Mathematical Methods Units 1 & 2

Chapter 3 — Quadratic functions

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Contents

Chapter 3 Quadratic functions — 3A Expanding algebraic expressions

Theory

Like terms have exactly the same pronumeral part, so they can be combined by adding their coefficients. For example, $3x$ and $5x$ are like terms ($3x+5x=8x$), and so are x^2 and $-2x^2$; but x^2 and x are **not** like terms.

The distributive law. To **expand** a bracket, multiply every term inside it by the term outside:

$$a(b+c)=ab+ac.$$

Product of two binomials. Multiply each term of the first bracket by each term of the second:

$$(a+b)(c+d)=ac+ad+bc+bd.$$

In particular $(x+p)(x+q)=x^2+(p+q)x+pq$.

Special products (worth memorising):

$$(a+b)^2=a^2+2ab+b^2,(a-b)^2=a^2-2ab+b^2,$$

$$(a+b)(a-b)=a^2-b^2\text{(difference of two squares)}.$$

Collecting like terms. After expanding, simplify by collecting like terms. Take care with signs: when a bracket is **subtracted**, the subtraction applies to *every* term inside it — $-(x-3)=-x+3$.

Worked examples

Example 1 — scaffolded

Expand and simplify $3(2x-5)+2(x+4)$.

Working	Comment
$3(2x-5)=6x-15$ and $2(x+4)=2x+8$.	Apply the distributive law to each bracket.
$=6x-15+2x+8$	Write all four terms.
$=8x-7$	Collect like terms: $6x+2x=8x$ and $-15+8=-7$.

Example 2 — scaffolded

Expand $(2x-3)(x+5)$.

Working	Comment
$(2x-3)(x+5)=2x\times x+2x\times 5-3\times x-3\times 5$	Multiply each term of the first bracket by each term of the second.
$=2x^2+10x-3x-15$	Simplify each product.
$=2x^2+7x-15$	Collect like terms: $10x-3x=7x$.

Example 3 — unscaffolded

Expand and simplify $(x+4)^2-(x-3)(x+2)$.

$(x+4)^2=x^2+8x+16$ and $(x-3)(x+2)=x^2-x-6$.

$$(x+4)^2-(x-3)(x+2)=(x^2+8x+16)-(x^2-x-6)=x^2+8x+16-x^2+x+6.$$

$\therefore=9x+22$ (note the sign change on every term of the subtracted bracket).

Example 4 — unscaffolded

Expand $(3x-2)(2x^2-x+4)$.

$$(3x-2)(2x^2-x+4)=3x(2x^2-x+4)-2(2x^2-x+4)=6x^3-3x^2+12x-4x^2+2x-8.$$

$\therefore =6x^3-7x^2+14x-8$.

Practice questions

Scaffolded

Q1. Expand $4(3x-2)$.

Q2. Expand $-2(5-3x)$.

Q3. Expand and simplify $5(x+2)-3(2x-1)$.

Q4. Expand $(x+6)(x-4)$.

Q5. Expand $(2x+1)(3x-5)$.

Q6. Expand $(x-7)^2$.

Unscaffolded

Expand and simplify each of the following.

Q7. $3(2x+7)$

Q8. $2(x-3)+4(2x+1)$

Q9. $6(2x-1)-3(x-4)$

Q10. $(x+3)(x+8)$

Q11. $(x-5)(x+2)$

Q12. $(2x-3)(4x+1)$

Q13. $(x+5)^2$

Q14. $(3x-2)^2$

Q15. $(x+7)(x-7)$

Q16. $(2x-5)(2x+5)$

Q17. $(x+4)^2-(x-2)^2$

Q18. $(x-1)(x^2+3x-2)$

Answers

Q1. $12x-8$. **Q2.** $6x-10$. **Q3.** $-x+13$. **Q4.** $x^2+2x-24$. **Q5.** $6x^2-7x-5$. **Q6.** $x^2-14x+49$. **Q7.** $6x+21$. **Q8.** $10x-2$. **Q9.** $9x+6$. **Q10.** $x^2+11x+24$. **Q11.** $x^2-3x-10$. **Q12.** $8x^2-10x-3$. **Q13.** $x^2+10x+25$. **Q14.** $9x^2-12x+4$. **Q15.** x^2-49. **Q16.** $4x^2-25$. **Q17.** $12x+12$. **Q18.** x^3+2x^2-5x+2.

Fully-worked solutions

Q1. $4(3x-2)=12x-8$.

Q2. $-2(5-3x)=-10+6x=6x-10$.

Q3. $5(x+2)-3(2x-1)=5x+10-6x+3=-x+13$.

Q4. $(x+6)(x-4)=x^2-4x+6x-24=x^2+2x-24$.

Q5. $(2x+1)(3x-5)=6x^2-10x+3x-5=6x^2-7x-5$.

Q6. $(x-7)^2=x^2-2(7)x+49=x^2-14x+49$.

Q7. $3(2x+7)=6x+21$.

Q8. $2(x-3)+4(2x+1)=2x-6+8x+4=10x-2$.

Q9. $6(2x-1)-3(x-4)=12x-6-3x+12=9x+6$.

Q10. $(x+3)(x+8)=x^2+8x+3x+24=x^2+11x+24$.

Q11. $(x-5)(x+2)=x^2+2x-5x-10=x^2-3x-10$.

Q12. $(2x-3)(4x+1)=8x^2+2x-12x-3=8x^2-10x-3$.

Q13. $(x+5)^2=x^2+2(5)x+25=x^2+10x+25$.

Q14. $(3x-2)^2=(3x)^2-2(3x)(2)+2^2=9x^2-12x+4$.

Q15. $(x+7)(x-7)=x^2-7^2=x^2-49$ (difference of two squares).

Q16. $(2x-5)(2x+5)=(2x)^2-5^2=4x^2-25$ (difference of two squares).

Q17. $(x+4)^2-(x-2)^2=(x^2+8x+16)-(x^2-4x+4)=8x+16+4x-4=12x+12$.

Q18. $(x-1)(x^2+3x-2)=x(x^2+3x-2)-1(x^2+3x-2)=x^3+3x^2-2x-x^2-3x+2=x^3+2x^2-5x+2$.

Chapter 3 Quadratic functions — 3B Factorising quadratic expressions

Theory

Factorising is the reverse of expanding: it rewrites an expression as a product of factors. There are a few standard types.

Common factor. Take out the highest common factor (HCF) of every term:

$$ab+ac=a(b+c), \text{e.g.}\ 6x^2+9x=3x(2x+3).$$

Always look for a common factor **first**.

Difference of two squares.

$$a^2-b^2=(a-b)(a+b).$$

Monic quadratic trinomials (x^2+bx+c). Find two numbers p and q with $p+q=b$ and $pq=c$; then $x^2+bx+c=(x+p)(x+q)$.

Non-monic quadratic trinomials (ax^2+bx+c, with $a\neq 1$). Find two numbers with product ac and sum b, split the middle term, then factorise by grouping.

Perfect square trinomials.

$$a^2+2ab+b^2=(a+b)^2, a^2-2ab+b^2=(a-b)^2.$$

You can always check a factorisation by expanding it — the result should be the original expression.

Worked examples

Example 1 — scaffolded

Factorise $12x^2-8x$.

Working	Comment
The HCF of $12x^2$ and $8x$ is $4x$.	Find the largest factor common to both terms (numbers and pronumerals).
$12x^2-8x=4x(3x-2)$	Divide each term by $4x$ and write it outside the bracket.
Check: $4x(3x-2)=12x^2-8x$.	Expand to confirm.

Example 2 — scaffolded

Factorise $x^2+7x+12$.

Working	Comment
This is monic ($a=1$). Look for p and q with $p+q=7$ and $pq=12$.	Sum is the coefficient of x; product is the constant.
$3+4=7$ and $3\times 4=12$, so $p=3$, $q=4$.	List factor pairs of 12 and pick the pair that adds to 7.
$x^2+7x+12=(x+3)(x+4)$	Write the factors.

Example 3 — unscaffolded

Factorise $9x^2-25$.

This is a difference of two squares, since $9x^2=(3x)^2$ and $25=5^2$.

$$9x^2 - 25 = (3x)^2 - 5^2 = (3x-5)(3x+5).$$

$\therefore 9x^2 - 25 = (3x-5)(3x+5)$.

Example 4 — unscaffolded

Factorise $6x^2 + 11x - 10$.

Here $a = 6$ and $c = -10$, so $ac = -60$. Two numbers with product -60 and sum 11 are 15 and -4. Split the middle term and factorise by grouping:

$$6x^2 + 11x - 10 = 6x^2 + 15x - 4x - 10 = 3x(2x+5) - 2(2x+5) = (2x+5)(3x-2).$$

$\therefore 6x^2 + 11x - 10 = (2x+5)(3x-2)$.

Practice questions

Scaffolded

Q1. Factorise $10x^2 + 15x$ by taking out the highest common factor.

Q2. Factorise $x^2 - 9x + 20$ (find two numbers with sum -9 and product 20).

Q3. Factorise $x^2 + 6x - 16$.

Q4. Factorise $x^2 - 36$ using the difference of two squares.

Q5. Factorise $2x^2 + 9x + 10$ by splitting the middle term.

Q6. Factorise $x^2 + 16x + 64$ (a perfect square).

Unscaffolded

Factorise each of the following completely.

Q7. $8x^2 - 12x$

Q8. $x^2 + 9x + 18$

Q9. $x^2 - 8x + 15$

Q10. $x^2 - 3x - 28$

Q11. $x^2 - 100$

Q12. $4x^2 - 81$

Q13. $3x^2 - 27$

Q14. $3x^2 + 10x + 8$

Q15. $6x^2 - 7x - 3$

Q16. $x^2 - 12x + 36$

Q17. $9x^2 + 24x + 16$

Q18. $2x^2 - 2x - 24$

Answers

Q1. $5x(2x+3)$. **Q2.** $(x-4)(x-5)$. **Q3.** $(x+8)(x-2)$. **Q4.** $(x-6)(x+6)$. **Q5.** $(2x+5)(x+2)$. **Q6.** $(x+8)^2$. **Q7.** $4x(2x-3)$. **Q8.** $(x+3)(x+6)$. **Q9.** $(x-3)(x-5)$. **Q10.** $(x-7)(x+4)$. **Q11.** $(x-10)(x+10)$. **Q12.** $(2x-9)(2x+9)$. **Q13.** $3(x-3)(x+3)$. **Q14.** $(3x+4)(x+2)$. **Q15.** $(2x-3)(3x+1)$. **Q16.** $(x-6)^2$. **Q17.** $(3x+4)^2$. **Q18.** $2(x-4)(x+3)$.

Fully-worked solutions

Q1. The HCF of $10x^2$ and $15x$ is $5x$: $10x^2+15x=5x(2x+3)$.

Q2. Two numbers with sum -9 and product 20 are -4 and -5: $x^2-9x+20=(x-4)(x-5)$.

Q3. Two numbers with sum 6 and product -16 are 8 and -2: $x^2+6x-16=(x+8)(x-2)$.

Q4. $x^2-36=x^2-6^2=(x-6)(x+6)$ (difference of two squares).

Q5. $ac=2\times 10=20$; two numbers with product 20 and sum 9 are 4 and 5:
$2x^2+9x+10=2x^2+4x+5x+10=2x(x+2)+5(x+2)=(2x+5)(x+2)$.

Q6. $x^2+16x+64=x^2+2(8)x+8^2=(x+8)^2$.

Q7. The HCF of $8x^2$ and $12x$ is $4x$: $8x^2-12x=4x(2x-3)$.

Q8. Sum 9, product 18: 3 and 6. $x^2+9x+18=(x+3)(x+6)$.

Q9. Sum -8, product 15: -3 and -5. $x^2-8x+15=(x-3)(x-5)$.

Q10. Sum -3, product -28: -7 and 4. $x^2-3x-28=(x-7)(x+4)$.

Q11. $x^2-100=x^2-10^2=(x-10)(x+10)$.

Q12. $4x^2-81=(2x)^2-9^2=(2x-9)(2x+9)$.

Q13. Take out the common factor 3 first, then difference of two squares: $3x^2-27=3(x^2-9)=3(x-3)(x+3)$.

Q14. $ac=3\times 8=24$; two numbers with product 24 and sum 10 are 6 and 4:
$3x^2+10x+8=3x^2+6x+4x+8=3x(x+2)+4(x+2)=(3x+4)(x+2)$.

Q15. $ac=6\times(-3)=-18$; two numbers with product -18 and sum -7 are -9 and 2:
$6x^2-7x-3=6x^2-9x+2x-3=3x(2x-3)+1(2x-3)=(2x-3)(3x+1)$.

Q16. $x^2-12x+36=x^2-2(6)x+6^2=(x-6)^2$.

Q17. $9x^2+24x+16=(3x)^2+2(3x)(4)+4^2=(3x+4)^2$.

Q18. Take out the common factor 2 first: $2x^2-2x-24=2(x^2-x-12)$. Two numbers with sum -1 and product -12 are -4 and 3, so $2(x^2-x-12)=2(x-4)(x+3)$.

Theory

A **quadratic equation** is one that can be written in the form $ax^2+bx+c=0$, where $a\neq 0$. Many quadratic equations can be solved by **factorising** and then using the null factor law.

The null factor law. If two quantities multiply to give zero, then at least one of them must be zero:

$$\text{if } a\times b=0\text{, then } a=0 \text{ or } b=0.$$

To solve a quadratic equation by factorising:

1. Rearrange the equation so that one side is **zero**.
2. Factorise the quadratic expression.
3. Apply the null factor law: set each factor equal to zero.
4. Solve each linear equation.

Important. The null factor law only works when one side is zero, so *always* write the equation as $\ldots=0$ **first**. In particular, do **not** divide both sides by x to solve $x^2=5x$ — that would lose the solution $x=0$. Instead, move everything to one side and take out a common factor.

Worked examples

Example 1 — scaffolded

Solve $x^2-4x-21=0$.

Working	Comment
$x^2-4x-21=0 \Rightarrow (x-7)(x+3)=0$	Factorise: find two numbers with product -21 and sum -4 — namely -7 and $+3$.
$x-7=0$ or $x+3=0$	Apply the null factor law to each factor.
$x=7$ or $x=-3$	Solve each linear equation.

Example 2 — scaffolded

Solve $2x^2-7x-4=0$.

Working	Comment
$2x^2-8x+x-4=0$	Split the middle term: product $2\times(-4)=-8$; two numbers with product -8 and sum -7 are -8 and $+1$.
$2x(x-4)+1(x-4)=0 \Rightarrow (2x+1)(x-4)=0$	Factorise by grouping.
$2x+1=0$ or $x-4=0$	Null factor law.
$x=-1/2$ or $x=4$	Solve each.

Example 3 — unscaffolded

Solve $x^2=5x$.

$$x^2=5x \Rightarrow x^2-5x=0 \Rightarrow x(x-5)=0.$$

So $x=0$ or $x-5=0$. $\therefore x=0$ or $x=5$.

(Do not divide by x — that would lose the solution $x=0$.)

Example 4 — unscaffolded

Solve $x^2+4x=12$.

$$x^2+4x=12 \Rightarrow x^2+4x-12=0 \Rightarrow (x+6)(x-2)=0.$$

$\therefore x=-6$ or $x=2$.

Practice questions

Scaffolded

Q1. Solve $x^2-x-20=0$: (a) factorise the left-hand side; (b) apply the null factor law; (c) state the two solutions.

Q2. Solve $x^2+3x-10=0$.

Q3. Solve $x^2-64=0$ (use the difference of two squares).

Q4. Solve $3x^2-4x-4=0$.

Q5. Solve $x^2=8x$ (first write the equation as $\ldots=0$; do not divide by x).

Q6. Solve $x^2+12x+36=0$ and explain why there is only one solution.

Unscaffolded

Solve each of the following equations.

Q7. $x^2+9x+20=0$

Q8. $x^2-2x-35=0$

Q9. $2x^2+5x-12=0$

Q10. $5x^2-13x-6=0$

Q11. $x^2-25=0$

Q12. $4x^2-9=0$

Q13. $x^2=7x$

Q14. $4x^2+12x=0$

Q15. $x^2=2x+8$

Q16. $x(x-9)=10$

Q17. $x^2-10x+25=0$

Q18. $(x+1)(x-6)=8$

Answers

Q1. $x=5$ or $x=-4$. **Q2.** $x=-5$ or $x=2$. **Q3.** $x=8$ or $x=-8$. **Q4.** $x=-2/3$ or $x=2$. **Q5.** $x=0$ or $x=8$. **Q6.** $x=-6$ (repeated). **Q7.** $x=-5$ or $x=-4$. **Q8.** $x=-5$ or $x=7$. **Q9.** $x=3/2$ or $x=-4$. **Q10.** $x=-2/5$ or $x=3$. **Q11.** $x=-5$ or $x=5$. **Q12.** $x=-3/2$ or $x=3/2$. **Q13.** $x=0$ or $x=7$. **Q14.** $x=0$ or $x=-3$. **Q15.** $x=-2$ or $x=4$. **Q16.** $x=10$ or $x=-1$. **Q17.** $x=5$ (repeated). **Q18.** $x=7$ or $x=-2$.

Fully-worked solutions

Q1. $x^2-x-20=0 \Rightarrow (x-5)(x+4)=0$, so $x=5$ or $x=-4$.

Q2. $x^2+3x-10=0 \Rightarrow (x+5)(x-2)=0$, so $x=-5$ or $x=2$.

Q3. $x^2-64=0 \Rightarrow (x-8)(x+8)=0$, so $x=8$ or $x=-8$.

Q4. $3x^2-4x-4=0 \Rightarrow (3x+2)(x-2)=0$, so $x=-2/3$ or $x=2$.

Q5. $x^2=8x \Rightarrow x^2-8x=0 \Rightarrow x(x-8)=0$, so $x=0$ or $x=8$.

Q6. $x^2+12x+36=0 \Rightarrow (x+6)^2=0$, so $x+6=0$ giving $x=-6$. The factor $(x+6)$ is repeated, so there is only one distinct solution.

Q7. $x^2+9x+20=0 \Rightarrow (x+4)(x+5)=0$, so $x=-4$ or $x=-5$.

Q8. $x^2-2x-35=0 \Rightarrow (x-7)(x+5)=0$, so $x=7$ or $x=-5$.

Q9. $2x^2+5x-12=0 \Rightarrow (2x-3)(x+4)=0$, so $x=3/2$ or $x=-4$.

Q10. $5x^2-13x-6=0 \Rightarrow (5x+2)(x-3)=0$, so $x=-2/5$ or $x=3$.

Q11. $x^2-25=0 \Rightarrow (x-5)(x+5)=0$, so $x=5$ or $x=-5$.

Q12. $4x^2-9=0 \Rightarrow (2x-3)(2x+3)=0$, so $x=3/2$ or $x=-3/2$.

Q13. $x^2=7x \Rightarrow x^2-7x=0 \Rightarrow x(x-7)=0$, so $x=0$ or $x=7$.

Q14. $4x^2+12x=0 \Rightarrow 4x(x+3)=0$, so $x=0$ or $x=-3$.

Q15. $x^2=2x+8 \Rightarrow x^2-2x-8=0 \Rightarrow (x-4)(x+2)=0$, so $x=4$ or $x=-2$.

Q16. $x(x-9)=10 \Rightarrow x^2-9x-10=0 \Rightarrow (x-10)(x+1)=0$, so $x=10$ or $x=-1$.

Q17. $x^2-10x+25=0 \Rightarrow (x-5)^2=0$, so $x=5$ (repeated).

Q18. $(x+1)(x-6)=8 \Rightarrow x^2-5x-6=8 \Rightarrow x^2-5x-14=0 \Rightarrow (x-7)(x+2)=0$, so $x=7$ or $x=-2$.

Chapter 3 Quadratic functions — 3D Graphs of quadratic functions

Theory

A quadratic function has a graph called a **parabola**. This section shows how to **sketch** a parabola — that is, show its key features (turning point and axis intercepts) accurately, though not to scale.

The basic parabola. The graph of $y=x^2$ is a parabola that is symmetric about the y-axis, opens upwards, and has its turning point (a minimum) at the origin $(0,0)$.

Turning-point form. Every quadratic can be written in the form

$$y=a(x-h)^2+k.$$

- The **turning point** is (h,k) and the **axis of symmetry** is the line $x=h$.
- If $a>0$ the parabola opens **upwards** (minimum turning point); if $a<0$ it opens **downwards** (maximum).
- $|a|$ is the **dilation** factor from the x-axis: if $|a|>1$ the parabola is narrower than $y=x^2$; if $0<|a|<1$ it is wider.

The six transformations of $y=x^2$. The graph of $y=a(x-h)^2+k$ is built up from $y=x^2$ by combining the following.

- **Translation up/down** (k): $y=x^2+k$ moves the graph k units vertically.
- **Translation left/right** (h): $y=(x-h)^2$ moves the graph h units horizontally (right if $h>0$, left if $h<0$).
- **Dilation from the x-axis** (factor a): $y=ax^2$ stretches the graph vertically — narrower when $a>1$.
- **Dilation from the y-axis**: replacing x by nx gives $y=(nx)^2$, a horizontal compression. *For a parabola this is equivalent to a dilation from the x-axis* — e.g. $y=(2x)^2=4x^2$.
- **Reflection in the x-axis**: $y=-f(x)$ flips the graph upside down (a minimum becomes a maximum).
- **Reflection in the y-axis**: $y=f(-x)$ replaces x by $-x$; the vertex (h,k) moves to $(-h,k)$.

Polynomial (general) form. For $y=ax^2+bx+c$:

- **y-intercept:** substitute $x=0$, giving $(0,c)$.
- **Axis of symmetry:** $x=-\frac{b}{2a}$; substitute to get the turning point.
- **x-intercepts:** solve $ax^2+bx+c=0$ by factorising, completing the square, or the quadratic formula. The **discriminant** $\Delta=b^2-4ac$ gives the number of x-intercepts: two if $\Delta>0$, one (a repeated root — the parabola touches the axis) if $\Delta=0$, none if $\Delta<0$.

Factored form. If $y=a(x-p)(x-q)$, the x-intercepts are $x=p$ and $x=q$, and the axis of symmetry is $x=\frac{p+q}{2}$.

To sketch a parabola: (1) decide the shape from the sign of a; (2) find the turning point; (3) find the y-intercept; (4) find the x-intercepts, if any; (5) mark and label these points and draw a smooth curve through them.

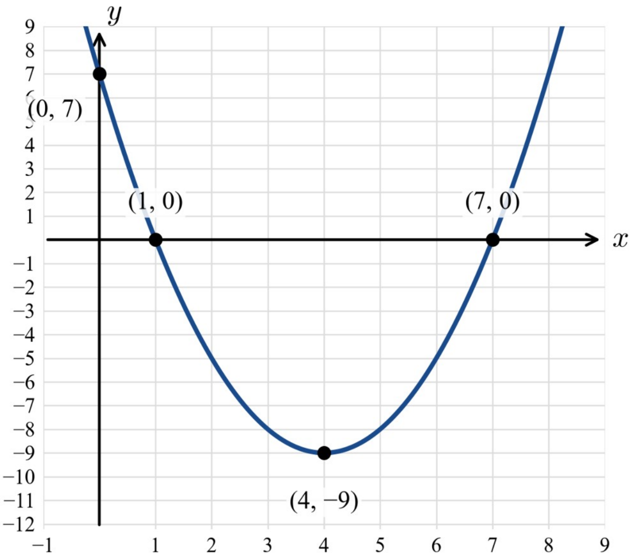
Worked examples

Example 1 — scaffolded

Sketch the graph of $y=(x-4)^2-9$, labelling the turning point and all axis intercepts.

Working	Comment
$a=1>0$, so the parabola has a **minimum** turning point.	The sign of the coefficient of x^2 gives the shape.

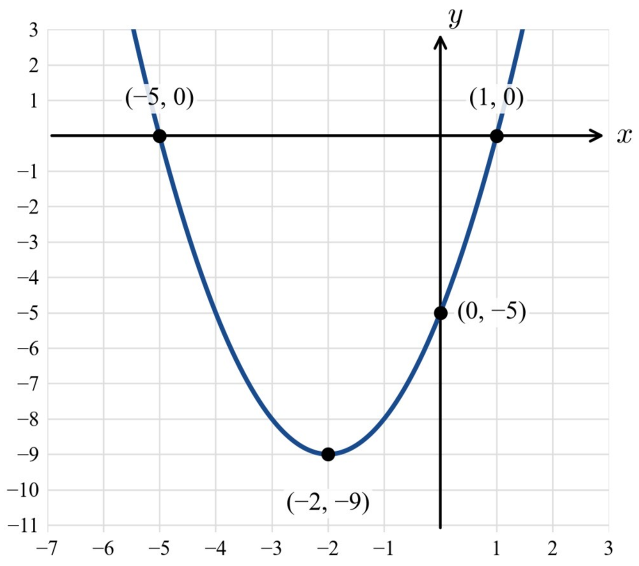
Working	Comment
Comparing with $y=a(x-h)^2+k$, $h=4$ and $k=-9$. Turning point $(4,-9)$; axis $x=4$.	In turning-point form the turning point is read straight off as (h,k).
y-intercept: put $x=0$: $y=(0-4)^2-9=16-9=7$, giving $(0,7)$.	Substitute $x=0$.
x-intercepts: put $y=0$: $(x-4)^2-9=0 \Rightarrow (x-4)^2=9 \Rightarrow x-4=\pm 3$, so $x=1$ or $x=7$, giving $(1,0)$ and $(7,0)$.	Keep the ± — a square root gives two solutions.



