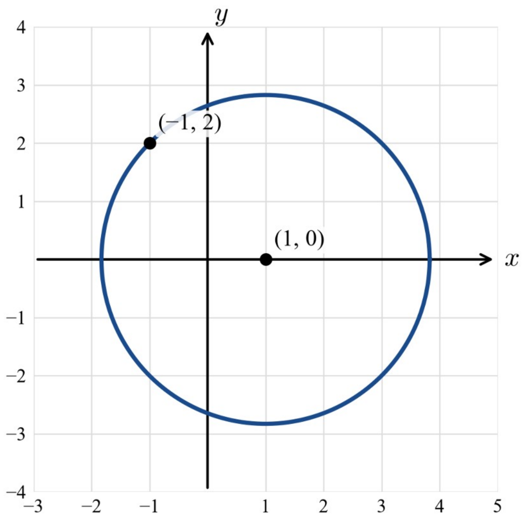
Q14. Centre $(2, -1)$, radius 5: $(x-2)^2 + (y+1)^2 = 25$.



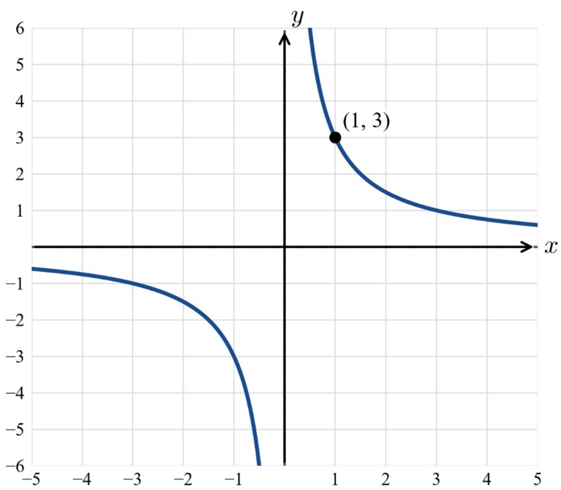
Q15. $r^2 = (0-(-3))^2 + (4-0)^2 = 9 + 16 = 25$. $\therefore (x+3)^2 + y^2 = 25$.



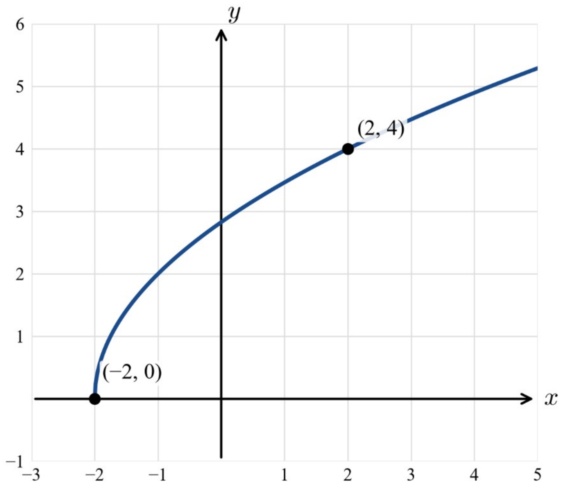
Q16. Centre $=\left(\dfrac{-1+3}{2},\dfrac{2+(-2)}{2}\right)=(1,0)$; $r^2=(1-(-1))^2+(0-2)^2=4+4=8$. $\therefore (x-1)^2+y^2=8$.



Q17. Asymptotes $x=0,\ y=0$ give $y=\dfrac{a}{x}$; $3=\dfrac{a}{1}$, so $a=3$. $\therefore y=\dfrac{3}{x}$.



Q18. $y=a\sqrt{x+2}$; $4=a\sqrt{2+2}=2a$, so $a=2$. $\therefore y=2\sqrt{x+2}$.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