Example 2 — scaffolded

Sketch the graph of $y=x^2+4x-5$.

Working	Comment
$a=1>0$, so the parabola has a **minimum**.	Shape from the coefficient of x^2.
y-intercept: $x=0 \Rightarrow y=-5$, giving $(0,-5)$.	For $y=ax^2+bx+c$ the y-intercept is $(0,c)$.
Axis of symmetry: $x=-\dfrac{b}{2a}=-\dfrac{4}{2\times 1}=-2$.	Here $a=1$, $b=4$.
Turning point: $y(-2)=(-2)^2+4(-2)-5=4-8-5=-9$, giving $(-2,-9)$.	Substitute the axis value into the rule.
x-intercepts: $x^2+4x-5=0 \Rightarrow (x+5)(x-1)=0$, so $x=-5$ or $x=1$, giving $(-5,0)$ and $(1,0)$.	Factorise, then apply the null factor law.



Example 3 — unscaffolded

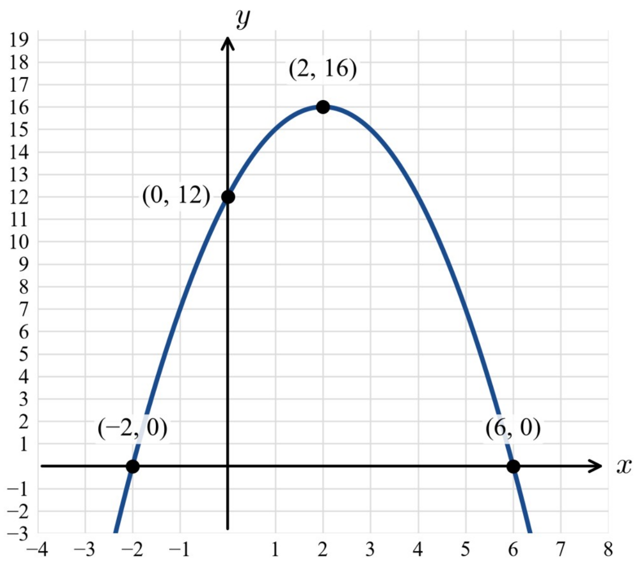
Sketch the graph of $y=-(x-2)^2+16$.

$a=-1<0$, so the parabola has a **maximum** turning point.

In the form $y=a(x-h)^2+k$, $h=2$ and $k=16$: turning point $(2,16)$, axis $x=2$.

y-intercept: $y=-(0-2)^2+16=-4+16=12$, giving $(0,12)$.

x-intercepts: $-(x-2)^2+16=0 \Rightarrow (x-2)^2=16 \Rightarrow x-2=\pm 4$, so $x=6$ or $x=-2$, giving $(6,0)$ and $(-2,0)$.



Example 4 — unscaffolded

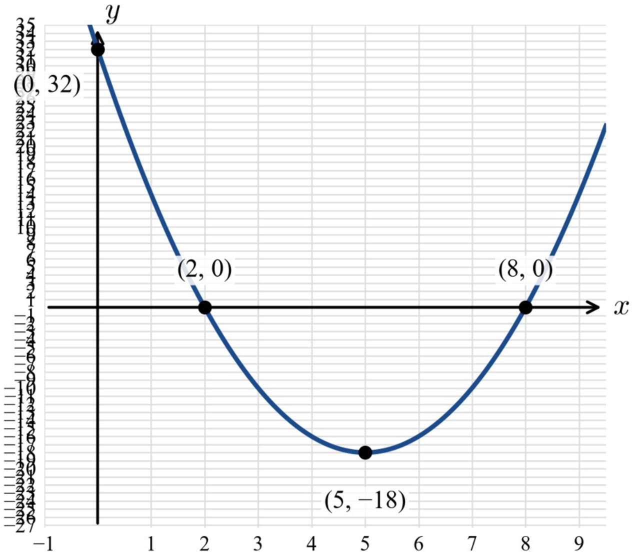
Sketch the graph of $y=2x^2-20x+32$.

$a=2>0$: **minimum**. y-intercept $(0,32)$.

Axis of symmetry: $x=-\frac{-20}{2\times 2}=\frac{20}{4}=5$.

Turning point: $y(5)=2(5)^2-20(5)+32=50-100+32=-18$, giving $(5,-18)$.

x-intercepts: $2x^2 - 20x + 32 = 0 \Rightarrow x^2 - 10x + 16 = 0 \Rightarrow (x-2)(x-8) = 0$, so $x = 2$ or $x = 8$, giving $(2, 0)$ and $(8, 0)$.



Practice questions

Scaffolded

Q1. For $y = (x-3)^2 - 16$: (a) write the turning point; (b) find the y-intercept; (c) find the x-intercepts; (d) sketch the graph, labelling all key features.

Q2. For $y = (x+3)^2 - 1$: (a) state the shape and turning point; (b) find the y-intercept; (c) find the x-intercepts; (d) sketch the graph.

Q3. For $y = x^2 - 6x + 5$: (a) find the axis of symmetry; (b) hence find the turning point; (c) find the x- and y-intercepts; (d) sketch the graph.

Q4. For $y = x^2 + 20x + 100$: (a) factorise the right-hand side; (b) explain why the parabola touches the x-axis at exactly one point; (c) state the turning point; (d) sketch the graph.

Q5. For $y = -(x+4)^2 + 9$: (a) state the shape and turning point; (b) find all axis intercepts; (c) sketch the graph.

Q6. For $y = 3(x-1)^2 - 3$: (a) state the turning point; (b) find all axis intercepts; (c) sketch the graph; (d) compared with $y = (x-1)^2 - 3$, how does the factor of 3 change the graph?

Unscaffolded

For each of the following: sketch the graph, labelling the turning point and all axis intercepts, **and** describe the transformation(s) applied to $y = x^2$ to obtain it.

Q7. $y = x^2 + 6$

Q8. $y = x^2 - 12$

Q9. $y = (x-8)^2$

Q10. $y = (x+11)^2$

Q11. $y = (x+3)^2 - 16$

Q12. $y = -x^2 + 10$

Q13. $y = -(x-2)^2 + 9$

Q14. $y = 2x^2 - 8$

Q15. $y = 1/2\,x^2 - 3$

Q16. $y = 4x^2 - 4$

Q17. $y = -2(x-4)^2 + 32$

Q18. The graph of $y = (x-5)^2 - 16$ is reflected in the y-axis. Find the rule of the image, then sketch it, labelling the turning point and all axis intercepts.

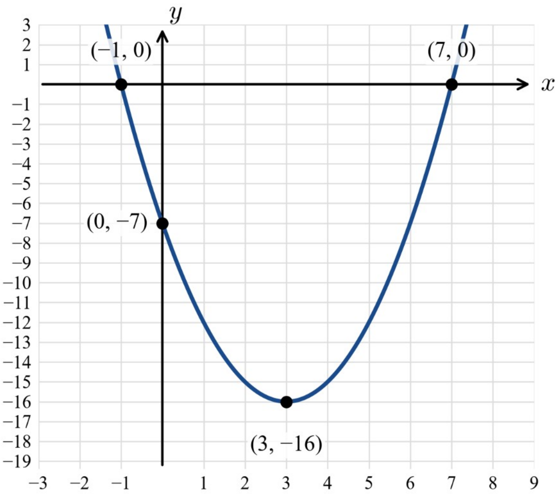
Answers

Use these to check your key features; full worked solutions follow.

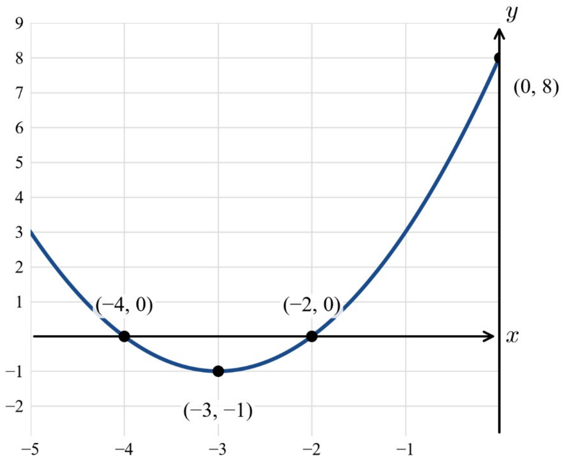
Q1. TP $(3,-16)$; x-int $(-1,0),(7,0)$; y-int $(0,-7)$. **Q2.** Minimum; TP $(-3,-1)$; x-int $(-4,0),(-2,0)$; y-int $(0,8)$. **Q3.** Axis $x=3$; TP $(3,-4)$; x-int $(1,0),(5,0)$; y-int $(0,5)$. **Q4.** $(x+10)^2$; touches at $(-10,0)$; TP $(-10,0)$; y-int $(0,100)$. **Q5.** Maximum; TP $(-4,9)$; x-int $(-7,0),(-1,0)$; y-int $(0,-7)$. **Q6.** TP $(1,-3)$; x-int $(0,0),(2,0)$; y-int $(0,0)$; three times as steep. **Q7.** TP $(0,6)$; no x-intercepts; y-int $(0,6)$; translation 6 up. **Q8.** TP $(0,-12)$; x-int $(-2\sqrt{3},0),(2\sqrt{3},0)$; y-int $(0,-12)$; translation 12 down. **Q9.** TP $(8,0)$; touches at $(8,0)$; y-int $(0,64)$; translation 8 right. **Q10.** TP $(-11,0)$; touches at $(-11,0)$; y-int $(0,121)$; translation 11 left. **Q11.** TP $(-3,-16)$; x-int $(-7,0),(1,0)$; y-int $(0,-7)$; 3 left and 16 down. **Q12.** Maximum; TP $(0,10)$; x-int $(-\sqrt{10},0),(\sqrt{10},0)$; y-int $(0,10)$; reflection in the x-axis, then 10 up. **Q13.** Maximum; TP $(2,9)$; x-int $(-1,0),(5,0)$; y-int $(0,5)$; reflection in the x-axis, then 2 right and 9 up. **Q14.** TP $(0,-8)$; x-int $(-2,0),(2,0)$; y-int $(0,-8)$; dilation factor 2 from the x-axis, then 8 down. **Q15.** TP $(0,-3)$; x-int $(-\sqrt{6},0),(\sqrt{6},0)$; y-int $(0,-3)$; dilation factor $1/2$ from the x-axis, then 3 down. **Q16.** TP $(0,-4)$; x-int $(-1,0),(1,0)$; y-int $(0,-4)$; dilation factor $1/2$ from the y-axis, then 4 down. **Q17.** Maximum; TP $(4,32)$; x-int $(0,0),(8,0)$; y-int $(0,0)$; reflection in the x-axis, dilation factor 2, then 4 right and 32 up. **Q18.** Image $y=(x+5)^2-16$; TP $(-5,-16)$; x-int $(-9,0),(-1,0)$; y-int $(0,9)$.

Fully-worked solutions

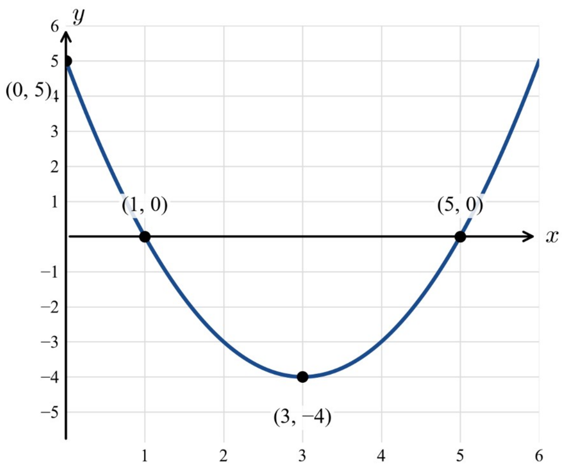
Q1. $y=(x-3)^2-16$. (a) Turning point $(3,-16)$. (b) $x=0 \Rightarrow y=(0-3)^2-16=9-16=-7$, giving $(0,-7)$. (c) $(x-3)^2-16=0 \Rightarrow (x-3)^2=16 \Rightarrow x-3=\pm 4$, so $x=7$ or $x=-1$, giving $(7,0)$ and $(-1,0)$.



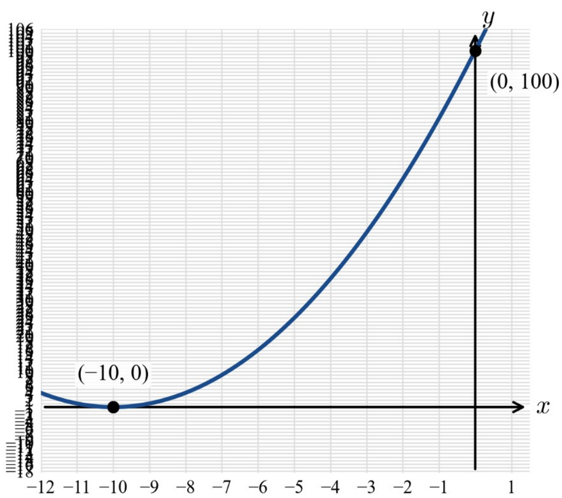
Q2. $y=(x+3)^2-1$. (a) $a=1>0$: **minimum**; turning point $(-3,-1)$. (b) $x=0 \Rightarrow y=9-1=8$, giving $(0,8)$. (c) $(x+3)^2=1 \Rightarrow x+3=\pm 1$, so $x=-2$ or $x=-4$, giving $(-2,0)$ and $(-4,0)$.



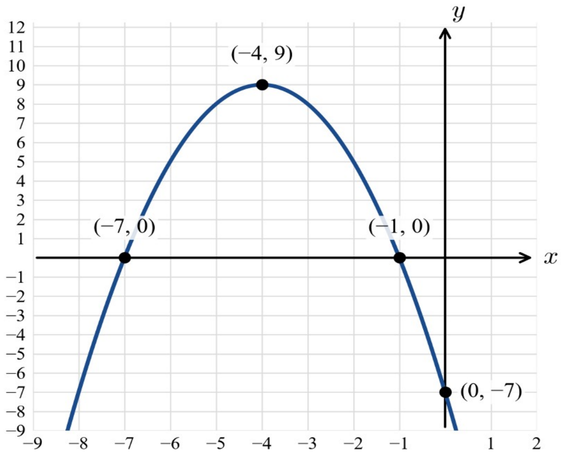
Q3. $y=x^2-6x+5$. (a) Axis: $x=-\frac{-6}{2\times1}=3$. (b) $y(3)=9-18+5=-4$, giving $(3,-4)$. (c) y-int $(0,5)$; $x^2-6x+5=(x-1)(x-5)=0$, so $(1,0)$ and $(5,0)$.



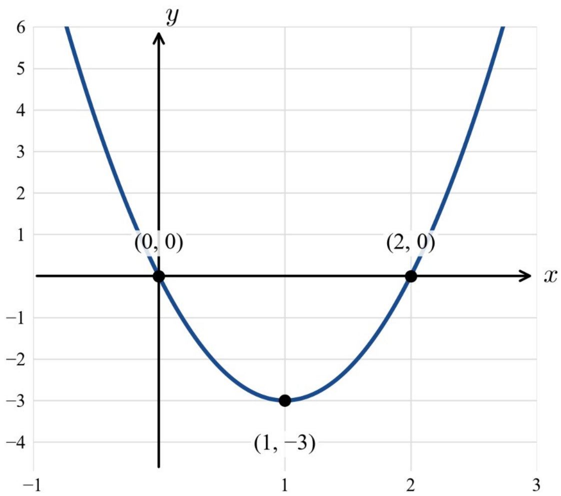
Q4. $y=x^2+20x+100$. (a) $x^2+20x+100=(x+10)^2$. (b) $y=(x+10)^2=0$ has the single repeated solution $x=-10$ (here $\Delta=20^2-4(1)(100)=0$), so the parabola **touches** the x-axis at one point. (c) Turning point $(-10,0)$.



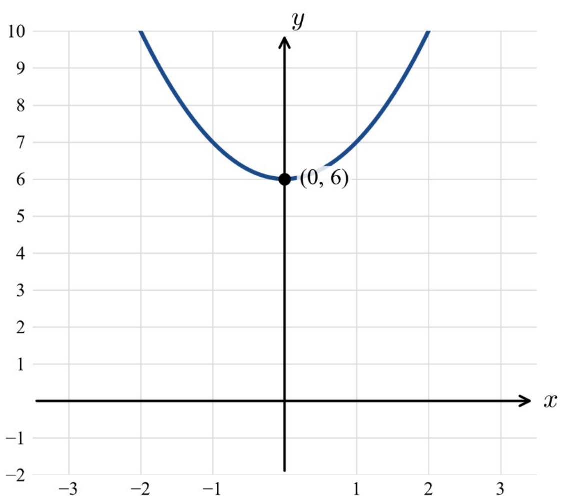
Q5. $y=-(x+4)^2+9$. (a) $a=-1<0$: **maximum**; turning point $(-4,9)$. (b) y-int $x=0\Rightarrow y=-16+9=-7$, giving $(0,-7)$; x-int $(x+4)^2=9\Rightarrow x+4=\pm3$, so $x=-1$ or $x=-7$, giving $(-1,0)$ and $(-7,0)$.



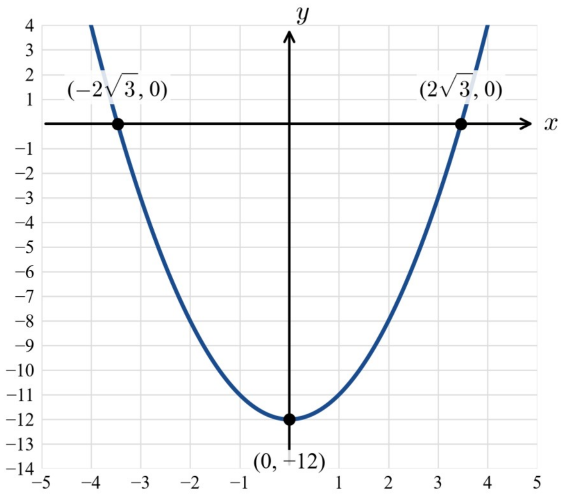
Q6. $y=3(x-1)^2-3$. (a) Turning point $(1,-3)$. (b) y-int $x=0 \Rightarrow y=3(1)-3=0$, giving $(0,0)$; x-int $3(x-1)^2=3 \Rightarrow (x-1)^2=1 \Rightarrow x=0$ or $x=2$, giving $(0,0)$ and $(2,0)$. (d) The factor of 3 is a dilation of factor 3 from the x-axis, so the parabola is **three times as steep** (narrower); the turning point is unchanged.



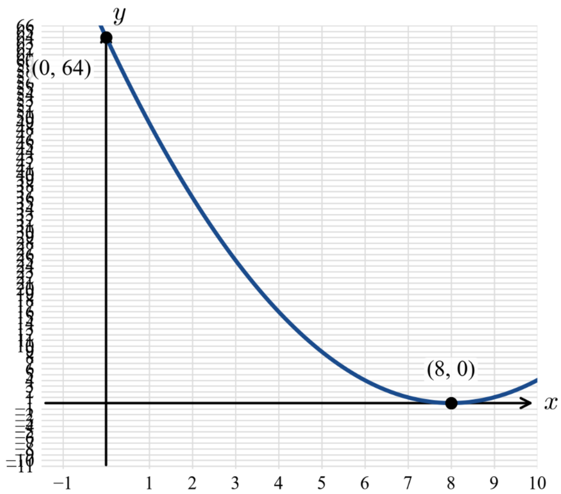
Q7. $y=x^2+6$. Turning point $(0,6)$; y-intercept $(0,6)$. $\Delta=-24<0$, so there are **no x-intercepts**. *Transformation:* translation 6 units up.



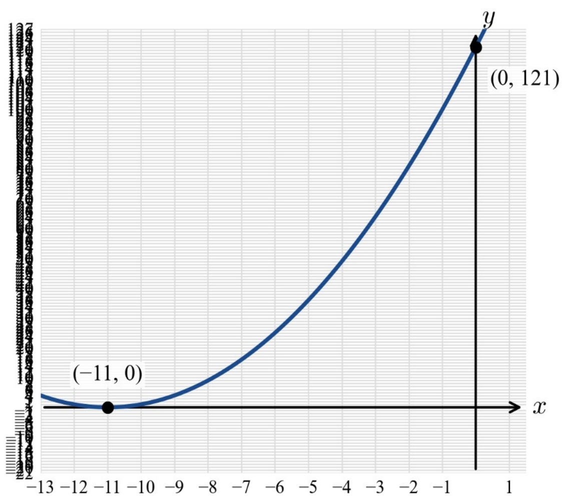
Q8. $y=x^2-12$. Turning point $(0,-12)$; y-intercept $(0,-12)$; $x^2-12=0 \Rightarrow x^2=12 \Rightarrow x=\pm 2\sqrt{3}$, giving $(-2\sqrt{3},0)$ and $(2\sqrt{3},0)$. *Transformation:* translation 12 units down.



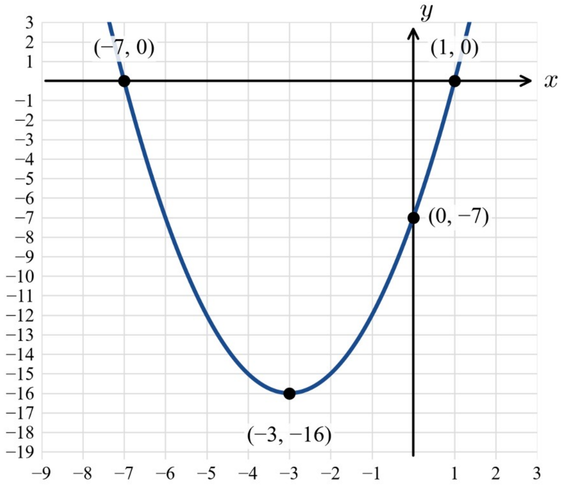
Q9. $y=(x-8)^2$. Turning point $(8,0)$; $\Delta=0$, so it **touches** at $(8,0)$; y-intercept $x=0 \Rightarrow y=64$, giving $(0,64)$. *Transformation:* translation 8 units right.



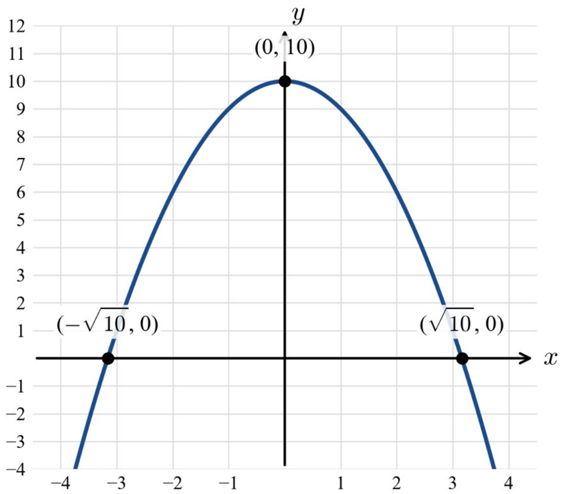
Q10. $y=(x+11)^2$. Turning point $(-11,0)$; touches at $(-11,0)$; y-intercept $x=0 \Rightarrow y=121$, giving $(0,121)$. *Transformation:* translation 11 units left.



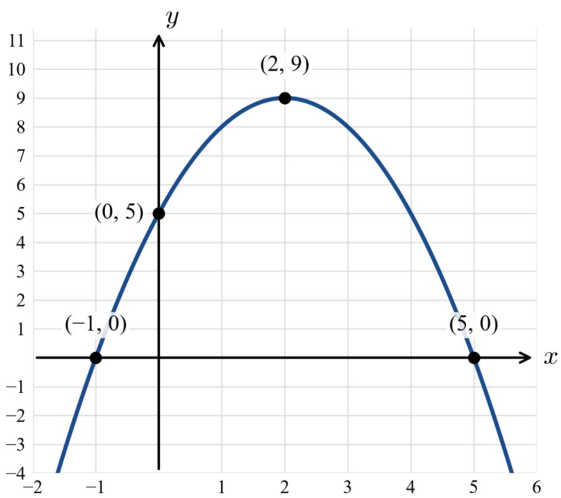
Q11. $y=(x+3)^2-16$. Turning point $(-3,-16)$; y-intercept $x=0 \Rightarrow y=9-16=-7$, giving $(0,-7)$; $(x+3)^2=16 \Rightarrow x+3=\pm 4$, so $x=1$ or $x=-7$, giving $(1,0)$ and $(-7,0)$. *Transformation:* translation 3 left and 16 down.



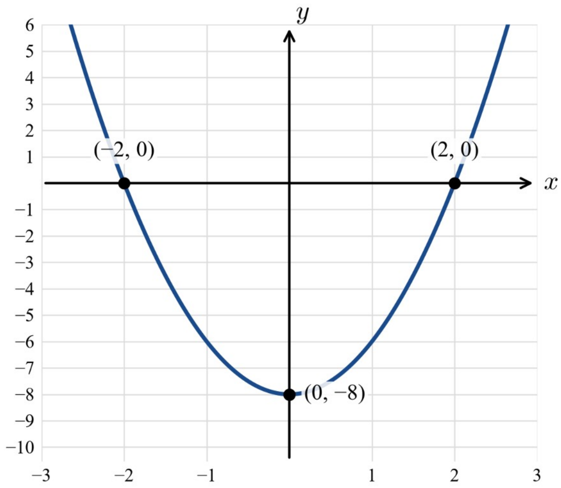
Q12. $y=-x^2+10$. $a=-1<0$: **maximum**; turning point $(0,10)$; y-intercept $(0,10)$; $-x^2+10=0 \Rightarrow x^2=10 \Rightarrow x=\pm\sqrt{10}$, giving $(-\sqrt{10},0)$ and $(\sqrt{10},0)$. *Transformation:* reflection in the x-axis, then translation 10 units up.



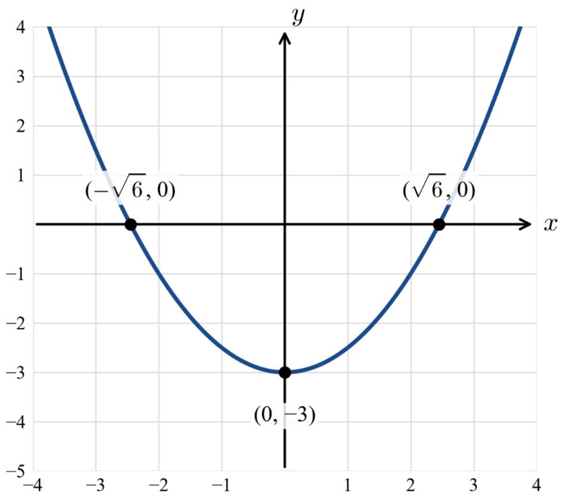
Q13. $y=-(x-2)^2+9$. $a=-1<0$: **maximum**; turning point $(2,9)$; y-intercept $x=0 \Rightarrow y=-4+9=5$, giving $(0,5)$; $(x-2)^2=9 \Rightarrow x-2=\pm 3$, so $x=5$ or $x=-1$, giving $(-1,0)$ and $(5,0)$. *Transformation:* reflection in the x-axis, then 2 right and 9 up.



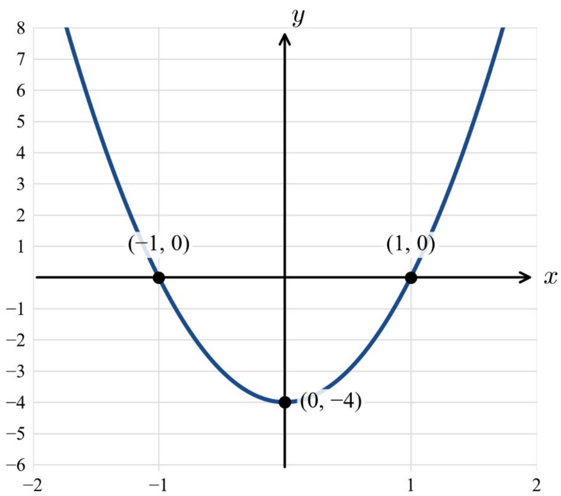
Q14. $y=2x^2-8$. Turning point $(0,-8)$; y-intercept $(0,-8)$; $2x^2-8=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$, giving $(-2,0)$ and $(2,0)$. *Transformation:* dilation of factor 2 from the x-axis (narrower), then 8 units down.



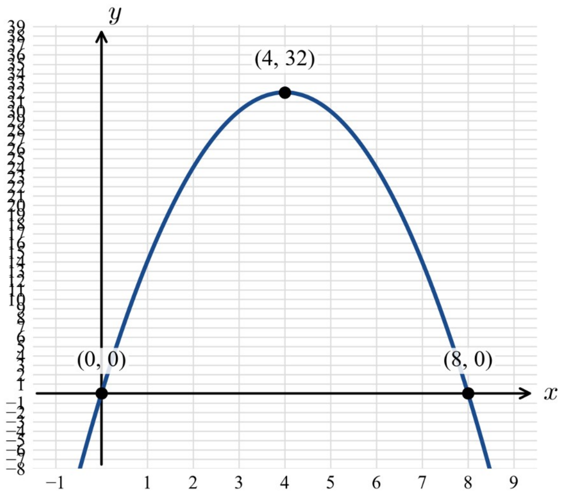
Q15. $y = 1/2\,x^2 - 3$. Turning point $(0, -3)$; y-intercept $(0, -3)$; $1/2\,x^2 - 3 = 0 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$, giving $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$. *Transformation:* dilation of factor $1/2$ from the x-axis (wider), then 3 units down.



Q16. $y = 4x^2 - 4$. Turning point $(0, -4)$; y-intercept $(0, -4)$; $4x^2 - 4 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, giving $(-1, 0)$ and $(1, 0)$. *Transformation:* since $4x^2 = (2x)^2$, a dilation of factor $1/2$ from the y-axis (equivalently factor 4 from the x-axis), then 4 units down.



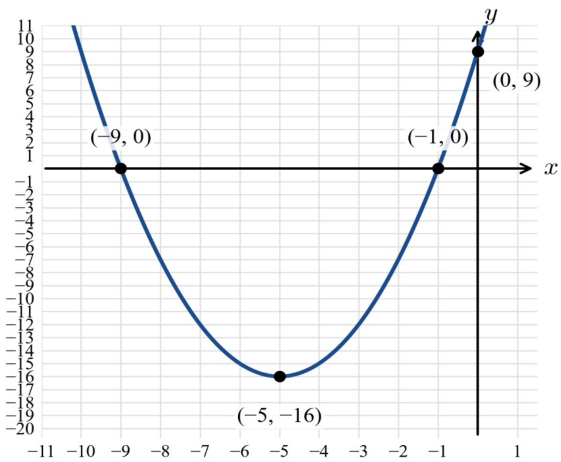
Q17. $y = -2(x-4)^2 + 32$. $a = -2 < 0$: **maximum**; turning point $(4, 32)$; y-intercept $x = 0 \Rightarrow y = -2(16) + 32 = 0$, giving $(0, 0)$; $-2(x-4)^2 + 32 = 0 \Rightarrow (x-4)^2 = 16 \Rightarrow x - 4 = \pm 4$, so $x = 8$ or $x = 0$, giving $(0, 0)$ and $(8, 0)$. *Transformation:* reflection in the x-axis, dilation of factor 2 from the x-axis, then 4 right and 32 up.



Q18. Reflecting $y=(x-5)^2-16$ in the y-axis replaces x with $-x$:

$$y=(-x-5)^2-16=(x+5)^2-16.$$

Turning point $(-5,-16)$; y-intercept $x=0 \Rightarrow y=25-16=9$, giving $(0,9)$; $(x+5)^2=16 \Rightarrow x+5=\pm 4$, so $x=-1$ or $x=-9$, giving $(-1,0)$ and $(-9,0)$.



Theory

Completing the square rewrites a quadratic in **turning-point form** $y=a(x-h)^2+k$, from which the turning point (h,k) can be read directly. It relies on the perfect-square pattern $x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$.

Monic quadratics ($a=1$): halve the coefficient of x, square it, then add and subtract it:

$$x^2+bx+c=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c.$$

Non-monic quadratics ($a\neq 1$): first factor a out of the x^2 and x terms, complete the square inside the bracket, then expand the a back through:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c.$$

Reading the graph. From $y=a(x-h)^2+k$:

- the **turning point** is (h,k) and the axis of symmetry is $x=h$;
- if $a>0$ the turning point is a **minimum**, the minimum value is k, and the range is $[k,\infty)$;
- if $a<0$ the turning point is a **maximum**, the maximum value is k, and the range is $(-\infty,k]$.

Completing the square works for **every** quadratic, even when it has no x-intercepts, so it is the most reliable way to locate a turning point.

Worked examples

Example 1 — scaffolded

Express $y=x^2+10x+21$ in turning-point form and state the coordinates of the turning point.

Working	Comment
Half of 10 is 5, and $5^2=25$.	Halve the coefficient of x and square it.
$x^2+10x+21=(x^2+10x+25)-25+21$	Add and subtract 25 so the value is unchanged.
$=(x+5)^2-4$	$x^2+10x+25=(x+5)^2$; then $-25+21=-4$.
Turning point $(-5,-4)$.	Read (h,k) from $y=(x-h)^2+k$.

Example 2 — scaffolded

Express $y=x^2-2x-7$ in turning-point form, and hence state the turning point and the range.

Working	Comment
Half of -2 is -1, and $(-1)^2=1$.	Halve the coefficient of x and square it.
$x^2-2x-7=(x^2-2x+1)-1-7=(x-1)^2-8$	Complete the square, then simplify $-1-7=-8$.
Turning point $(1,-8)$; $a=1>0$, so it is a minimum.	Minimum value -8.
Range: $[-8,\infty)$.	For a minimum the range is $[k,\infty)$.

Example 3 — unscaffolded

Complete the square for $y=2x^2-12x+7$ and state the turning point.

$$2x^2-12x+7=2(x^2-6x)+7=2\left((x-3)^2-9\right)+7=2(x-3)^2-18+7.$$

$\therefore y=2(x-3)^2-11$, so the turning point is $(3,-11)$.

Example 4 — unscaffolded

Complete the square for $y=x^2-3x+1$, and hence state the turning point and the minimum value.

Half of -3 is $-\frac{3}{2}$, and $\left(-\frac{3}{2}\right)^2=\frac{9}{4}$.

$$x^2-3x+1=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+1=\left(x-\frac{3}{2}\right)^2-\frac{5}{4}.$$

$\therefore$ turning point $\left(\frac{3}{2},-\frac{5}{4}\right)$; the minimum value is $-\frac{5}{4}$.

Practice questions

Scaffolded

Q1. Express $y=x^2+8x+3$ in turning-point form and state the turning point.

Q2. Express $y=x^2-10x+2$ in turning-point form and state the turning point.

Q3. Express $y=x^2+2x-5$ in turning-point form and state the turning point.

Q4. Express $y=x^2-6x+11$ in turning-point form, state the turning point, and explain why the graph has no x-intercepts.

Q5. Express $y=x^2+5x+4$ in turning-point form and state the turning point.

Q6. Express $y=2x^2+8x+5$ in turning-point form and state the turning point.

Unscaffolded

Express each of the following in turning-point form by completing the square, and hence state the coordinates of the turning point. Where indicated, also state the maximum or minimum value and the range.

Q7. $y=x^2+4x+2$ (state the range)

Q8. $y=x^2-8x+10$

Q9. $y=x^2+x-3$

Q10. $y=x^2-5x+2$

Q11. $y=x^2-7x+3$

Q12. $y=x^2+6x+13$ (state the minimum value and the range)

Q13. $y=2x^2-4x-3$

Q14. $y=3x^2+12x+5$

Q15. $y = 2x^2 + 6x + 1$

Q16. $y = -x^2 + 4x + 1$ (state the maximum value and the range)

Q17. $y = -x^2 + 2x + 5$

Q18. $y = -2x^2 + 8x - 1$ (state the maximum value and the range)

Answers

Q1. $(x+4)^2-13$; TP $(-4,-13)$. **Q2.** $(x-5)^2-23$; TP $(5,-23)$. **Q3.** $(x+1)^2-6$; TP $(-1,-6)$. **Q4.** $(x-3)^2+2$; TP $(3,2)$; minimum value $2>0$, so no x-intercepts. **Q5.** $\left(x+\frac{5}{2}\right)^2-\frac{9}{4}$; TP $\left(-\frac{5}{2},-\frac{9}{4}\right)$. **Q6.** $2(x+2)^2-3$; TP $(-2,-3)$. **Q7.** $(x+2)^2-2$; TP $(-2,-2)$; range $[-2,\infty)$. **Q8.** $(x-4)^2-6$; TP $(4,-6)$. **Q9.** $\left(x+\frac{1}{2}\right)^2-\frac{13}{4}$; TP $\left(-\frac{1}{2},-\frac{13}{4}\right)$. **Q10.** $\left(x-\frac{5}{2}\right)^2-\frac{17}{4}$; TP $\left(\frac{5}{2},-\frac{17}{4}\right)$. **Q11.** $\left(x-\frac{7}{2}\right)^2-\frac{37}{4}$; TP $\left(\frac{7}{2},-\frac{37}{4}\right)$. **Q12.** $(x+3)^2+4$; TP $(-3,4)$; minimum value 4; range $[4,\infty)$. **Q13.** $2(x-1)^2-5$; TP $(1,-5)$. **Q14.** $3(x+2)^2-7$; TP $(-2,-7)$. **Q15.** $2\left(x+\frac{3}{2}\right)^2-\frac{7}{2}$; TP $\left(-\frac{3}{2},-\frac{7}{2}\right)$. **Q16.** $-(x-2)^2+5$; TP $(2,5)$; maximum value 5; range $(-\infty,5]$. **Q17.** $-(x-1)^2+6$; TP $(1,6)$. **Q18.** $-2(x-2)^2+7$; TP $(2,7)$; maximum value 7; range $(-\infty,7]$.

Fully-worked solutions

Q1. $x^2+8x+3=(x^2+8x+16)-16+3=(x+4)^2-13$. $\therefore$ TP $(-4,-13)$.

Q2. $x^2-10x+2=(x^2-10x+25)-25+2=(x-5)^2-23$. $\therefore$ TP $(5,-23)$.

Q3. $x^2+2x-5=(x^2+2x+1)-1-5=(x+1)^2-6$. $\therefore$ TP $(-1,-6)$.

Q4. $x^2-6x+11=(x^2-6x+9)-9+11=(x-3)^2+2$, so TP $(3,2)$. Since the minimum value is $2>0$, the graph lies entirely above the x-axis and has **no x-intercepts**.

Q5. Half of 5 is $\frac{5}{2}$, and $\left(\frac{5}{2}\right)^2=\frac{25}{4}$, so $x^2+5x+4=\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+4=\left(x+\frac{5}{2}\right)^2-\frac{9}{4}$. $\therefore$ TP $\left(-\frac{5}{2},-\frac{9}{4}\right)$.

Q6. $2x^2+8x+5=2(x^2+4x)+5=2\left((x+2)^2-4\right)+5=2(x+2)^2-3$. $\therefore$ TP $(-2,-3)$.

Q7. $x^2+4x+2=(x^2+4x+4)-4+2=(x+2)^2-2$, so TP $(-2,-2)$. Since $a=1>0$, the range is $[-2,\infty)$.

Q8. $x^2-8x+10=(x^2-8x+16)-16+10=(x-4)^2-6$. $\therefore$ TP $(4,-6)$.

Q9. Half of 1 is $\frac{1}{2}$, and $\left(\frac{1}{2}\right)^2=\frac{1}{4}$, so $x^2+x-3=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-3=\left(x+\frac{1}{2}\right)^2-\frac{13}{4}$. $\therefore$ TP $\left(-\frac{1}{2},-\frac{13}{4}\right)$.

Q10. Half of -5 is $-\frac{5}{2}$, and $\left(\frac{5}{2}\right)^2=\frac{25}{4}$, so $x^2-5x+2=\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+2=\left(x-\frac{5}{2}\right)^2-\frac{17}{4}$. $\therefore$ TP $\left(\frac{5}{2},-\frac{17}{4}\right)$.

Q11. Half of -7 is $-\frac{7}{2}$, and $\left(\frac{7}{2}\right)^2=\frac{49}{4}$, so $x^2-7x+3=\left(x-\frac{7}{2}\right)^2-\frac{49}{4}+3=\left(x-\frac{7}{2}\right)^2-\frac{37}{4}$. $\therefore$ TP $\left(\frac{7}{2},-\frac{37}{4}\right)$.

Q12. $x^2+6x+13=(x^2+6x+9)-9+13=(x+3)^2+4$, so TP $(-3,4)$. Minimum value 4; range $[4,\infty)$.

Q13. $2x^2-4x-3=2(x^2-2x)-3=2\left((x-1)^2-1\right)-3=2(x-1)^2-5$. $\therefore$ TP $(1,-5)$.

Q14. $3x^2+12x+5=3(x^2+4x)+5=3\left((x+2)^2-4\right)+5=3(x+2)^2-7$. $\therefore$ TP $(-2,-7)$.

Q15. $2x^2+6x+1=2(x^2+3x)+1=2\left(\left(x+\frac{3}{2}\right)^2-\frac{9}{4}\right)+1=2\left(x+\frac{3}{2}\right)^2-\frac{9}{2}+1=2\left(x+\frac{3}{2}\right)^2-\frac{7}{2}$. $\therefore$ TP $\left(-\frac{3}{2},-\frac{7}{2}\right)$.

Q16. $-x^2+4x+1=-(x^2-4x)+1=-\left((x-2)^2-4\right)+1=-(x-2)^2+5$, so TP $(2,5)$. Since $a=-1<0$, the maximum value is 5 and the range is $(-\infty,5]$.

Q17. $-x^2+2x+5=-(x^2-2x)+5=-\left((x-1)^2-1\right)+5=-(x-1)^2+6$. $\therefore$ TP $(1,6)$.

Q18. $-2x^2+8x-1=-2(x^2-4x)-1=-2\left((x-2)^2-4\right)-1=-2(x-2)^2+7$, so TP $(2,7)$. Since $a=-2<0$, the maximum value is 7 and the range is $(-\infty,7]$.

Theory

This section sketches a parabola given in **polynomial form** $y=ax^2+bx+c$. (In 3D the quadratic was given in turning-point form; here we must find the key features from the coefficients.)

Shape. The sign of a gives the shape: $a>0$ opens **upwards** (minimum), $a<0$ opens **downwards** (maximum).

y-intercept. Substitute $x=0$: the y-intercept is $(0,c)$.

Axis of symmetry and turning point. The axis of symmetry is

$$x=-\frac{b}{2a}.$$

Substituting this value of x into the rule gives the y-coordinate of the turning point.

x-intercepts. Substitute $y=0$ and solve $ax^2+bx+c=0$ by factorising or with the quadratic formula. The **discriminant** $\Delta=b^2-4ac$ gives the number of x-intercepts: two if $\Delta>0$, one (the parabola **touches** the axis) if $\Delta=0$, and none if $\Delta<0$.

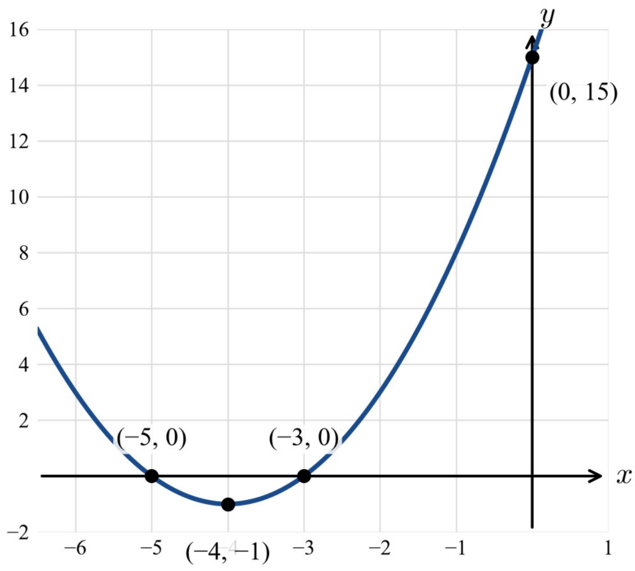
To sketch: (1) shape from a; (2) y-intercept $(0,c)$; (3) axis of symmetry and turning point; (4) x-intercepts, if any; (5) mark and label all key points and draw a smooth curve. When $\Delta<0$ there are no x-intercepts, so sketch using the turning point and y-intercept.

Worked examples

Example 1 — scaffolded

Sketch the graph of $y=x^2+8x+15$, labelling the turning point and all axis intercepts.

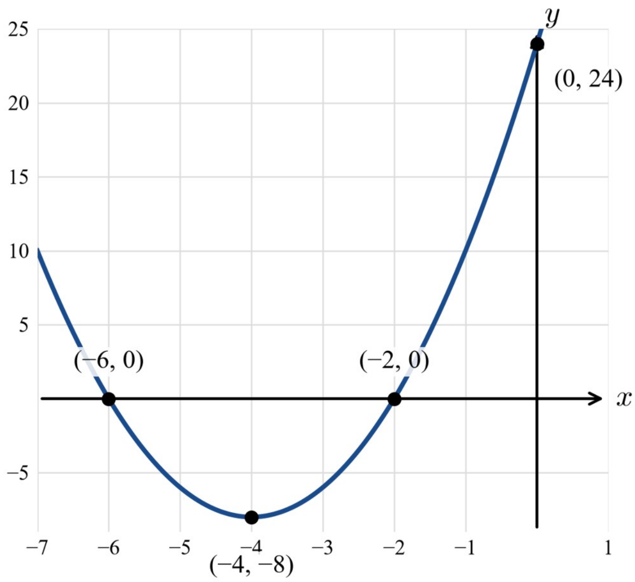
Working	Comment
$a=1>0$, so the parabola has a **minimum**.	Shape from the coefficient of x^2.
y-intercept $(0,15)$.	Put $x=0$: $y=c=15$.
Axis of symmetry: $x=-\dfrac{b}{2a}=-\dfrac{8}{2\times 1}=-4$.	Here $a=1$, $b=8$.
Turning point: $y(-4)=16-32+15=-1$, giving $(-4,-1)$.	Substitute $x=-4$ into the rule.
x-intercepts: $x^2+8x+15=(x+3)(x+5)=0$, so $x=-3$ or $x=-5$, giving $(-3,0)$ and $(-5,0)$.	Factorise and apply the null factor law.



Example 2 — scaffolded

Sketch the graph of $y = 2x^2 + 16x + 24$.

Working	Comment
$a = 2 > 0$: **minimum**. y-intercept $(0, 24)$.	Shape from a; y-intercept is $(0, c)$.
Axis of symmetry: $x = -\dfrac{16}{2 \times 2} = -4$.	$a = 2$, $b = 16$.
Turning point: $y(-4) = 2(-4)^2 + 16(-4) + 24 = 32 - 64 + 24 = -8$, giving $(-4, -8)$.	Substitute $x = -4$.
x-intercepts: $2x^2 + 16x + 24 = 0 \Rightarrow x^2 + 8x + 12 = (x+6)(x+2) = 0$, so $x = -6$ or $x = -2$.	Divide by 2 first, then factorise.



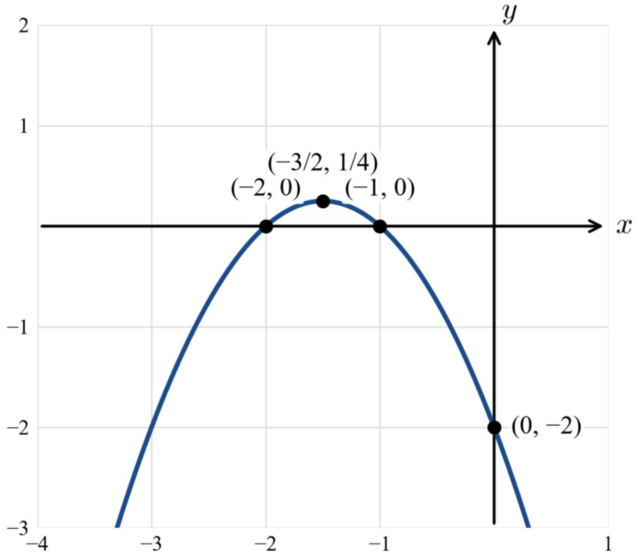
Example 3 — unscaffolded

Sketch the graph of $y = -x^2 - 3x - 2$.

$a = -1 < 0$: **maximum**. y-intercept $(0, -2)$.

Axis of symmetry: $x=-\dfrac{-3}{2\times(-1)}=-\dfrac{3}{2}$. Turning point: $y(-3/2)=-9/4+9/2-2=1/4$, giving $(-3/2,1/4)$.

x-intercepts: $-x^2-3x-2=0\Rightarrow x^2+3x+2=(x+1)(x+2)=0$, so $x=-1$ or $x=-2$, giving $(-1,0)$ and $(-2,0)$.

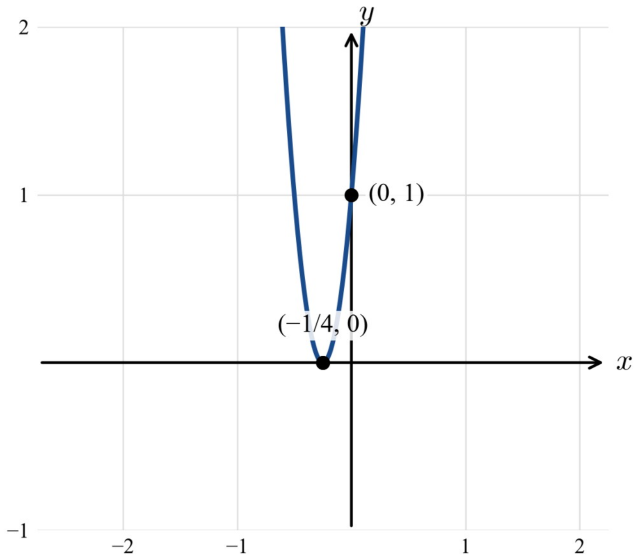


Example 4 — unscaffolded

Sketch the graph of $y=16x^2+8x+1$.

$a=16>0$: **minimum**. y-intercept $(0,1)$. Axis of symmetry $x=-\dfrac{8}{2\times16}=-\dfrac{1}{4}$; turning point $y(-1/4)=1-2+1=0$, giving $(-1/4,0)$.

$\Delta=8^2-4(16)(1)=0$, so there is one x-intercept — the parabola **touches** the x-axis at $(-1/4,0)$ (which is also the turning point).



Practice questions

Scaffolded

Q1. For $y=x^2-6x+8$: (a) state the shape and y-intercept; (b) find the axis of symmetry and turning point; (c) find the x-intercepts; (d) sketch the graph.

Q2. For $y=x^2+10x+16$: (a) state the shape and y-intercept; (b) find the turning point; (c) find the x-intercepts; (d) sketch the graph.

Q3. For $y=2x^2+14x+20$: (a) find the y-intercept; (b) find the axis of symmetry and turning point; (c) find the x-intercepts; (d) sketch the graph.

Q4. For $y=-x^2+5x-4$: (a) state the shape; (b) find the turning point; (c) find all axis intercepts; (d) sketch the graph.

Q5. For $y=25x^2-10x+1$: (a) evaluate the discriminant; (b) explain what it tells you about the x-intercepts; (c) state the turning point; (d) sketch the graph.

Q6. For $y=x^2-4x+6$: (a) show that $\Delta<0$; (b) find the turning point and y-intercept; (c) sketch the graph.

Unscaffolded

Sketch each of the following, labelling the turning point and all axis intercepts.

Q7. $y=x^2-9x+14$ **Q8.** $y=x^2+9x+14$ **Q9.** $y=x^2-9x+18$

Q10. $y=2x^2-18x+16$ **Q11.** $y=3x^2+27x+24$ **Q12.** $y=-x^2-7x-6$

Q13. $y=-x^2+7x-6$ **Q14.** $y=25x^2+10x+1$ **Q15.** $y=4x^2+4x+1$

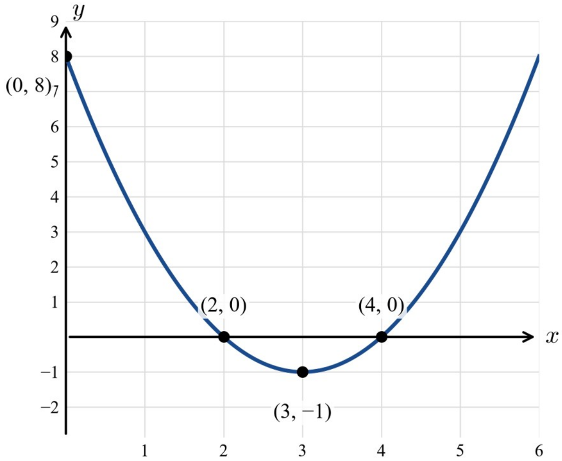
Q16. $y=3x^2+2$ **Q17.** $y=x^2-2x+2$ **Q18.** $y=x^2+4x-1$

Answers

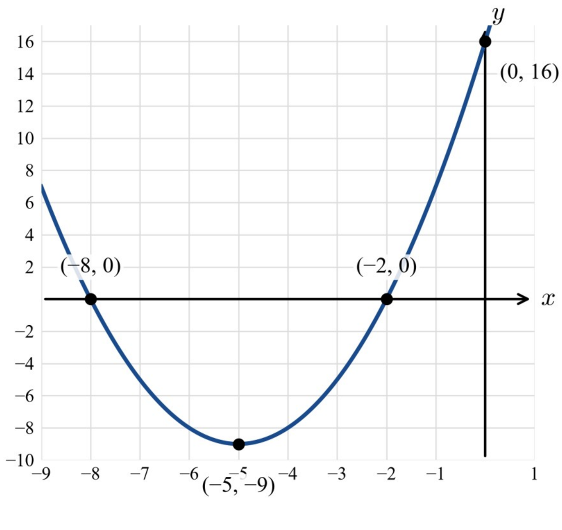
Q1. TP $(3,-1)$; x-int $(2,0),(4,0)$; y-int $(0,8)$. **Q2.** TP $(-5,-9)$; x-int $(-8,0),(-2,0)$; y-int $(0,16)$. **Q3.** TP $(-7/2,-9/2)$; x-int $(-5,0),(-2,0)$; y-int $(0,20)$. **Q4.** Maximum; TP $(5/2,9/4)$; x-int $(1,0),(4,0)$; y-int $(0,-4)$. **Q5.** $\Delta=0$; touches at $(1/5,0)$; TP $(1/5,0)$; y-int $(0,1)$. **Q6.** $\Delta=-8<0$; no x-intercepts; TP $(2,2)$; y-int $(0,6)$. **Q7.** TP $(9/2,-25/4)$; x-int $(2,0),(7,0)$; y-int $(0,14)$. **Q8.** TP $(-9/2,-25/4)$; x-int $(-7,0),(-2,0)$; y-int $(0,14)$. **Q9.** TP $(9/2,-9/4)$; x-int $(3,0),(6,0)$; y-int $(0,18)$. **Q10.** TP $(9/2,-49/2)$; x-int $(1,0),(8,0)$; y-int $(0,16)$. **Q11.** TP $(-9/2,-147/4)$; x-int $(-8,0),(-1,0)$; y-int $(0,24)$. **Q12.** Maximum; TP $(-7/2,25/4)$; x-int $(-6,0),(-1,0)$; y-int $(0,-6)$. **Q13.** Maximum; TP $(7/2,25/4)$; x-int $(1,0),(6,0)$; y-int $(0,-6)$. **Q14.** Touches at $(-1/5,0)$; TP $(-1/5,0)$; y-int $(0,1)$. **Q15.** Touches at $(-1/2,0)$; TP $(-1/2,0)$; y-int $(0,1)$. **Q16.** TP $(0,2)$; no x-intercepts; y-int $(0,2)$. **Q17.** TP $(1,1)$; no x-intercepts; y-int $(0,2)$. **Q18.** TP $(-2,-5)$; x-int $(-2-\sqrt{5},0),(-2+\sqrt{5},0)$; y-int $(0,-1)$.

Fully-worked solutions

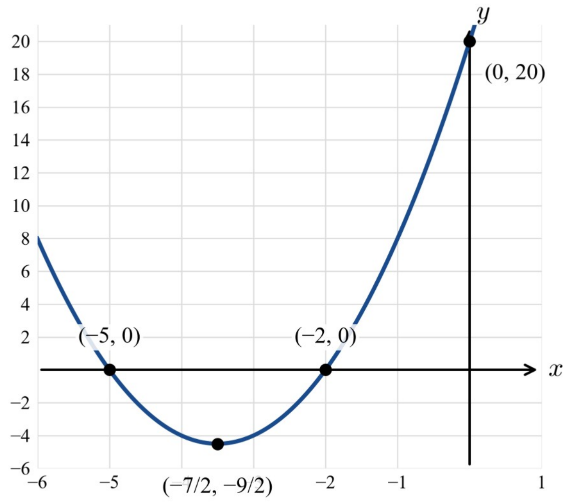
Q1. $y=x^2-6x+8$. Minimum; y-int $(0,8)$. Axis $x=-\dfrac{-6}{2}=3$; TP $y(3)=9-18+8=-1$, giving $(3,-1)$. $x^2-6x+8=(x-2)(x-4)=0$, so $(2,0)$ and $(4,0)$.



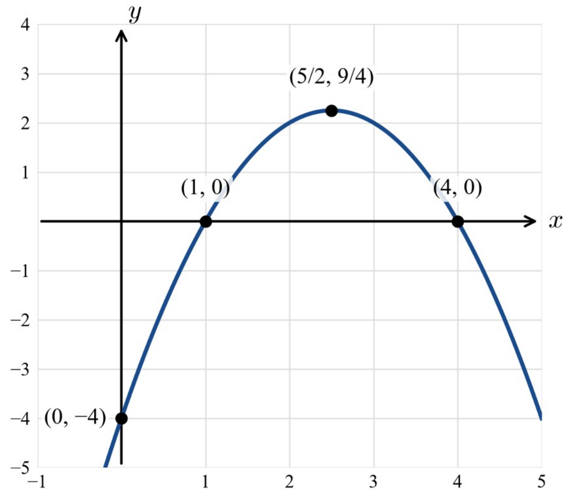
Q2. $y=x^2+10x+16$. Minimum; y-int $(0,16)$. Axis $x=-5$; TP $y(-5)=25-50+16=-9$, giving $(-5,-9)$. $(x+8)(x+2)=0$, so $(-8,0)$ and $(-2,0)$.



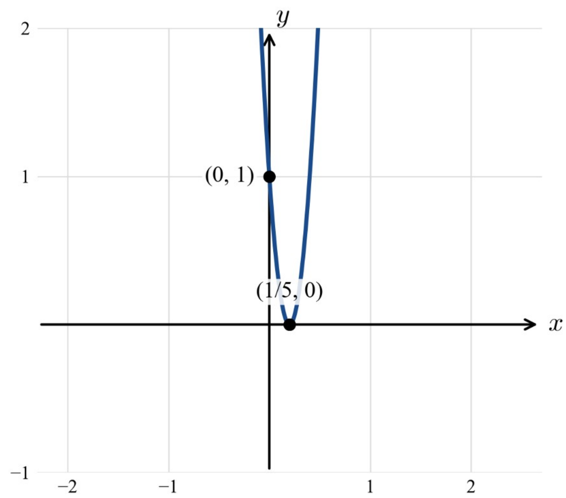
Q3. $y=2x^2+14x+20$. y-int $(0,20)$. Axis $x=-\dfrac{14}{2\times 2}=-\dfrac{7}{2}$; TP $y(-7/2)=49/2-49+20=-9/2$, giving $(-7/2,-9/2)$. $2x^2+14x+20=0 \Rightarrow x^2+7x+10=(x+5)(x+2)=0$, so $(-5,0)$ and $(-2,0)$.



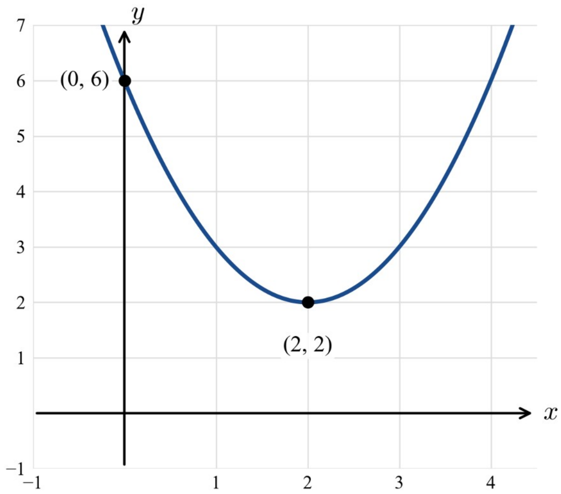
Q4. $y=-x^2+5x-4$. $a=-1<0$: **maximum**. y-int $(0,-4)$. Axis $x=-\dfrac{5}{2\times(-1)}=\dfrac{5}{2}$; TP $y(5/2)=-25/4+25/2-4=9/4$, giving $(5/2,9/4)$. $-x^2+5x-4=0\Rightarrow x^2-5x+4=(x-1)(x-4)=0$, so $(1,0)$ and $(4,0)$.



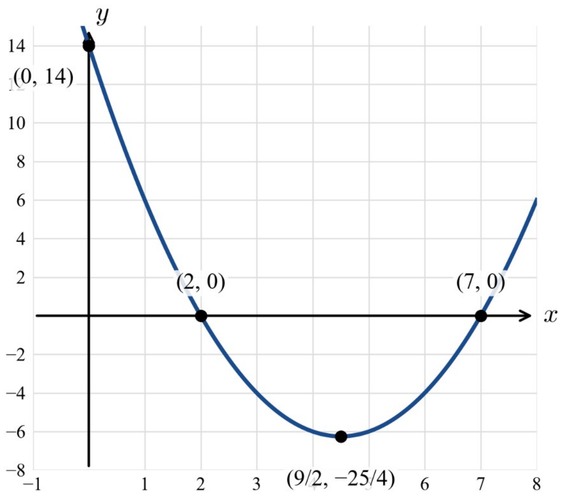
Q5. $y=25x^2-10x+1$. (a) $\Delta=(-10)^2-4(25)(1)=100-100=0$. (b) $\Delta=0$ means one repeated x-intercept — the parabola touches the axis. (c) $(5x-1)^2=0\Rightarrow x=1/5$; TP $(1/5,0)$; y-int $(0,1)$.



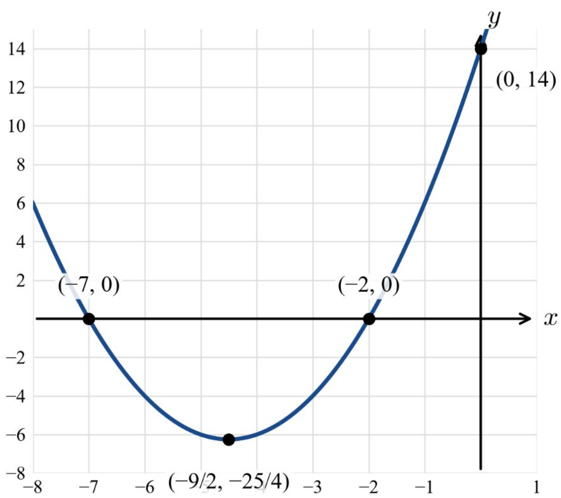
Q6. $y=x^2-4x+6$. (a) $\Delta=(-4)^2-4(1)(6)=16-24=-8<0$, so there are no x-intercepts. (b) Axis $x=2$; TP $y(2)=4-8+6=2$, giving $(2,2)$; y-int $(0,6)$.



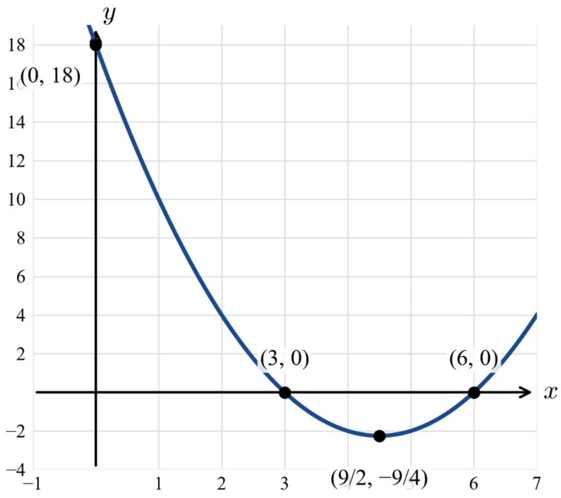
Q7. $y=x^2-9x+14$. Minimum; y-int $(0,14)$; axis $x=\dfrac{9}{2}$; TP $(9/2,-25/4)$; $(x-2)(x-7)=0$, so $(2,0)$ and $(7,0)$.



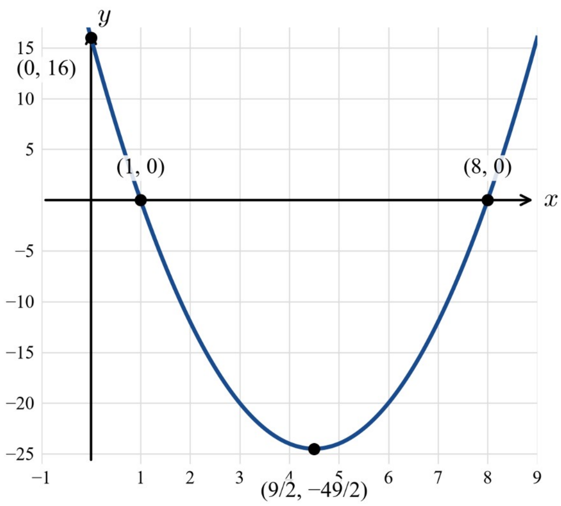
Q8. $y=x^2+9x+14$. Minimum; y-int $(0,14)$; axis $x=-\dfrac{9}{2}$; TP $(-9/2,-25/4)$; $(x+7)(x+2)=0$, so $(-7,0)$ and $(-2,0)$.



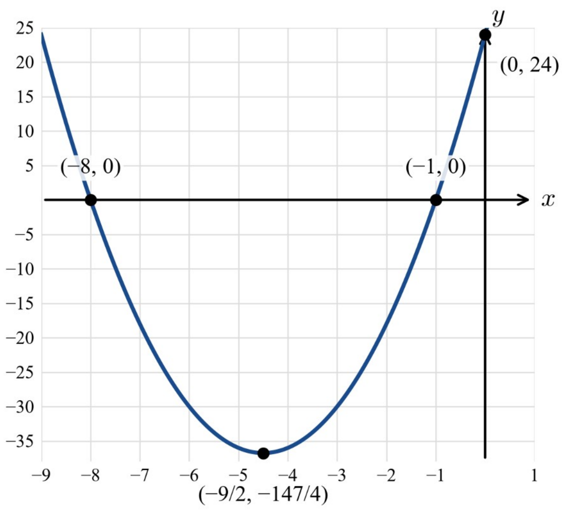
Q9. $y=x^2-9x+18$. Minimum; y-int $(0,18)$; axis $x=\dfrac{9}{2}$; TP $(9/2,-9/4)$; $(x-3)(x-6)=0$, so $(3,0)$ and $(6,0)$.



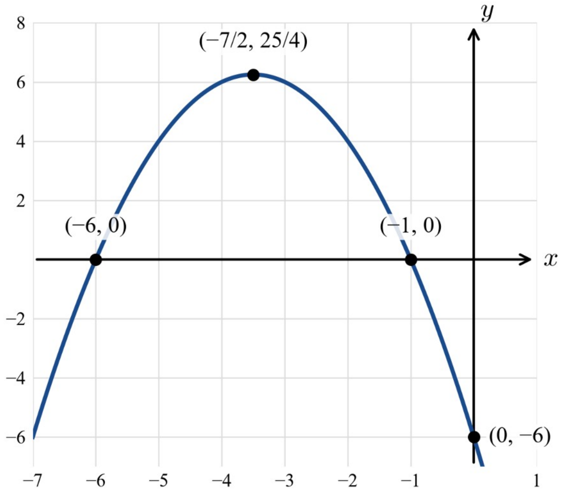
Q10. $y = 2x^2 - 18x + 16$. Minimum; y-int $(0, 16)$; axis $x = \dfrac{18}{2\times 2} = \dfrac{9}{2}$; TP $(9/2, -49/2)$;
$2x^2 - 18x + 16 = 0 \Rightarrow x^2 - 9x + 8 = (x-1)(x-8) = 0$, so $(1, 0)$ and $(8, 0)$.



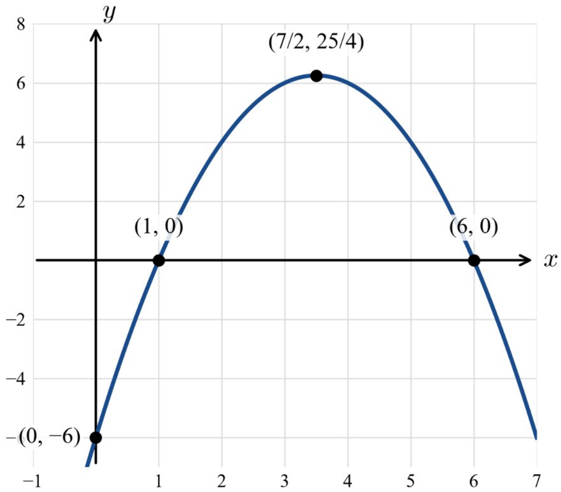
Q11. $y = 3x^2 + 27x + 24$. Minimum; y-int $(0, 24)$; axis $x = -\dfrac{27}{2\times 3} = -\dfrac{9}{2}$; TP $(-9/2, -147/4)$;
$3x^2 + 27x + 24 = 0 \Rightarrow x^2 + 9x + 8 = (x+8)(x+1) = 0$, so $(-8, 0)$ and $(-1, 0)$.



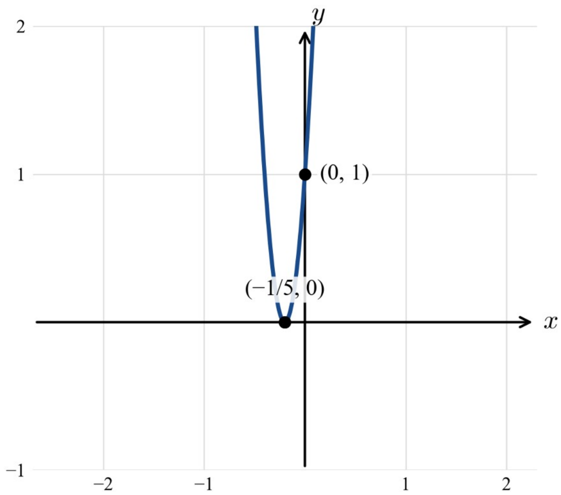
Q12. $y = -x^2 - 7x - 6$. $a < 0$: **maximum**; y-int $(0, -6)$; axis $x = -\dfrac{7}{2}$; TP $(-7/2, 25/4)$;
$-x^2 - 7x - 6 = 0 \Rightarrow x^2 + 7x + 6 = (x+6)(x+1) = 0$, so $(-6, 0)$ and $(-1, 0)$.



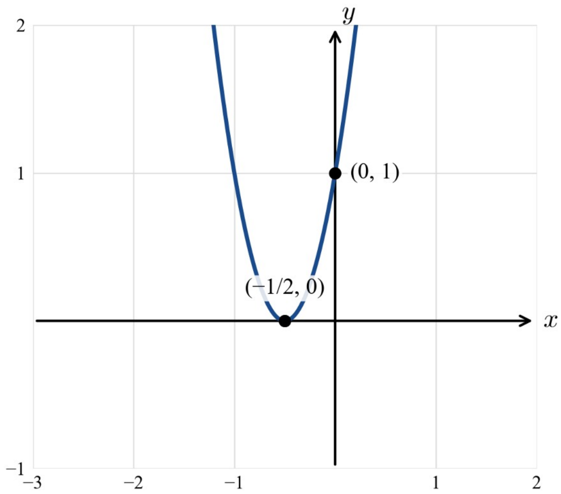
Q13. $y=-x^2+7x-6$. $a<0$: **maximum**; y-int $(0,-6)$; axis $x=\dfrac{7}{2}$; TP $(7/2, 25/4)$;
$-x^2+7x-6=0 \Rightarrow x^2-7x+6=(x-1)(x-6)=0$, so $(1,0)$ and $(6,0)$.



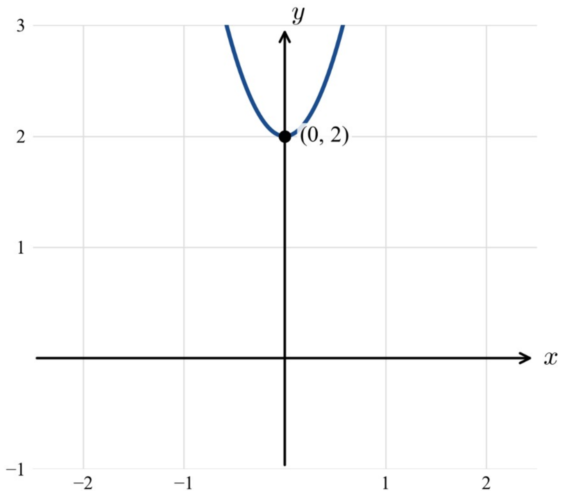
Q14. $y=25x^2+10x+1=(5x+1)^2$. $\Delta=0$: touches at $(-1/5,0)$; TP $(-1/5,0)$; y-int $(0,1)$.



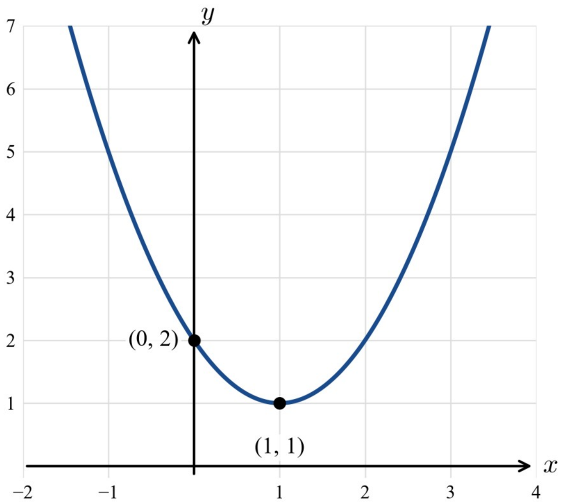
Q15. $y=4x^2+4x+1=(2x+1)^2$. $\Delta=0$: touches at $(-1/2,0)$; TP $(-1/2,0)$; y-int $(0,1)$.



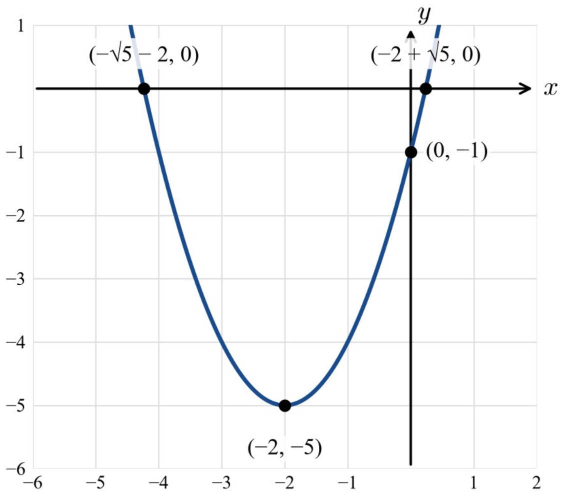
Q16. $y=3x^2+2$. $\Delta=0-24=-24<0$: no x-intercepts; TP $(0,2)$; y-int $(0,2)$.



Q17. $y=x^2-2x+2$. $\Delta=(-2)^2-4(1)(2)=-4<0$: no x-intercepts; axis $x=1$; TP $(1,1)$; y-int $(0,2)$.



Q18. $y=x^2+4x-1$. Minimum; y-int $(0,-1)$; axis $x=-2$; TP $(-2,-5)$. Using the quadratic formula, $x=\dfrac{-4\pm\sqrt{4^2-4(1)(-1)}}{2}=\dfrac{-4\pm\sqrt{20}}{2}=\dfrac{-4\pm2\sqrt{5}}{2}=-2\pm\sqrt{5}$, so the x-intercepts are $(-2-\sqrt{5},0)$ and $(-2+\sqrt{5},0)$.



y
x
$(-\sqrt{5}-2, 0)$
$(-2+\sqrt{5}, 0)$
$(0, -1)$
$(-2, -5)$
1
−1
−2
−3
−4
−5
−6
−6
−5
−4
−3
−2
−1
1
2

Theory

A **quadratic inequality** has the form $ax^2+bx+c>0$ (or with ≥, <, ≤). Solving it means finding all values of x that make it true.

Method — read the sign from the graph.

1. Find the x-intercepts of the parabola $y=ax^2+bx+c$ by solving $ax^2+bx+c=0$ (usually by factorising).
2. Sketch the parabola — it opens **upwards** if $a>0$ and **downwards** if $a<0$.
3. The parabola is **above** the x-axis where $y>0$ and **below** it where $y<0$. Read the required region straight off the sketch.

For an upright parabola ($a>0$) with x-intercepts $p<q$:

- $y>0$ (above the axis) for $x<p$ or $x>q$ — **outside** the intercepts;
- $y<0$ (below the axis) for $p<x<q$ — **between** the intercepts.

Use ≤ or ≥ to include the endpoints.

Negative leading coefficient. Either sketch the downward parabola directly, or multiply both sides by -1 and **reverse** the inequality sign, then solve as an upright parabola.

Special cases (the discriminant). If the parabola has no x-intercepts ($\Delta<0$) it lies entirely on one side of the axis — for example $x^2+1>0$ is true for **all real** x, while $x^2+1<0$ has **no solution**. If it touches the axis ($\Delta=0$), then $(x-3)^2\geq 0$ holds for all x, but $(x-3)^2>0$ holds for all x **except** $x=3$.

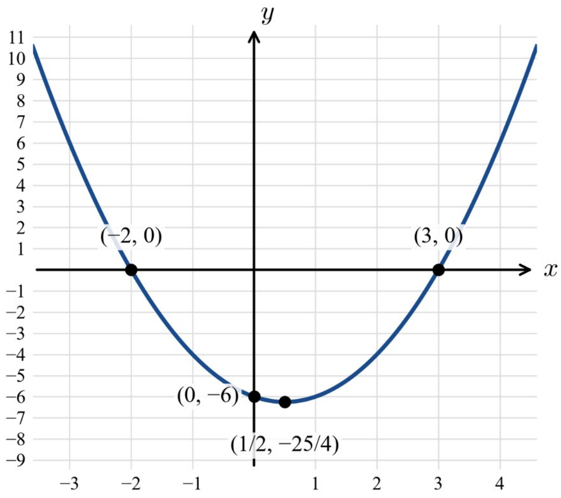
Give answers in set-builder notation (Australian colon form), e.g. $\{x : x<-2\}\cup\{x : x>3\}$, or in interval notation.

Worked examples

Example 1 — scaffolded

Solve $x^2-x-6>0$.

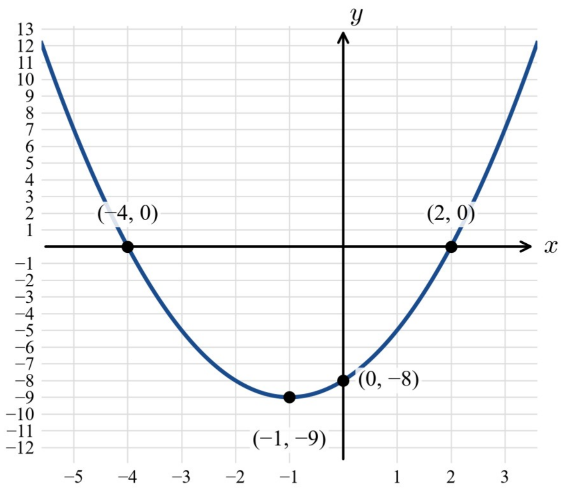
Working	Comment
$x^2-x-6=0 \Rightarrow (x-3)(x+2)=0$, so $x=-2$ or $x=3$.	Find the x-intercepts of $y=x^2-x-6$.
$a=1>0$, so the parabola opens **upwards**.	Shape from the coefficient of x^2.
We want $y>0$: the parts of the parabola **above** the x-axis, which are **outside** the intercepts.	Compare the sketch with the required sign.
$\therefore \{x : x<-2\}\cup\{x : x>3\}$.	Read off the region.



Example 2 — scaffolded

Solve $x^2+2x-8\le 0$.

Working	Comment
$x^2+2x-8=0 \Rightarrow (x+4)(x-2)=0$, so $x=-4$ or $x=2$.	Find the x-intercepts.
$a=1>0$: opens **upwards**.	Shape.
We want $y\le 0$: on or **below** the axis, which is **between** the intercepts (endpoints included).	≤ includes the intercepts.
$\therefore \{x:-4\le x\le 2\}$.	Read off the region.

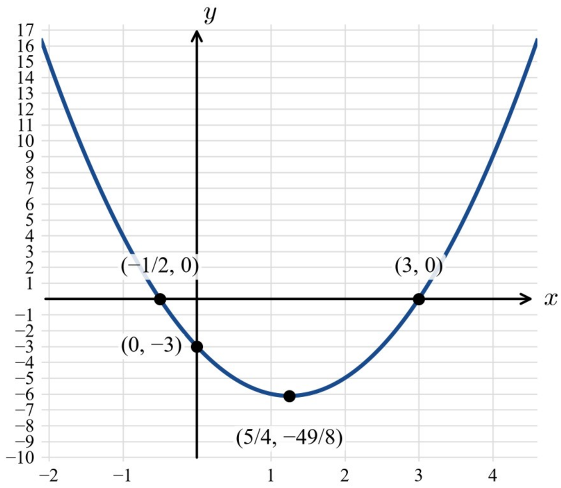


Example 3 — unscaffolded

Solve $2x^2-5x-3<0$.

$2x^2-5x-3=(2x+1)(x-3)=0 \Rightarrow x=-1/2$ or $x=3$. The parabola opens upwards ($a=2>0$), so $y<0$ **between** the intercepts.

$\therefore \{x:-1/2<x<3\}$.

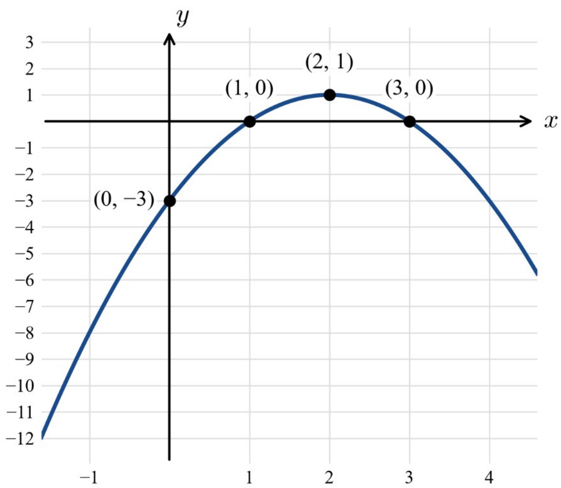


Example 4 — unscaffolded

Solve $-x^2+4x-3\geq 0$.

Multiply both sides by -1 and **reverse** the sign: $x^2-4x+3\leq 0$. Then $(x-1)(x-3)\leq 0$, with intercepts $x=1$ and $x=3$; the upright parabola is on or below the axis **between** them.

$\therefore \{x:1\leq x\leq 3\}$.



Practice questions

Scaffolded

Q1. Solve $x^2-9>0$: (a) find the x-intercepts of $y=x^2-9$; (b) state the shape; (c) hence write the solution set.

Q2. Solve $x^2-7x+10\leq 0$: (a) factorise; (b) sketch the parabola; (c) hence solve.

Q3. Solve $x^2+4x<0$: (a) factorise (common factor); (b) find the intercepts; (c) hence solve.

Q4. Solve $x^2-x-12\geq 0$.

Q5. Solve $3x^2+5x-2<0$.

Q6. Solve $x^2-6x+9>0$. (Hint: the right-hand side is a perfect square.)

Unscaffolded

Solve each of the following, giving your answer in set-builder notation.

Q7. $x^2 - x - 2 > 0$

Q8. $x^2 + 5x + 6 \leq 0$

Q9. $x^2 - 16 < 0$

Q10. $x^2 + 2x - 15 \geq 0$

Q11. $2x^2 + 5x - 3 > 0$

Q12. $3x^2 - 10x - 8 \leq 0$

Q13. $-x^2 + x + 6 > 0$

Q14. $-x^2 - 2x + 8 \leq 0$

Q15. $x^2 + 1 > 0$

Q16. $x^2 + 4 < 0$

Q17. $x^2 - 6x + 9 \geq 0$

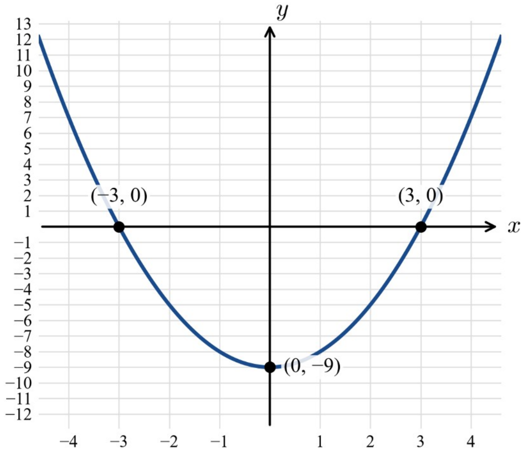
Q18. $x^2 < 0$

Answers

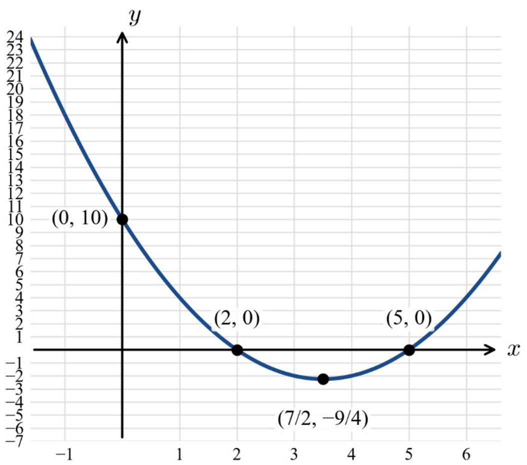
Q1. $\{x : x < -3\} \cup \{x : x > 3\}$. **Q2.** $\{x : 2 \le x \le 5\}$. **Q3.** $\{x : -4 < x < 0\}$. **Q4.** $\{x : x \le -3\} \cup \{x : x \ge 4\}$. **Q5.** $\{x : -2 < x < 1/3\}$. **Q6.** $\{x : x \ne 3\}$ (all real x except 3). **Q7.** $\{x : x < -1\} \cup \{x : x > 2\}$. **Q8.** $\{x : -3 \le x \le -2\}$. **Q9.** $\{x : -4 < x < 4\}$. **Q10.** $\{x : x \le -5\} \cup \{x : x \ge 3\}$. **Q11.** $\{x : x < -3\} \cup \{x : x > 1/2\}$. **Q12.** $\{x : -2/3 \le x \le 4\}$. **Q13.** $\{x : -2 < x < 3\}$. **Q14.** $\{x : x \le -4\} \cup \{x : x \ge 2\}$. **Q15.** all real x ($x \in \mathbb{R}$). **Q16.** no solution. **Q17.** all real x ($x \in \mathbb{R}$). **Q18.** no solution.

Fully-worked solutions

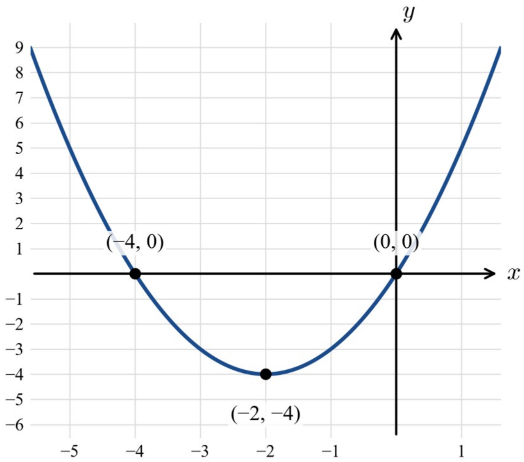
Q1. $x^2 - 9 = (x-3)(x+3) = 0 \Rightarrow x = \pm 3$; upright parabola, want $y > 0$ (outside). $\therefore \{x : x < -3\} \cup \{x : x > 3\}$.



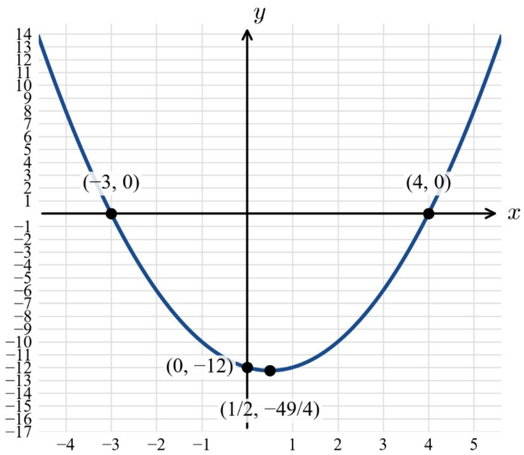
Q2. $x^2 - 7x + 10 = (x-2)(x-5) = 0 \Rightarrow x = 2$ or 5; upright, want $y \le 0$ (between, inclusive). $\therefore \{x : 2 \le x \le 5\}$.



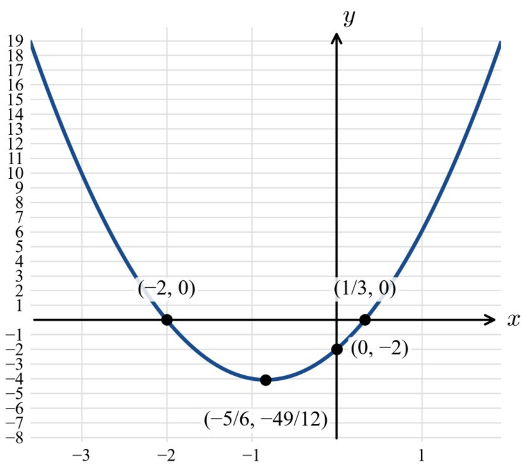
Q3. $x^2 + 4x = x(x+4) = 0 \Rightarrow x = 0$ or $x = -4$; upright, want $y < 0$ (between). $\therefore \{x : -4 < x < 0\}$.



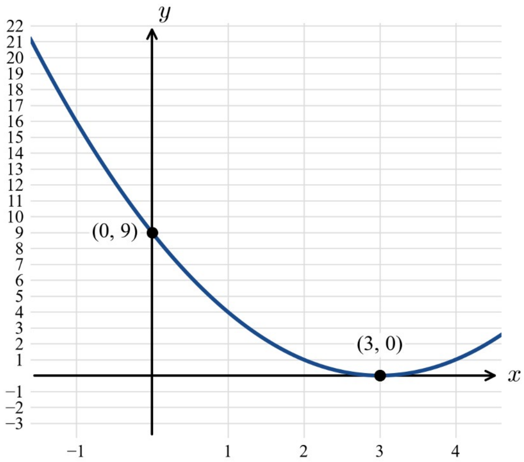
Q4. $x^2 - x - 12 = (x-4)(x+3) = 0 \Rightarrow x = -3$ or 4; upright, want $y \geq 0$ (outside, inclusive). $\therefore \{x : x \leq -3\} \cup \{x : x \geq 4\}$.



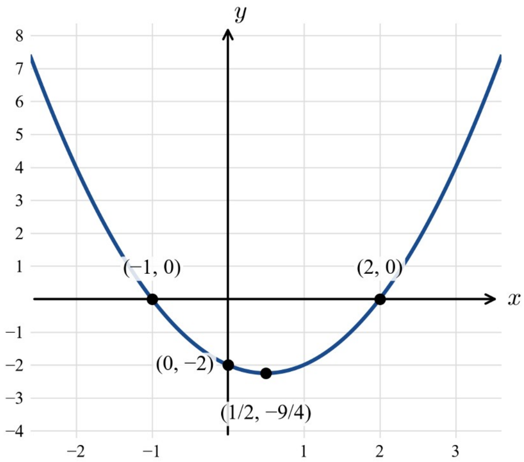
Q5. $3x^2 + 5x - 2 = (3x-1)(x+2) = 0 \Rightarrow x = -2$ or $x = 1/3$; upright, want $y < 0$ (between). $\therefore \{x : -2 < x < 1/3\}$.



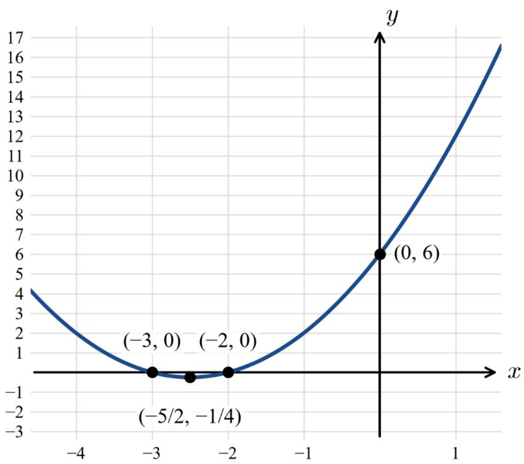
Q6. $x^2 - 6x + 9 = (x-3)^2 = 0 \Rightarrow x = 3$ (repeated); the parabola touches the axis at $x = 3$ and is positive everywhere else. $\therefore \{x : x \neq 3\}$.



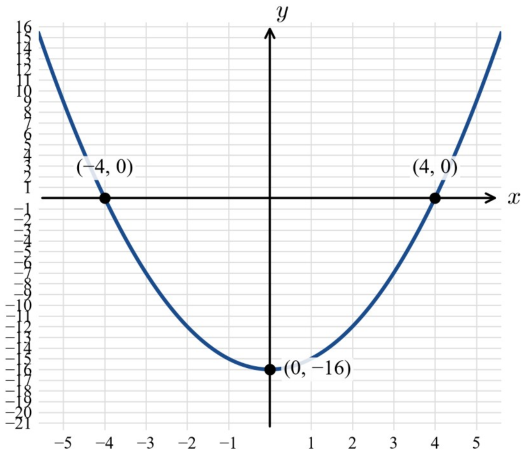
Q7. $x^2 - x - 2 = (x-2)(x+1) = 0 \Rightarrow x = -1$ or 2; upright, want $y > 0$ (outside). $\therefore \{x : x < -1\} \cup \{x : x > 2\}$.



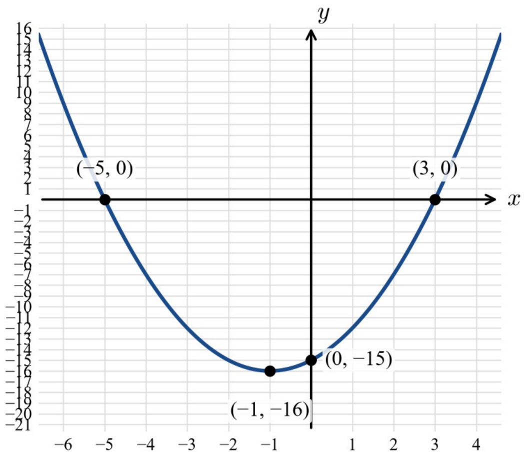
Q8. $x^2 + 5x + 6 = (x+2)(x+3) = 0 \Rightarrow x = -3$ or -2; upright, want $y \le 0$ (between, inclusive). $\therefore \{x : -3 \le x \le -2\}$.



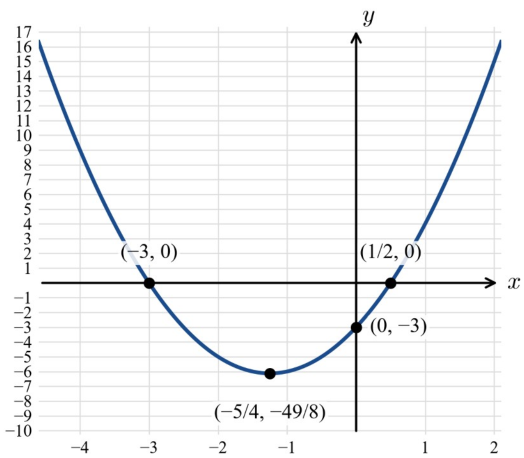
Q9. $x^2 - 16 = (x-4)(x+4) = 0 \Rightarrow x = \pm 4$; upright, want $y < 0$ (between). $\therefore \{x : -4 < x < 4\}$.



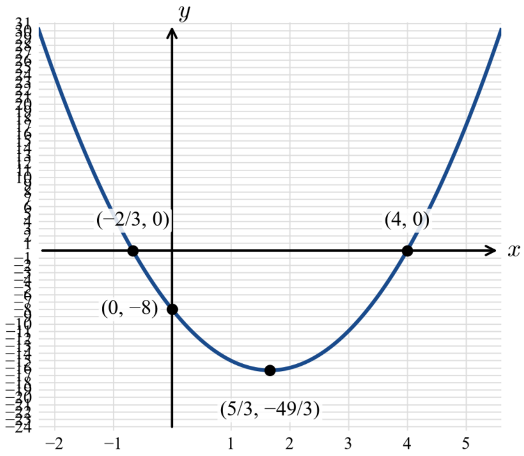
Q10. $x^2+2x-15=(x+5)(x-3)=0 \Rightarrow x=-5$ or 3; upright, want $y \geq 0$ (outside, inclusive).
$\therefore \{x : x \leq -5\} \cup \{x : x \geq 3\}$.



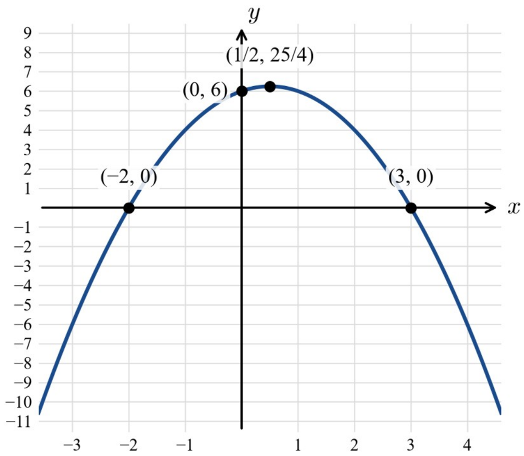
Q11. $2x^2+5x-3=(2x-1)(x+3)=0 \Rightarrow x=-3$ or $x=1/2$; upright, want $y>0$ (outside).
$\therefore \{x : x<-3\} \cup \{x : x>1/2\}$.



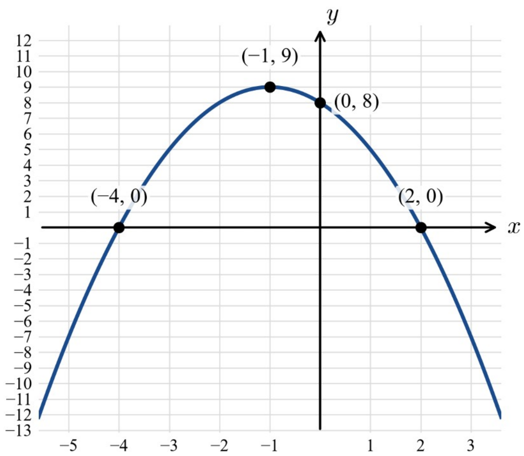
Q12. $3x^2-10x-8=(3x+2)(x-4)=0 \Rightarrow x=-2/3$ or 4; upright, want $y \leq 0$ (between, inclusive).
$\therefore \{x : -2/3 \leq x \leq 4\}$.



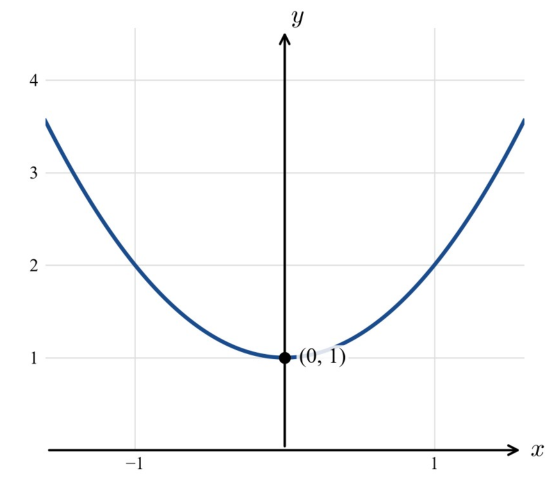
Q13. $-x^2+x+6>0$. Multiply by -1 and reverse: $x^2-x-6<0 \Rightarrow (x-3)(x+2)<0$, intercepts -2 and 3; want the between region. $\therefore \{x:-2<x<3\}$.



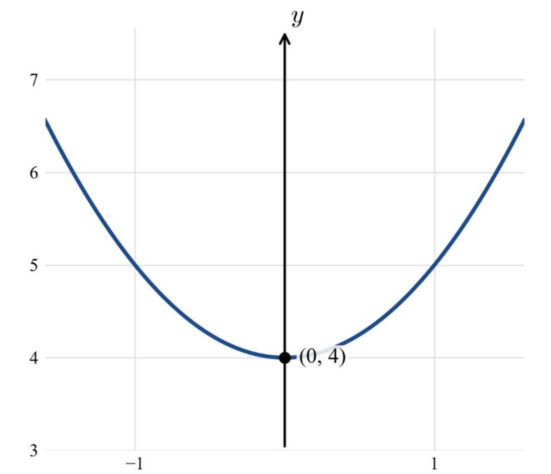
Q14. $-x^2-2x+8\leq 0$. Multiply by -1 and reverse: $x^2+2x-8\geq 0 \Rightarrow (x+4)(x-2)\geq 0$, intercepts -4 and 2; want the outside region (inclusive). $\therefore \{x:x\leq -4\}\cup\{x:x\geq 2\}$.



Q15. $\Delta=0^2-4(1)(1)=-4<0$ and $a>0$, so the parabola lies entirely above the x-axis. $\therefore x^2+1>0$ for **all real** x.

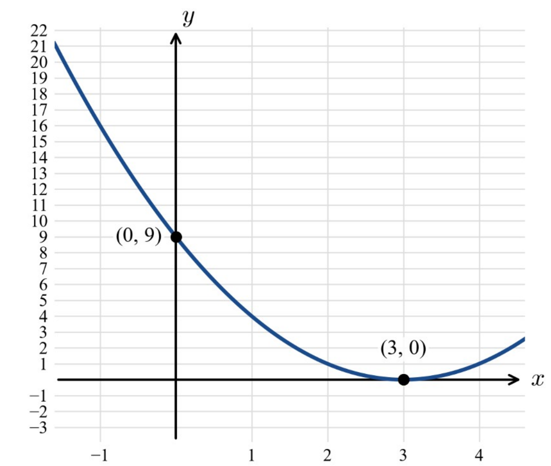


Q16. As in Q15, x^2+4 is always positive, so it can never be negative. $\therefore$ **no solution**.

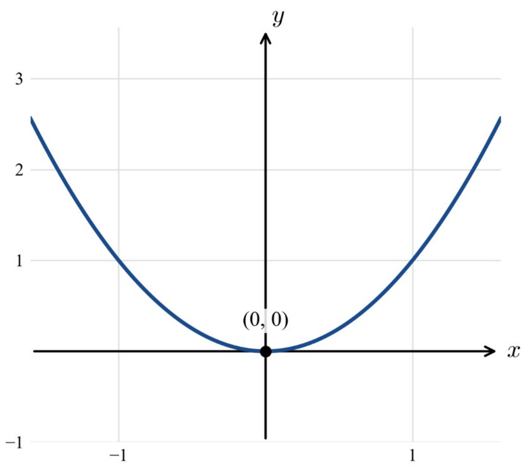


x

Q17. $x^2-6x+9=(x-3)^2\geq 0$ for every value of x (a square is never negative). $\therefore$ **all real** x.



Q18. $x^2\geq 0$ for every x, so $x^2<0$ is never true. $\therefore$ **no solution**.



y
3
2
1
(0, 0)
x
−1
−1
1

Theory

Not every quadratic factorises neatly. The **general quadratic formula** solves *any* quadratic equation. For $a x^2 + b x + c = 0$ (with $a \neq 0$),

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When to use it. Use the formula when factorising is difficult or impossible. First rearrange the equation into the form $a x^2 + b x + c = 0$, then carefully identify a, b and c *including their signs*.

The discriminant. The quantity under the square root, $\Delta = b^2 - 4ac$, is the **discriminant**. It tells you how many real solutions there are:

- $\Delta > 0$: two distinct real solutions;
- $\Delta = 0$: one (repeated) real solution, $x = -\frac{b}{2a}$;
- $\Delta < 0$: no real solutions (the square root of a negative number is not real).

Exact answers and surds. Unless a decimal approximation is asked for, leave answers in **exact** (surd) form. Simplify surds where possible — for example $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$ — and then cancel any common factor between the numerator and denominator.

Worked examples

Example 1 — scaffolded

Solve $x^2 + 5x + 3 = 0$, giving exact answers.

Working	Comment
$a = 1$, $b = 5$, $c = 3$.	Compare with $a x^2 + b x + c = 0$.
$\Delta = b^2 - 4ac = 5^2 - 4(1)(3) = 25 - 12 = 13$.	$\Delta > 0$, so there are two real solutions.
$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{13}}{2 \times 1}$.	Substitute into the formula.
$\therefore x = \frac{-5 + \sqrt{13}}{2}$ or $x = \frac{-5 - \sqrt{13}}{2}$.	$\sqrt{13}$ does not simplify — leave the answer in exact form.

Example 2 — scaffolded

Solve $2x^2 - 4x - 3 = 0$, giving exact answers.

Working	Comment
$a = 2$, $b = -4$, $c = -3$.	Take care with the negative signs.
$\Delta = (-4)^2 - 4(2)(-3) = 16 + 24 = 40$.	Compute the discriminant.
$x = \frac{-(-4) \pm \sqrt{40}}{2 \times 2} = \frac{4 \pm \sqrt{40}}{4}$.	Substitute; $-(-4) = 4$.
$\sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}$, so $x = \frac{4 \pm 2\sqrt{10}}{4}$.	Simplify the surd.

Working	Comment
$\therefore x=\dfrac{2\pm\sqrt{10}}{2}$.	Divide the numerator and denominator by 2.

Example 3 — unscaffolded

Solve $3x^2-2x-4=0$, giving exact answers.

$a=3$, $b=-2$, $c=-4$. $\Delta=(-2)^2-4(3)(-4)=4+48=52$.

$$x=\frac{-(-2)\pm\sqrt{52}}{2\times3}=\frac{2\pm\sqrt{52}}{6}.$$

Since $\sqrt{52}=\sqrt{4\times13}=2\sqrt{13}$, $x=\dfrac{2\pm2\sqrt{13}}{6}$.

$\therefore x=\dfrac{1+\sqrt{13}}{3}$ or $x=\dfrac{1-\sqrt{13}}{3}$.

Example 4 — unscaffolded

Solve $x^2=6x-7$, giving exact answers.

Rearrange into $ax^2+bx+c=0$: $x^2-6x+7=0$, so $a=1$, $b=-6$, $c=7$.

$\Delta=(-6)^2-4(1)(7)=36-28=8$.

$$x=\frac{-(-6)\pm\sqrt{8}}{2\times1}=\frac{6\pm\sqrt{8}}{2}.$$

Since $\sqrt{8}=2\sqrt{2}$, $x=\dfrac{6\pm2\sqrt{2}}{2}=3\pm\sqrt{2}$.

$\therefore x=3+\sqrt{2}$ or $x=3-\sqrt{2}$.

Example 5 — unscaffolded

Verify by substitution that $x=3+\sqrt{2}$ is a solution of $x^2-6x+7=0$.

First expand the square: $(3+\sqrt{2})^2=3^2+2(3)(\sqrt{2})+(\sqrt{2})^2=9+6\sqrt{2}+2=11+6\sqrt{2}$.

Substitute $x=3+\sqrt{2}$ into the left-hand side:

$$x^2-6x+7=(11+6\sqrt{2})-6(3+\sqrt{2})+7.$$

Expand and collect the terms: $11+6\sqrt{2}-18-6\sqrt{2}+7=(11-18+7)+(6\sqrt{2}-6\sqrt{2})=0$.

The left-hand side equals 0, so $x=3+\sqrt{2}$ satisfies the equation, as required. (The conjugate $x=3-\sqrt{2}$ checks the same way.)

Practice questions

Scaffolded

Q1. Solve $x^2+3x+1=0$. (a) State a, b and c. (b) Find Δ. (c) Use the formula to find the exact solutions.

Q2. Solve $x^2-5x+5=0$. (a) Find Δ. (b) Hence find the exact solutions.

Q3. Solve $2x^2+3x-1=0$, giving exact answers.

Q4. Solve $x^2-4x-1=0$, giving exact answers. (Simplify the surd.)

Q5. Solve $3x^2-5x+1=0$, giving exact answers.

Q6. Solve $x^2+2x-6=0$, giving exact answers. (Simplify the surd.)

Unscaffolded

Solve each of the following using the quadratic formula, giving exact answers and simplifying any surds.

Q7. $x^2-7x+12=0$

Q8. $2x^2-5x-3=0$

Q9. $x^2-6x+4=0$

Q10. $2x^2-8x+3=0$

Q11. $x^2+3x-5=0$

Q12. $x^2-x-1=0$

Q13. $9x^2-6x+1=0$

Q14. $x^2+2x+5=0$

Q15. $x^2=4x+6$

Q16. $2x^2+1=6x$

Q17. $3x^2+4x-2=0$

Q18. $(x-2)(x+1)=1$

Q19. A solution of a quadratic is $x=1+\sqrt{2}$. By substituting and expanding by hand, find the exact value of x^2-2x.

Answers

Q1. $x=\frac{-3\pm\sqrt{5}}{2}$. **Q2.** $x=\frac{5\pm\sqrt{5}}{2}$. **Q3.** $x=\frac{-3\pm\sqrt{17}}{4}$. **Q4.** $x=2\pm\sqrt{5}$. **Q5.** $x=\frac{5\pm\sqrt{13}}{6}$. **Q6.** $x=-1\pm\sqrt{7}$. **Q7.** $x=3$ or $x=4$. **Q8.** $x=3$ or $x=-\frac{1}{2}$. **Q9.** $x=3\pm\sqrt{5}$. **Q10.** $x=\frac{4\pm\sqrt{10}}{2}$. **Q11.** $x=\frac{-3\pm\sqrt{29}}{2}$. **Q12.** $x=\frac{1\pm\sqrt{5}}{2}$. **Q13.** $x=\frac{1}{3}$ (repeated). **Q14.** No real solutions ($\Delta=-16<0$). **Q15.** $x=2\pm\sqrt{10}$. **Q16.** $x=\frac{3\pm\sqrt{7}}{2}$. **Q17.** $x=\frac{-2\pm\sqrt{10}}{3}$. **Q18.** $x=\frac{1\pm\sqrt{13}}{2}$. **Q19.** $x^2-2x=1$.

Fully-worked solutions

Q1. $x^2+3x+1=0$. (a) $a=1$, $b=3$, $c=1$. (b) $\Delta=3^2-4(1)(1)=9-4=5$. (c) $x=\frac{-3\pm\sqrt{5}}{2}$.

Q2. $x^2-5x+5=0$. $\Delta=(-5)^2-4(1)(5)=25-20=5$, so $x=\frac{-(-5)\pm\sqrt{5}}{2}=\frac{5\pm\sqrt{5}}{2}$.

Q3. $2x^2+3x-1=0$. $\Delta=3^2-4(2)(-1)=9+8=17$, so $x=\frac{-3\pm\sqrt{17}}{2\times 2}=\frac{-3\pm\sqrt{17}}{4}$.

Q4. $x^2-4x-1=0$. $\Delta=(-4)^2-4(1)(-1)=16+4=20$, so $x=\frac{4\pm\sqrt{20}}{2}$. Since $\sqrt{20}=2\sqrt{5}$, $x=\frac{4\pm 2\sqrt{5}}{2}=2\pm\sqrt{5}$.

Q5. $3x^2-5x+1=0$. $\Delta=(-5)^2-4(3)(1)=25-12=13$, so $x=\frac{5\pm\sqrt{13}}{2\times 3}=\frac{5\pm\sqrt{13}}{6}$.

Q6. $x^2+2x-6=0$. $\Delta=2^2-4(1)(-6)=4+24=28$, so $x=\frac{-2\pm\sqrt{28}}{2}$. Since $\sqrt{28}=2\sqrt{7}$, $x=\frac{-2\pm 2\sqrt{7}}{2}=-1\pm\sqrt{7}$.

Q7. $x^2-7x+12=0$. $\Delta=(-7)^2-4(1)(12)=49-48=1$, so $x=\frac{7\pm\sqrt{1}}{2}=\frac{7\pm 1}{2}$, giving $x=4$ or $x=3$.

Q8. $2x^2-5x-3=0$. $\Delta=(-5)^2-4(2)(-3)=25+24=49$, so $x=\frac{5\pm\sqrt{49}}{4}=\frac{5\pm 7}{4}$, giving $x=3$ or $x=-\frac{1}{2}$.

Q9. $x^2-6x+4=0$. $\Delta=(-6)^2-4(1)(4)=36-16=20$, so $x=\frac{6\pm\sqrt{20}}{2}=\frac{6\pm 2\sqrt{5}}{2}=3\pm\sqrt{5}$.

Q10. $2x^2-8x+3=0$. $\Delta=(-8)^2-4(2)(3)=64-24=40$, so $x=\frac{8\pm\sqrt{40}}{4}=\frac{8\pm 2\sqrt{10}}{4}=\frac{4\pm\sqrt{10}}{2}$.

Q11. $x^2+3x-5=0$. $\Delta=3^2-4(1)(-5)=9+20=29$, so $x=\frac{-3\pm\sqrt{29}}{2}$ ($\sqrt{29}$ does not simplify).

Q12. $x^2-x-1=0$. $\Delta=(-1)^2-4(1)(-1)=1+4=5$, so $x=\frac{1\pm\sqrt{5}}{2}$.

Q13. $9x^2-6x+1=0$. $\Delta=(-6)^2-4(9)(1)=36-36=0$, so there is one repeated solution $x=\frac{6}{2\times 9}=\frac{1}{3}$.

Q14. $x^2+2x+5=0$. $\Delta=2^2-4(1)(5)=4-20=-16<0$. $\therefore$ there are **no real solutions**.

Q15. $x^2=4x+6 \Rightarrow x^2-4x-6=0$. $\Delta=(-4)^2-4(1)(-6)=16+24=40$, so $x=\frac{4\pm\sqrt{40}}{2}=\frac{4\pm2\sqrt{10}}{2}=2\pm\sqrt{10}$.

Q16. $2x^2+1=6x \Rightarrow 2x^2-6x+1=0$. $\Delta=(-6)^2-4(2)(1)=36-8=28$, so $x=\frac{6\pm\sqrt{28}}{4}=\frac{6\pm2\sqrt{7}}{4}=\frac{3\pm\sqrt{7}}{2}$.

Q17. $3x^2+4x-2=0$. $\Delta=4^2-4(3)(-2)=16+24=40$, so $x=\frac{-4\pm\sqrt{40}}{6}=\frac{-4\pm2\sqrt{10}}{6}=\frac{-2\pm\sqrt{10}}{3}$.

Q18. $(x-2)(x+1)=1 \Rightarrow x^2-x-2=1 \Rightarrow x^2-x-3=0$. $\Delta=(-1)^2-4(1)(-3)=1+12=13$, so $x=\frac{1\pm\sqrt{13}}{2}$.

Q19. $x=1+\sqrt{2}$. Expand the square first: $x^2=(1+\sqrt{2})^2=1^2+2(1)(\sqrt{2})+(\sqrt{2})^2=1+2\sqrt{2}+2=3+2\sqrt{2}$. Then substitute into x^2-2x: $x^2-2x=(3+2\sqrt{2})-2(1+\sqrt{2})=3+2\sqrt{2}-2-2\sqrt{2}=1$. The surd terms cancel, leaving the exact value $x^2-2x=1$.

Theory

For the quadratic equation $ax^2+bx+c=0$ (with $a\neq 0$), the expression under the square root in the quadratic formula is called the **discriminant**, written with a capital delta:

$$\Delta=b^2-4ac.$$

The discriminant tells us the **number and nature of the solutions** without solving the equation:

- $\Delta>0$: **two distinct real solutions** (the parabola crosses the x-axis twice).
- $\Delta=0$: **one repeated real solution** (the parabola touches the x-axis at its turning point).
- $\Delta<0$: **no real solutions** (the parabola does not meet the x-axis).

When $\Delta>0$, the *type* of solution depends on whether Δ is a perfect square:

- Δ a **perfect square** $(1,4,9,16,\ldots)$: the solutions are **rational**, so the quadratic factorises over the rational numbers.
- Δ **positive but not a perfect square**: the solutions are **irrational** (surds).

Using the discriminant to find parameter values. Because Δ controls how many solutions occur, an equation involving an unknown coefficient can be analysed by writing Δ in terms of that unknown and applying the appropriate condition ($\Delta=0$, $\Delta>0$ or $\Delta<0$). A related application: a line meets a parabola at exactly one point precisely when the quadratic formed by equating them has $\Delta=0$.

Worked examples

Example 1 — scaffolded

Use the discriminant to determine the number and nature of the solutions of $2x^2-5x+1=0$.

Working	Comment
Here $a=2$, $b=-5$, $c=1$.	Identify the coefficients.
$\Delta=b^2-4ac=(-5)^2-4(2)(1)=25-8=17$.	Substitute into $\Delta=b^2-4ac$.
$\Delta=17>0$, so there are **two distinct real solutions**.	Sign of Δ gives the number of solutions.
17 is not a perfect square, so the solutions are **irrational**.	A non-square positive Δ means surd solutions.

Example 2 — scaffolded

Find the value of k for which $x^2+6x+k=0$ has exactly one solution.

Working	Comment
Here $a=1$, $b=6$, $c=k$.	Identify the coefficients; c is the unknown.
$\Delta=6^2-4(1)(k)=36-4k$.	Write the discriminant in terms of k.
Exactly one solution requires $\Delta=0$: $36-4k=0$.	One repeated solution $\Leftrightarrow$ $\Delta=0$.
$\therefore k=9$.	Solve for k.

Example 3 — unscaffolded

Show that $x^2+2x+5=0$ has no real solutions.

$\Delta=2^2-4(1)(5)=4-20=-16$.

Since $\Delta = -16 < 0$, the equation has **no real solutions**.

$\therefore$ as $a = 1 > 0$ and the parabola never meets the x-axis, $x^2 + 2x + 5 > 0$ for all real x.

Example 4 — unscaffolded

For what values of m does $x^2 + 5x + m = 0$ have no real solutions?

$\Delta = 5^2 - 4(1)(m) = 25 - 4m$.

No real solutions requires $\Delta < 0$: $25 - 4m < 0 \Rightarrow 4m > 25$.

$\therefore m > \frac{25}{4}$.

Practice questions

Scaffolded

Q1. For $x^2 - 7x + 10 = 0$: (a) find the discriminant Δ; (b) state the number and nature of the solutions.

Q2. For $3x^2 - 6x + 3 = 0$: (a) find Δ; (b) state the number of solutions.

Q3. For $2x^2 + x + 3 = 0$: (a) find Δ; (b) state the number of solutions.

Q4. For $x^2 + 6x + 7 = 0$: (a) find Δ; (b) state the number and nature of the solutions.

Q5. Find the value(s) of k for which $x^2 + kx + 16 = 0$ has exactly one solution.

Q6. Find the values of p for which $x^2 + 4x + p = 0$ has two distinct real solutions.

Unscaffolded

For **Q7–Q12**, find the discriminant and state the number and nature of the solutions.

Q7. $x^2 - 3x - 10 = 0$

Q8. $x^2 - 6x + 9 = 0$

Q9. $x^2 + x + 1 = 0$

Q10. $2x^2 - 7x + 3 = 0$

Q11. $x^2 - 2x - 4 = 0$

Q12. $4x^2 - 12x + 9 = 0$

Q13. Find the value(s) of k for which $x^2 + kx + 9 = 0$ has exactly one solution.

Q14. Find the values of m for which $x^2 - 4x + m = 0$ has two distinct real solutions.

Q15. Find the values of p ($p \neq 0$) for which $px^2 + 3x + 1 = 0$ has no real solutions.

Q16. Show that $x^2 - 2x + 5 > 0$ for all real values of x.

Q17. Find the values of k for which $x^2 + kx + 4 = 0$ has two distinct real solutions.

Q18. Find the value of k for which the line $y = x + k$ touches the parabola $y = x^2$ (that is, meets it at exactly one point).

Answers

Q1. $\Delta=9$; two distinct rational solutions. **Q2.** $\Delta=0$; one repeated solution. **Q3.** $\Delta=-23$; no real solutions. **Q4.** $\Delta=8$; two distinct irrational solutions. **Q5.** $k=\pm 8$. **Q6.** $p<4$. **Q7.** $\Delta=49$; two distinct rational. **Q8.** $\Delta=0$; one repeated. **Q9.** $\Delta=-3$; no real solutions. **Q10.** $\Delta=25$; two distinct rational. **Q11.** $\Delta=20$; two distinct irrational. **Q12.** $\Delta=0$; one repeated. **Q13.** $k=\pm 6$. **Q14.** $m<4$. **Q15.** $p>\frac{9}{4}$. **Q16.** $\Delta=-16<0$ and $a>0$, so always positive. **Q17.** $k<-4$ or $k>4$. **Q18.** $k=-\frac{1}{4}$.

Fully-worked solutions

Q1. $\Delta=(-7)^2-4(1)(10)=49-40=9$. Since $\Delta=9>0$ and 9 is a perfect square, there are **two distinct rational solutions** (indeed $x^2-7x+10=(x-2)(x-5)$).

Q2. $\Delta=(-6)^2-4(3)(3)=36-36=0$. Since $\Delta=0$, there is **one repeated solution**.

Q3. $\Delta=1^2-4(2)(3)=1-24=-23$. Since $\Delta=-23<0$, there are **no real solutions**.

Q4. $\Delta=6^2-4(1)(7)=36-28=8$. Since $\Delta=8>0$ but 8 is not a perfect square, there are **two distinct irrational solutions** (namely $x=-3\pm\sqrt{2}$).

Q5. Here $\Delta=k^2-4(1)(16)=k^2-64$. Exactly one solution requires $\Delta=0$: $k^2-64=0 \Rightarrow k^2=64$. $\therefore k=\pm 8$.

Q6. Here $\Delta=4^2-4(1)(p)=16-4p$. Two distinct solutions require $\Delta>0$: $16-4p>0 \Rightarrow 4p<16$. $\therefore p<4$.

Q7. $\Delta=(-3)^2-4(1)(-10)=9+40=49$. As $\Delta=49>0$ is a perfect square, **two distinct rational solutions** ($x^2-3x-10=(x-5)(x+2)$).

Q8. $\Delta=(-6)^2-4(1)(9)=36-36=0$. **One repeated solution** ($x^2-6x+9=(x-3)^2$).

Q9. $\Delta=1^2-4(1)(1)=1-4=-3<0$. **No real solutions.**

Q10. $\Delta=(-7)^2-4(2)(3)=49-24=25$. As $\Delta=25>0$ is a perfect square, **two distinct rational solutions** ($2x^2-7x+3=(2x-1)(x-3)$, so $x=1/2$ or $x=3$).

Q11. $\Delta=(-2)^2-4(1)(-4)=4+16=20$. As $\Delta=20>0$ is not a perfect square, **two distinct irrational solutions** ($x=1\pm\sqrt{5}$).

Q12. $\Delta=(-12)^2-4(4)(9)=144-144=0$. **One repeated solution** ($4x^2-12x+9=(2x-3)^2$).

Q13. $\Delta=k^2-4(1)(9)=k^2-36$. One solution requires $\Delta=0$: $k^2=36$. $\therefore k=\pm 6$.

Q14. $\Delta=(-4)^2-4(1)(m)=16-4m$. Two distinct solutions require $\Delta>0$: $16-4m>0 \Rightarrow 4m<16$. $\therefore m<4$.

Q15. With $a=p$, $b=3$, $c=1$: $\Delta=3^2-4(p)(1)=9-4p$. No real solutions requires $\Delta<0$: $9-4p<0 \Rightarrow 4p>9$. $\therefore p>\frac{9}{4}$ (which also satisfies $p\neq 0$).

Q16. For $y=x^2-2x+5$, $\Delta=(-2)^2-4(1)(5)=4-20=-16<0$, so the graph has no x-intercepts. Since $a=1>0$ the parabola opens upwards and therefore lies entirely above the x-axis. $\therefore x^2-2x+5>0$ for all real x.

Q17. $\Delta=k^2-4(1)(4)=k^2-16$. Two distinct solutions require $\Delta>0$: $k^2-16>0 \Rightarrow k^2>16$. $\therefore k<-4$ or $k>4$.

Q18. The line touches the parabola where $x^2 = x + k$, i.e. $x^2 - x - k = 0$. This has exactly one solution when $\Delta = 0$: $\Delta = (-1)^2 - 4(1)(-k) = 1 + 4k$, so $1 + 4k = 0$. $\therefore k = -\frac{1}{4}$.

Theory

To solve a **linear** equation and a **quadratic** equation simultaneously means finding the point(s) (x, y) that satisfy both — geometrically, the point(s) where the **line** and the **parabola** intersect.

Method (substitution).

1. Make y the subject of each equation (if it is not already).
2. Set the two expressions for y equal. This gives a **quadratic equation in** x.
3. Rearrange to the form $ax^2+bx+c=0$ and solve for x.
4. Substitute each x-value back into the **linear** equation (it is simpler) to find y.
5. Each (x, y) pair is a point of intersection.

Number of solutions. After substitution, the discriminant $\Delta=b^2-4ac$ of the resulting quadratic tells you how the line and parabola meet:

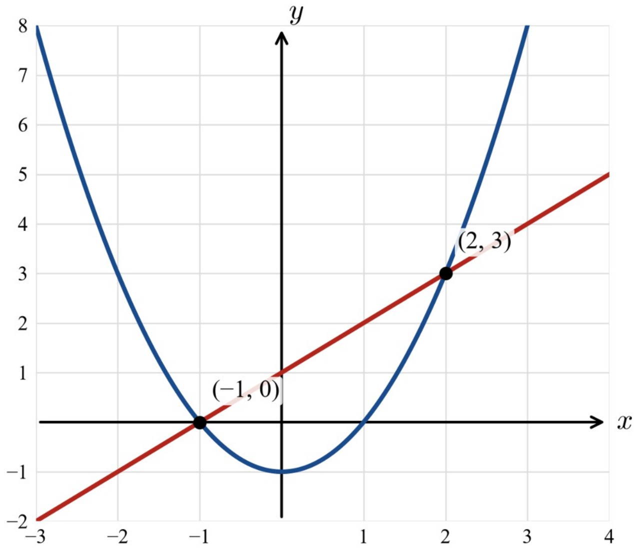
- $\Delta>0$: **two** points — the line cuts the parabola.
- $\Delta=0$: **one** point — the line is a **tangent** to the parabola.
- $\Delta<0$: **no** points — the line does not meet the parabola.

Worked examples

Example 1 — scaffolded

Solve $y=x+1$ and $y=x^2-1$ simultaneously, and find the points of intersection.

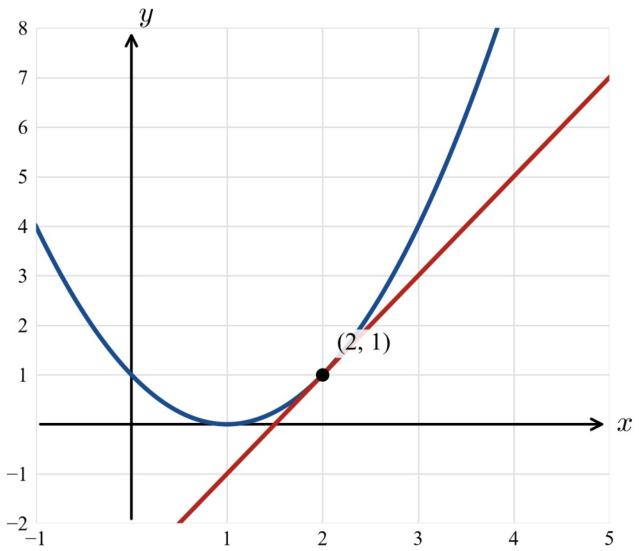
Working	Comment
$x^2-1=x+1$	Both expressions equal y, so set them equal.
$x^2-x-2=0$	Rearrange to a quadratic $=0$.
$(x-2)(x+1)=0 \Rightarrow x=2$ or $x=-1$	Factorise; apply the null factor law.
$x=2$: $y=2+1=3$; $x=-1$: $y=-1+1=0$	Substitute into the line $y=x+1$.
Points of intersection $(2,3)$ and $(-1,0)$.	$\Delta=9>0$: two points.



Example 2 — scaffolded

Solve $y=2x-3$ and $y=x^2-2x+1$ simultaneously, and describe how the line and parabola meet.

Working	Comment
$x^2-2x+1=2x-3$	Set the expressions for y equal.
$x^2-4x+4=0$	Rearrange to a quadratic $=0$.
$(x-2)^2=0 \Rightarrow x=2$	A repeated root ($\Delta=0$).
$y=2(2)-3=1$	Substitute into the line.
One point $(2,1)$; since $\Delta=0$ the line is a **tangent** to the parabola.	$\Delta=0 \Rightarrow$ tangent.

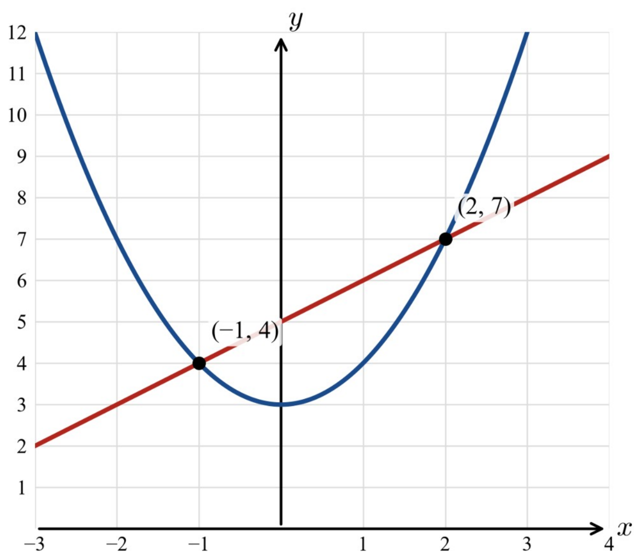


Example 3 — unscaffolded

Solve $y=x+5$ and $y=x^2+3$ simultaneously.

$$x^2+3=x+5 \Rightarrow x^2-x-2=0 \Rightarrow (x-2)(x+1)=0 \Rightarrow x=2 \text{ or } x=-1.$$

Substituting into $y=x+5$: $x=2 \Rightarrow y=7$; $x=-1 \Rightarrow y=4$. $\therefore$ the points of intersection are $(2,7)$ and $(-1,4)$.

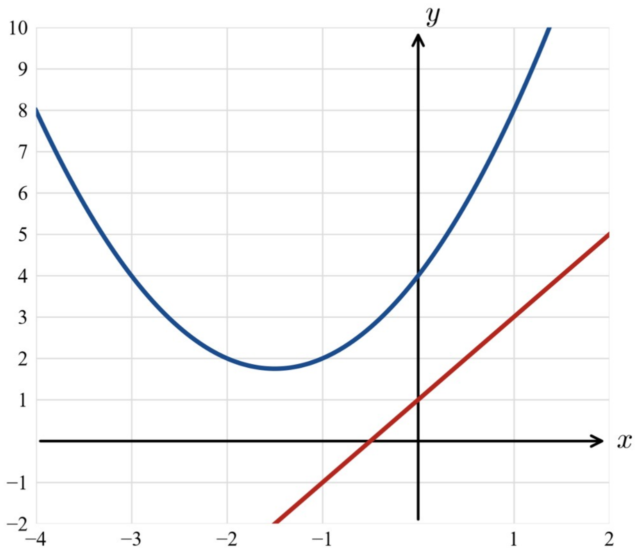


Example 4 — unscaffolded

Show that the line $y=2x+1$ does not meet the parabola $y=x^2+3x+4$.

$$x^2+3x+4=2x+1 \Rightarrow x^2+x+3=0.$$

$\Delta=1^2-4(1)(3)=1-12=-11<0$. There are no real solutions, $\therefore$ the line does not intersect the parabola.



Practice questions

Scaffolded

For each pair: (a) form the quadratic in x; (b) solve for x; (c) find the corresponding y-values; (d) state the point(s) of intersection (and say if the line is a tangent).

Q1. $y=x+3$ and $y=x^2+1$.

Q2. $y=-x+4$ and $y=x^2-2$.

Q3. $y=4x-4$ and $y=x^2$.

Q4. $y=2x$ and $y=x^2-3$.

Q5. $y=x-6$ and $y=x^2-4x$.

Q6. $y=-2x-1$ and $y=x^2$.

Unscaffolded

Solve each pair simultaneously and state the point(s) of intersection (or show there are none). Where the line is a tangent, say so.

Q7. $y=x+2$ and $y=x^2$.

Q8. $y=3x-2$ and $y=x^2$.

Q9. $y=2x+3$ and $y=x^2$.

Q10. $y=x-1$ and $y=x^2-3x+2$.

Q11. $y=-x+1$ and $y=x^2+2x-3$.

Q12. $y=6x-9$ and $y=x^2$.

Q13. $y=-4x-4$ and $y=x^2$.

Q14. $y = x - 4$ and $y = x^2$.

Q15. $y = 2x - 5$ and $y = x^2 + 1$.

Q16. $y = x + 4$ and $y = x^2 - 2x$.

Q17. Find the value of k for which the line $y = x + k$ is a tangent to the parabola $y = x^2$.

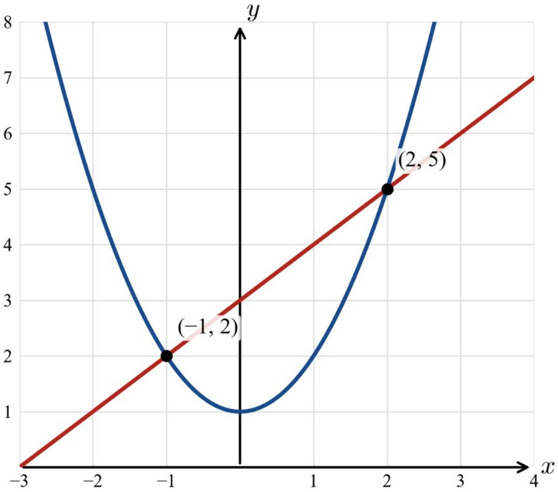
Q18. For what values of k does the line $y = x + k$ meet the parabola $y = x^2$ at two distinct points?

Answers

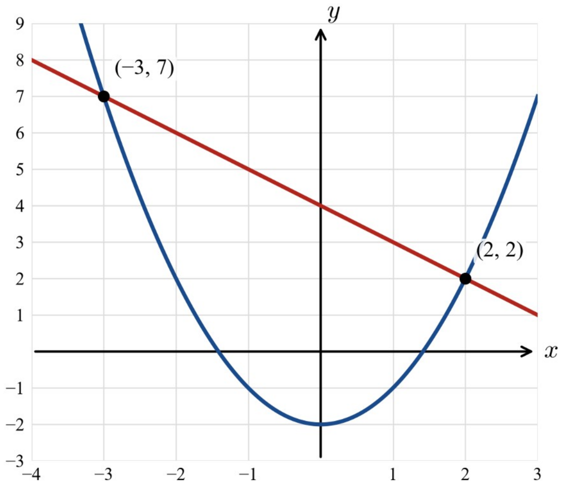
Q1. $(-1,2),(2,5)$. **Q2.** $(-3,7),(2,2)$. **Q3.** tangent at $(2,4)$. **Q4.** $(-1,-2),(3,6)$. **Q5.** $(2,-4),(3,-3)$. **Q6.** tangent at $(-1,1)$. **Q7.** $(-1,1),(2,4)$. **Q8.** $(1,1),(2,4)$. **Q9.** $(-1,1),(3,9)$. **Q10.** $(1,0),(3,2)$. **Q11.** $(-4,5),(1,0)$. **Q12.** tangent at $(3,9)$. **Q13.** tangent at $(-2,4)$. **Q14.** no intersection ($\Delta=-15$). **Q15.** no intersection ($\Delta=-20$). **Q16.** $(-1,3),(4,8)$. **Q17.** $k=-1/4$. **Q18.** $k>-1/4$.

Fully-worked solutions

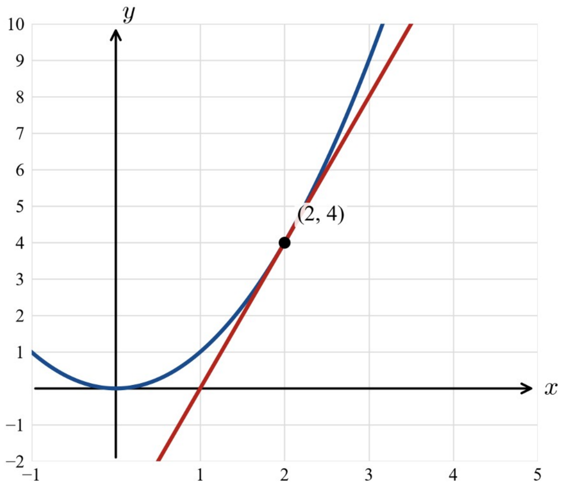
Q1. $x^2+1=x+3 \Rightarrow x^2-x-2=0 \Rightarrow (x-2)(x+1)=0 \Rightarrow x=2$ or -1. Into $y=x+3$: $(2,5)$ and $(-1,2)$.



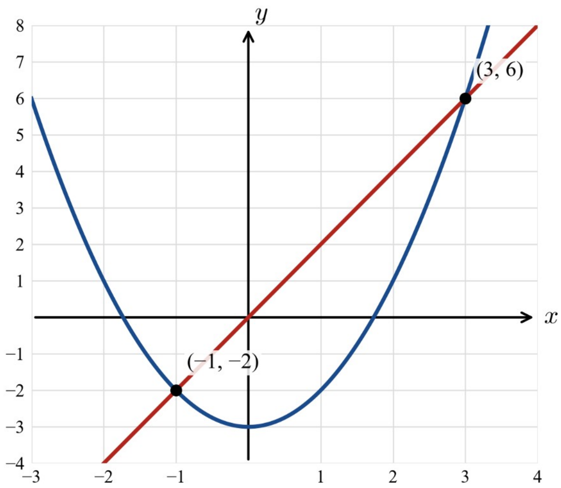
Q2. $x^2-2=-x+4 \Rightarrow x^2+x-6=0 \Rightarrow (x+3)(x-2)=0 \Rightarrow x=-3$ or 2. Into $y=-x+4$: $(-3,7)$ and $(2,2)$.



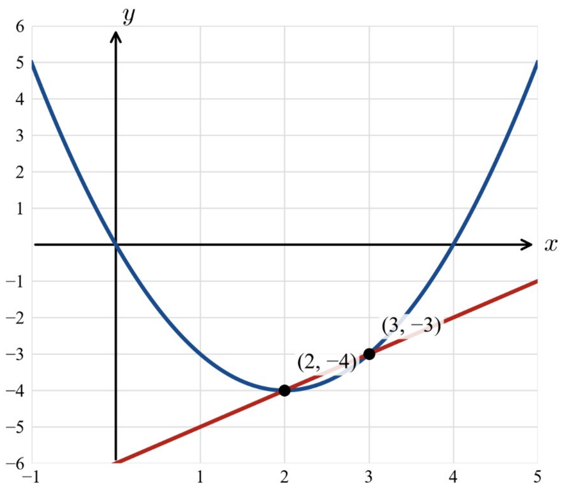
Q3. $x^2=4x-4 \Rightarrow x^2-4x+4=0 \Rightarrow (x-2)^2=0 \Rightarrow x=2$. Into $y=4x-4$: $y=4$. $\Delta=0$, $\therefore$ the line is a **tangent** at $(2,4)$.



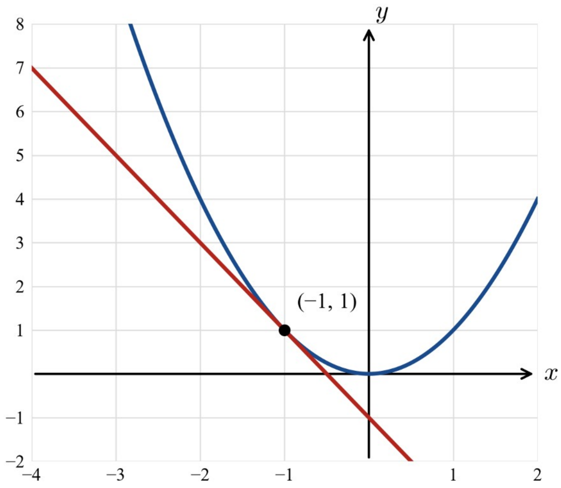
Q4. $x^2-3=2x \Rightarrow x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0 \Rightarrow x=3$ or -1. Into $y=2x$: $(3,6)$ and $(-1,-2)$.



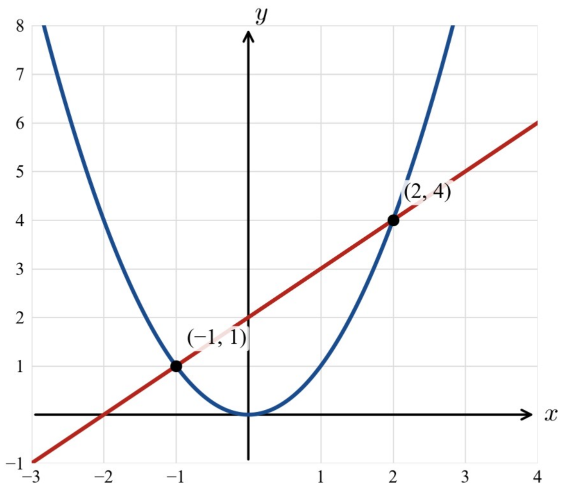
Q5. $x^2-4x=x-6 \Rightarrow x^2-5x+6=0 \Rightarrow (x-2)(x-3)=0 \Rightarrow x=2$ or 3. Into $y=x-6$: $(2,-4)$ and $(3,-3)$.



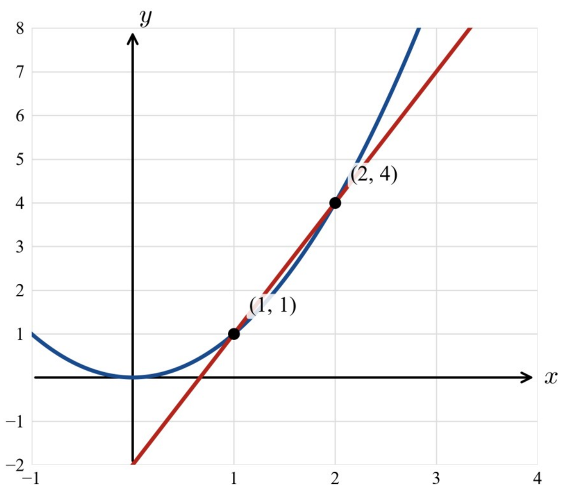
Q6. $x^2=-2x-1 \Rightarrow x^2+2x+1=0 \Rightarrow (x+1)^2=0 \Rightarrow x=-1$. Into $y=-2x-1$: $y=1$. $\Delta=0$, $\therefore$ the line is a **tangent** at $(-1,1)$.



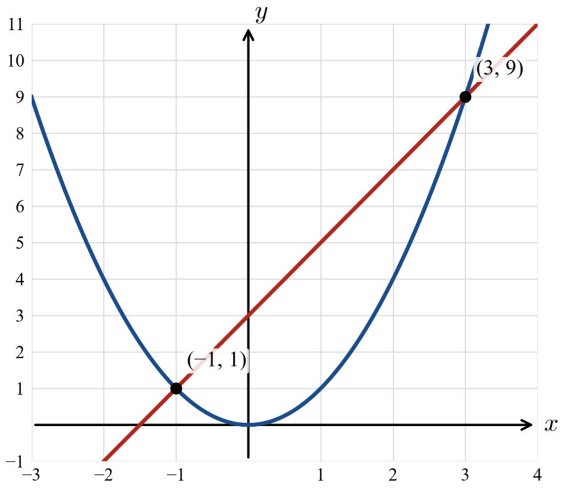
Q7. $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2 \text{ or } -1$. Into $y = x + 2$: $(2, 4)$ and $(-1, 1)$.



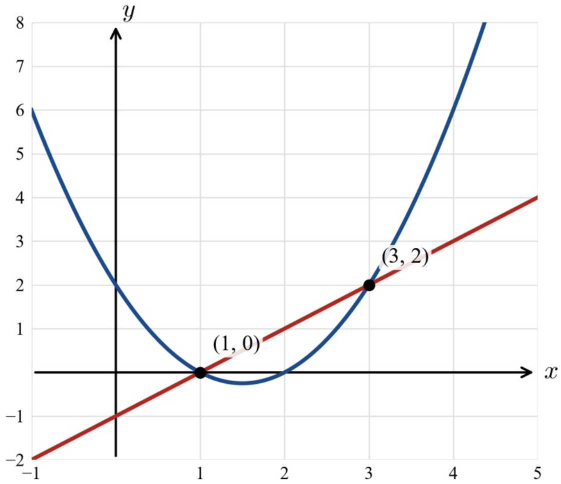
Q8. $x^2 = 3x - 2 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x = 1 \text{ or } 2$. Into $y = 3x - 2$: $(1, 1)$ and $(2, 4)$.



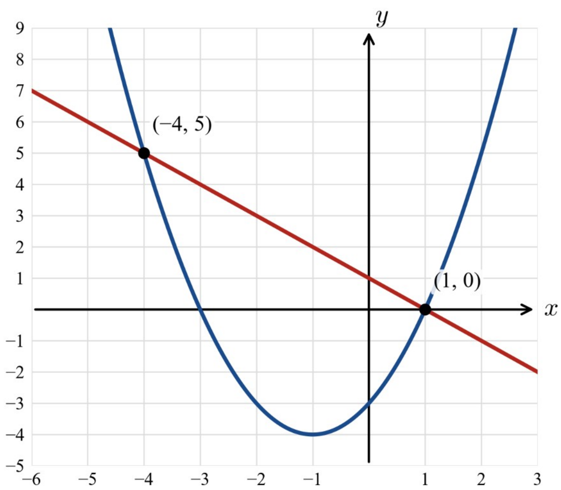
Q9. $x^2 = 2x + 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3 \text{ or } -1$. Into $y = 2x + 3$: $(3, 9)$ and $(-1, 1)$.



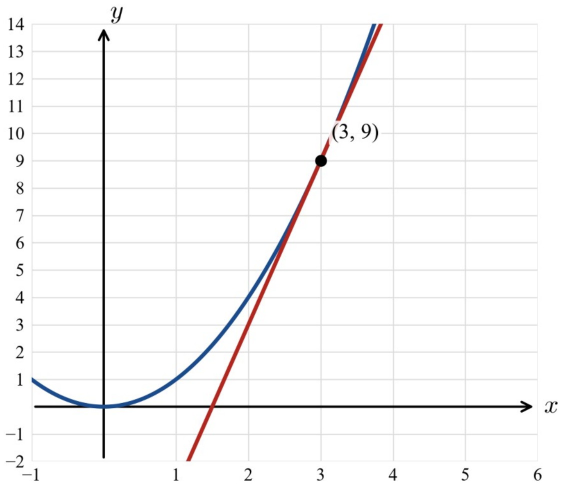
Q10. $x^2-3x+2=x-1 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0 \Rightarrow x=1$ or 3. Into $y=x-1$: $(1,0)$ and $(3,2)$.



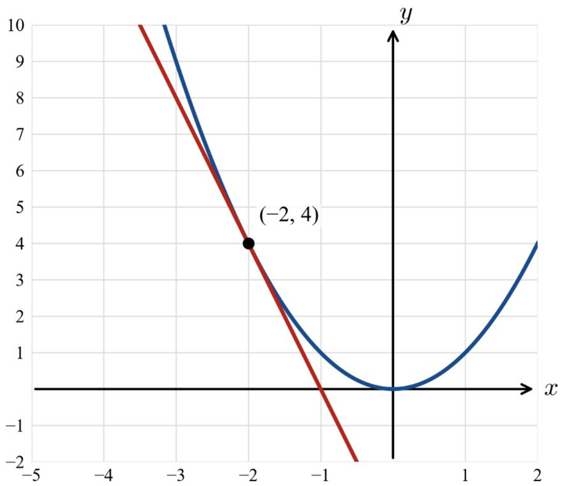
Q11. $x^2+2x-3=-x+1 \Rightarrow x^2+3x-4=0 \Rightarrow (x+4)(x-1)=0 \Rightarrow x=-4$ or 1. Into $y=-x+1$: $(-4,5)$ and $(1,0)$.



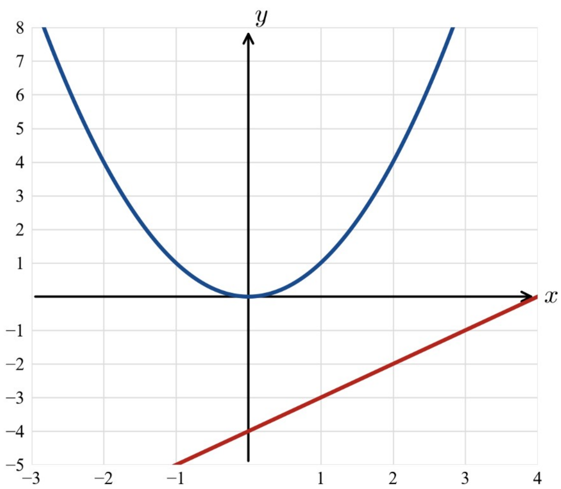
Q12. $x^2=6x-9 \Rightarrow x^2-6x+9=0 \Rightarrow (x-3)^2=0 \Rightarrow x=3$. Into $y=6x-9$: $y=9$. $\Delta=0$, $\therefore$ **tangent** at $(3,9)$.



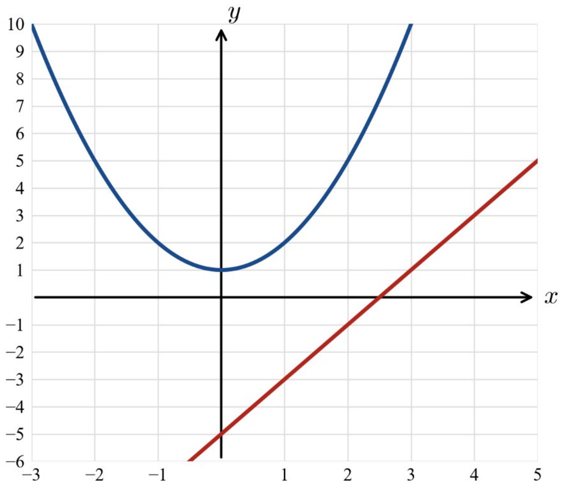
Q13. $x^2 = -4x - 4 \Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0 \Rightarrow x = -2$. Into $y = -4x - 4$: $y = 4$. $\Delta = 0$, $\therefore$ **tangent** at $(-2, 4)$.



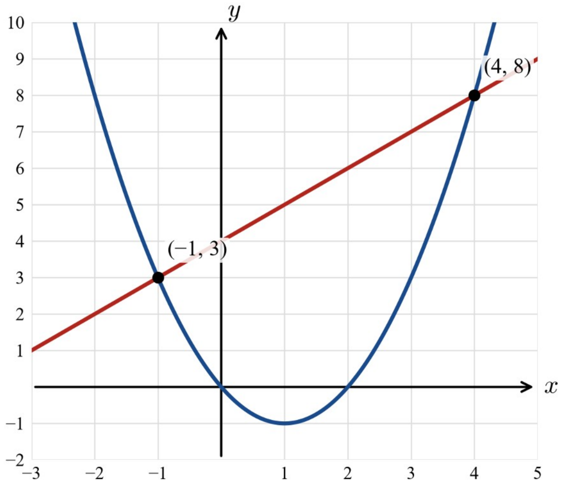
Q14. $x^2 = x - 4 \Rightarrow x^2 - x + 4 = 0$. $\Delta = (-1)^2 - 4(1)(4) = -15 < 0$, so there are no real solutions $\therefore$ the line does not meet the parabola.



Q15. $x^2 + 1 = 2x - 5 \Rightarrow x^2 - 2x + 6 = 0$. $\Delta = (-2)^2 - 4(1)(6) = -20 < 0$, so there are no real solutions $\therefore$ no intersection.



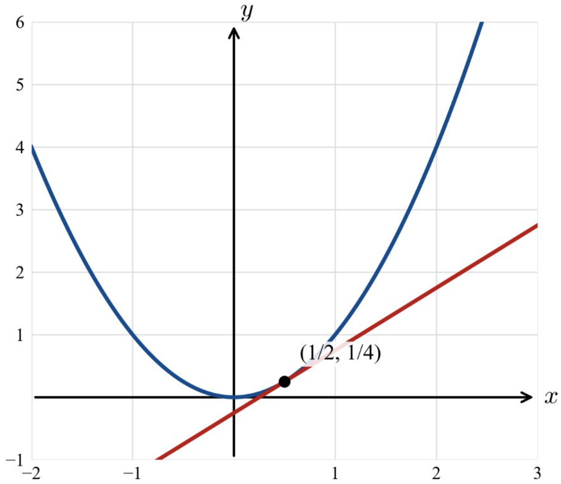
Q16. $x^2 - 2x = x + 4 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or -1. Into $y = x + 4$: $(4, 8)$ and $(-1, 3)$.



Q17. Substituting the line into the parabola: $x^2 = x + k \Rightarrow x^2 - x - k = 0$. The line is a tangent when this has one solution, i.e. $\Delta = 0$:

$$\Delta = (-1)^2 - 4(1)(-k) = 1 + 4k = 0 \Rightarrow k = -1/4.$$

(The point of contact is then $(1/2, 1/4)$.)



Q18. From $x^2 - x - k = 0$, $\Delta = 1 + 4k$. There are two distinct intersection points when $\Delta > 0$:

$$1 + 4k > 0 \Rightarrow k > -1/4.$$

(The line is a tangent when $k = -1/4$, and misses the parabola when $k < -1/4$.)

Theory

A **family** of quadratics is a set of parabolas that share a common feature, described by a rule containing a parameter. For example, every quadratic with turning point $(2,3)$ belongs to the family $y=a(x-2)^2+3$, $a\neq 0$; each value of a gives a different member.

To find the rule of a particular quadratic, choose the form that matches the information given.

Turning point given — use turning-point form

$$y=a(x-h)^2+k,$$

substituting the turning point (h,k), then use one more point to find a.

x**-intercepts given** — use factored form

$$y=a(x-p)(x-q),$$

substituting the intercepts $x=p$ and $x=q$, then use one more point to find a.

Three points given — use polynomial form

$$y=ax^2+bx+c,$$

substituting each point to obtain three simultaneous equations, then solve for a, b and c.

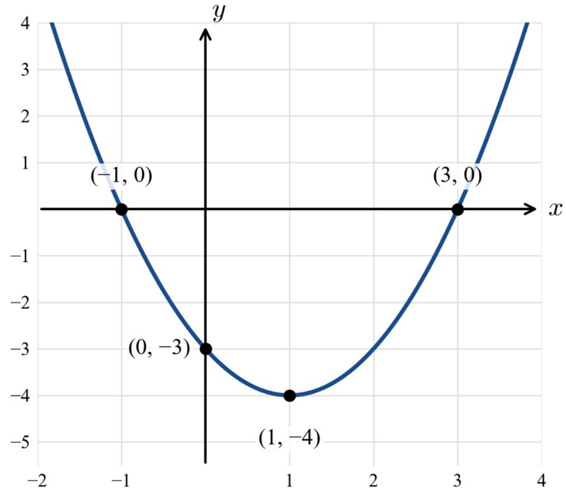
In every case, exactly the right number of conditions is needed: two for turning-point or factored form (the point plus the given feature), and three for polynomial form.

Worked examples

Example 1 — scaffolded

Find the rule of the quadratic with turning point $(1,-4)$ whose graph passes through $(3,0)$.

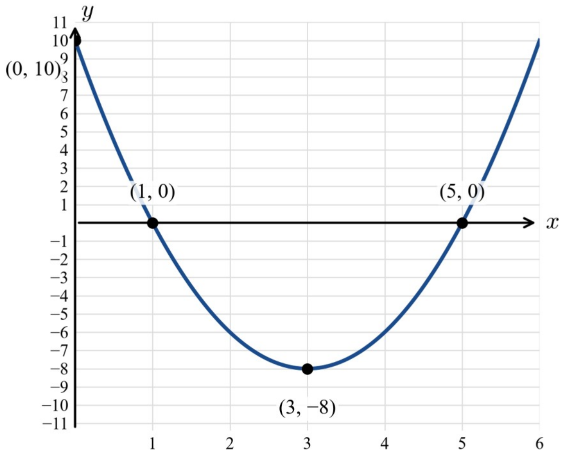
Working	Comment
The turning point is given, so use $y=a(x-h)^2+k$ with $h=1$, $k=-4$: $y=a(x-1)^2-4$.	Turning-point form is most efficient when the turning point is known.
Substitute $(3,0)$: $0=a(3-1)^2-4=4a-4$.	Use the extra point to form an equation in a.
$4a=4\Rightarrow a=1$.	Solve for a.
$\therefore y=(x-1)^2-4$.	State the rule (here $=x^2-2x-3$).



Example 2 — scaffolded

Find the rule of the quadratic with x-intercepts 1 and 5 whose graph passes through $(2,-6)$.

Working	Comment
The x-intercepts are given, so use $y=a(x-p)(x-q)$ with $p=1$, $q=5$: $y=a(x-1)(x-5)$.	Factored form is most efficient when the intercepts are known.
Substitute $(2,-6)$: $-6=a(2-1)(2-5)=a(1)(-3)=-3a$.	Use the extra point.
$-3a=-6 \Rightarrow a=2$.	Solve for a.
$\therefore y=2(x-1)(x-5)=2x^2-12x+10$.	Expand to give polynomial form.



Example 3 — unscaffolded

Find the rule $y=ax^2+bx+c$ of the quadratic passing through $(0,3)$, $(1,6)$ and $(-1,2)$.

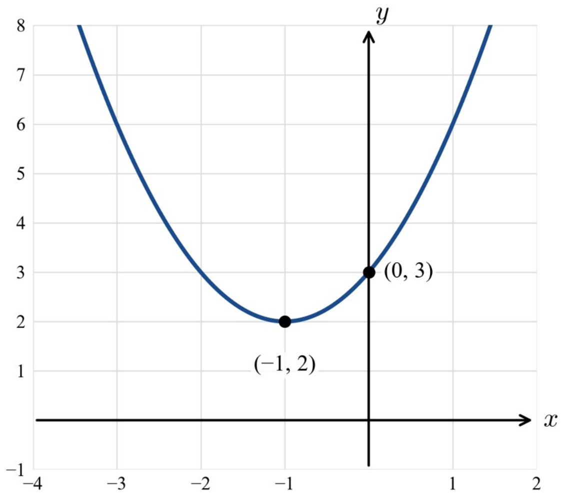
From $(0,3)$: $c=3$.

From $(1,6)$: $a+b+c=6 \Rightarrow a+b=3$.

From $(-1,2)$: $a-b+c=2 \Rightarrow a-b=-1$.

Adding the last two equations: $2a=2 \Rightarrow a=1$, so $b=2$.

$\therefore y=x^2+2x+3$.

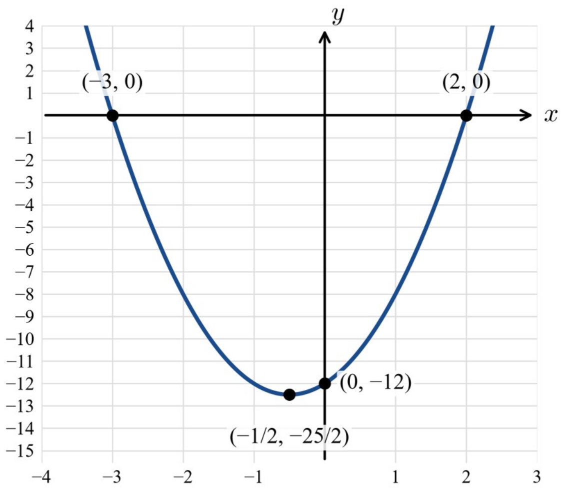


Example 4 — unscaffolded

Every quadratic with x-intercepts -3 and 2 has the form $y=a(x+3)(x-2)$. Find the member of this family whose graph passes through $(1,-8)$.

Substitute $(1,-8)$: $-8=a(1+3)(1-2)=a(4)(-1)=-4a$, so $a=2$.

$\therefore y=2(x+3)(x-2)=2x^2+2x-12$.



Practice questions

Scaffolded

Q1. A quadratic has turning point $(2,-1)$ and passes through $(0,3)$. (a) Write the rule in the form $y=a(x-h)^2+k$. (b) Substitute the given point to find a. (c) State the rule.

Q2. A quadratic has turning point $(-1,4)$ and passes through $(1,0)$. Find its rule in turning-point form.

Q3. A quadratic has x-intercepts -4 and 2 and passes through $(0,-8)$. (a) Write the rule in the form $y=a(x-p)(x-q)$. (b) Find a. (c) State the rule.

Q4. A quadratic has x-intercepts -1 and 3 and passes through $(1,-8)$. Find its rule.

Q5. A quadratic passes through $(0,1)$, $(1,0)$ and $(2,3)$. (a) Use $(0,1)$ to find c. (b) Form two equations from the other points. (c) Solve for a and b, and state the rule.

Q6. Every quadratic with turning point $(3,3)$ has the form $y=a(x-3)^2+3$. Find the member passing through $(5,7)$.

Unscaffolded

Find the rule of each quadratic, choosing the most efficient form.

Q7. Turning point $(1,2)$; passes through $(2,5)$.

Q8. Turning point $(-2,-1)$; passes through $(0,3)$.

Q9. x-intercepts 2 and 6; passes through $(4,-4)$.

Q10. x-intercepts -5 and -1; passes through $(-3,4)$.

Q11. Passes through $(0,-2)$, $(1,0)$ and $(3,10)$.

Q12. Passes through $(1,4)$, $(2,3)$ and $(3,0)$.

Q13. Describe the family of all quadratics with turning point $(0,5)$, then find the member passing through $(2,-3)$.

Q14. Describe the family of all quadratics with x-intercepts -2 and 3, then find the member with y-intercept -12.

Q15. Turning point $(2,-3)$; passes through $(0,5)$.

Q16. A quadratic has the same shape as $y=2x^2$ (the same dilation) and turning point $(1,-3)$. Find its rule.

Q17. x-intercepts -3 and 5; passes through $(1,-16)$.

Q18. Passes through $(-2,-3)$, $(0,1)$ and $(2,9)$.

Answers

Q1. $y=(x-2)^2-1$. **Q2.** $y=-(x+1)^2+4$. **Q3.** $y=(x+4)(x-2)=x^2+2x-8$. **Q4.** $y=2(x+1)(x-3)=2x^2-4x-6$. **Q5.** $y=2x^2-3x+1$. **Q6.** $y=(x-3)^2+3$. **Q7.** $y=3(x-1)^2+2$. **Q8.** $y=(x+2)^2-1$. **Q9.** $y=(x-2)(x-6)=x^2-8x+12$. **Q10.** $y=-(x+5)(x+1)=-x^2-6x-5$. **Q11.** $y=x^2+x-2$. **Q12.** $y=-x^2+2x+3$. **Q13.** Family $y=ax^2+5$; member $y=-2x^2+5$. **Q14.** Family $y=a(x+2)(x-3)$; member $y=2x^2-2x-12$. **Q15.** $y=2(x-2)^2-3=2x^2-8x+5$. **Q16.** $y=2(x-1)^2-3=2x^2-4x-1$. **Q17.** $y=(x+3)(x-5)=x^2-2x-15$. **Q18.** $y=1/2x^2+3x+1$.

Fully-worked solutions

Q1. Turning-point form: $y=a(x-2)^2-1$. Substitute $(0,3)$: $3=a(-2)^2-1=4a-1\Rightarrow 4a=4\Rightarrow a=1$. $\therefore y=(x-2)^2-1$.

Q2. $y=a(x+1)^2+4$. Substitute $(1,0)$: $0=a(2)^2+4=4a+4\Rightarrow a=-1$. $\therefore y=-(x+1)^2+4$.

Q3. Factored form: $y=a(x+4)(x-2)$. Substitute $(0,-8)$: $-8=a(4)(-2)=-8a\Rightarrow a=1$. $\therefore y=(x+4)(x-2)=x^2+2x-8$.

Q4. $y=a(x+1)(x-3)$. Substitute $(1,-8)$: $-8=a(2)(-2)=-4a\Rightarrow a=2$. $\therefore y=2(x+1)(x-3)=2x^2-4x-6$.

Q5. $y=ax^2+bx+c$. From $(0,1)$: $c=1$. From $(1,0)$: $a+b+1=0\Rightarrow a+b=-1$. From $(2,3)$: $4a+2b+1=3\Rightarrow 2a+b=1$. Subtracting: $a=2$, so $b=-3$. $\therefore y=2x^2-3x+1$.

Q6. $y=a(x-3)^2+3$. Substitute $(5,7)$: $7=a(2)^2+3=4a+3\Rightarrow a=1$. $\therefore y=(x-3)^2+3$.

Q7. $y=a(x-1)^2+2$. Substitute $(2,5)$: $5=a(1)^2+2=a+2\Rightarrow a=3$. $\therefore y=3(x-1)^2+2$.

Q8. $y=a(x+2)^2-1$. Substitute $(0,3)$: $3=a(2)^2-1=4a-1\Rightarrow a=1$. $\therefore y=(x+2)^2-1$.

Q9. $y=a(x-2)(x-6)$. Substitute $(4,-4)$: $-4=a(2)(-2)=-4a\Rightarrow a=1$. $\therefore y=(x-2)(x-6)=x^2-8x+12$.

Q10. $y=a(x+5)(x+1)$. Substitute $(-3,4)$: $4=a(2)(-2)=-4a\Rightarrow a=-1$. $\therefore y=-(x+5)(x+1)=-x^2-6x-5$.

Q11. $y=ax^2+bx+c$. From $(0,-2)$: $c=-2$. From $(1,0)$: $a+b-2=0\Rightarrow a+b=2$. From $(3,10)$: $9a+3b-2=10\Rightarrow 3a+b=4$. Subtracting: $2a=2\Rightarrow a=1$, $b=1$. $\therefore y=x^2+x-2$.

Q12. From $(1,4)$: $a+b+c=4$. From $(2,3)$: $4a+2b+c=3$. From $(3,0)$: $9a+3b+c=0$. Subtracting in pairs: $3a+b=-1$ and $5a+b=-3$, so $2a=-2\Rightarrow a=-1$, $b=2$, and $c=4-a-b=3$. $\therefore y=-x^2+2x+3$.

Q13. All quadratics with turning point $(0,5)$ have the form $y=ax^2+5$. Substitute $(2,-3)$: $-3=4a+5\Rightarrow 4a=-8\Rightarrow a=-2$. $\therefore y=-2x^2+5$.

Q14. All quadratics with x-intercepts -2 and 3 have the form $y=a(x+2)(x-3)$. The y-intercept is at $x=0$: $-12=a(2)(-3)=-6a\Rightarrow a=2$. $\therefore y=2(x+2)(x-3)=2x^2-2x-12$.

Q15. $y=a(x-2)^2-3$. Substitute $(0,5)$: $5=a(-2)^2-3=4a-3\Rightarrow 4a=8\Rightarrow a=2$. $\therefore y=2(x-2)^2-3=2x^2-8x+5$.

Q16. Same shape as $y=2x^2$ means $a=2$; turning point $(1,-3)$ gives $y=2(x-1)^2-3$. $\therefore y=2(x-1)^2-3=2x^2-4x-1$.

Q17. $y=a(x+3)(x-5)$. Substitute $(1,-16)$: $-16=a(4)(-4)=-16a \Rightarrow a=1$.
$\therefore y=(x+3)(x-5)=x^2-2x-15$.

Q18. $y=ax^2+bx+c$. From $(0,1)$: $c=1$. From $(-2,-3)$: $4a-2b+1=-3 \Rightarrow 2a-b=-2$. From $(2,9)$: $4a+2b+1=9 \Rightarrow 2a+b=4$. Adding: $4a=2 \Rightarrow a=1/2$, so $b=3$. $\therefore y=1/2x^2+3x+1$.

Theory

Many real situations — the flight of a ball, the area of an enclosure, a business's revenue — can be **modelled** by a quadratic. Because a quadratic has a single turning point, these models are especially useful for finding a **maximum** or **minimum** value.

A method for quadratic modelling.

1. **Define the variable(s)**, with units (e.g. let x be the width in metres).
2. **Form the model** — a quadratic equation or function — from the information given, using a constraint where needed to reduce to one variable.
3. **Solve or optimise.** To solve, set the quadratic to a value and use factorising, the quadratic formula, or completing the square. To optimise, find the turning point: the maximum or minimum occurs at $x=-\dfrac{b}{2a}$ (a maximum if $a<0$, a minimum if $a>0$).
4. **Interpret** the answer in context and **state units**.
5. **Check the domain.** Reject solutions that make no sense in context (for example a negative time, length, or price).

Common models.

- **Height–time (projectile):** height h is a quadratic in time t with a negative leading coefficient. The maximum height is the turning point; the object is at ground level when $h=0$.
- **Area:** use a constraint (such as a fixed length of fencing) to write the area as a quadratic in one length, then find its maximum.
- **Revenue and profit:** revenue = price × quantity; when quantity depends linearly on price, revenue is a quadratic in price.

Worked examples

Example 1 — scaffolded

A ball is thrown vertically upward. Its height h metres above the ground after t seconds is $h=-5t^2+40t$. (a) When does the ball return to the ground? (b) Find the maximum height reached and the time at which it occurs.

Working	Comment
The coefficient of t^2 is $-5<0$, so the graph is a parabola with a **maximum**.	Read the shape from the leading coefficient.
(a) At ground level $h=0$: $-5t^2+40t=0 \Rightarrow -5t(t-8)=0$, so $t=0$ or $t=8$.	Take out the common factor $-5t$, then apply the null factor law.
The ball leaves the ground at $t=0$ and returns at $t=8$. ∴ it returns after 8 s.	Interpret in context.
(b) Maximum at the turning point: $t=-\dfrac{40}{2\times(-5)}=4$.	Axis of symmetry $t=-\dfrac{b}{2a}$.
$h(4)=-5(4)^2+40(4)=-80+160=80$.	Substitute $t=4$.
∴ the maximum height is 80 m, reached after 4 s.	State the answer with units.

Example 2 — scaffolded

A rectangular enclosure is built against a straight wall, using 48 m of fencing for the other three sides. Let the width (each side perpendicular to the wall) be x m. (a) Show that the area is $A=48x-2x^2$. (b) Find the width that maximises the area, and the maximum area.

Working	Comment
The two widths and one length use all the fencing: $2x+\ell=48$, so $\ell=48-2x$.	The wall forms the fourth side, so it needs no fencing.
(a) $A=x\ell=x(48-2x)=48x-2x^2$.	Area = width × length.
(b) $a=-2<0$, so the maximum is at $x=-\dfrac{48}{2\times(-2)}=12$.	Turning point of the area parabola.
$A(12)=48(12)-2(12)^2=576-288=288$.	Substitute $x=12$.
∴ the area is greatest (288 m²) when $x=12$ m and $\ell=24$ m.	Domain $0<x<24$; state units.

Example 3 — unscaffolded

A cinema finds that if it charges $\$p$ per ticket it sells $(300-20p)$ tickets per screening. Find the ticket price that maximises revenue, and the maximum revenue.

Revenue $R=p(300-20p)=300p-20p^2$ dollars.

Since $a=-20<0$, the maximum is at the turning point:

$$p=-\frac{300}{2\times(-20)}=\frac{300}{40}=7.5.$$

$R(7.5)=7.5(300-20\times7.5)=7.5(300-150)=7.5\times150=1125$.

∴ the revenue is greatest at a ticket price of $\$7.50$, giving a maximum revenue of $\$1125$ per screening. (Sensible domain $0\le p\le15$.)

Example 4 — unscaffolded

The product of two consecutive positive even integers is 168. Find the integers.

Let the integers be x and $x+2$. Then

$$x(x+2)=168\Rightarrow x^2+2x-168=0\Rightarrow(x-12)(x+14)=0,$$

so $x=12$ or $x=-14$. Since the integers are positive, $x=12$.

∴ the integers are 12 and 14 (check: $12\times14=168$).

Practice questions

Scaffolded

Q1. A stone is thrown upward; its height h metres after t seconds is $h=-5t^2+50t$. (a) When does it return to the ground? (b) Find the maximum height and the time at which it occurs.

Q2. A rectangle has a perimeter of 28 m. Let its width be x m. (a) Show that the area is $A=14x-x^2$. (b) Find the maximum area and the dimensions that give it.

Q3. A rectangular garden bed is fenced on three sides (a wall forms the fourth side) using 32 m of fencing, with width x m. (a) Write the area A in terms of x. (b) Find the width that maximises the area and state the maximum area.

Q4. Two numbers add to 22. Let one of them be x. (a) Write their product P in terms of x. (b) Find the maximum possible product.

Q5. A stall sells $(500-25p)$ drinks per day at $\$p$ each. (a) Write the daily revenue R in terms of p. (b) Find the price that maximises revenue and the maximum revenue.

Q6. A ball is thrown from a 15 m high balcony; its height is $h=-5t^2+10t+15$ metres after t seconds. (a) State the initial height. (b) When does the ball hit the ground? (c) Find the maximum height.

Unscaffolded

For each problem, define your variable(s), form a quadratic model, solve, and interpret the answer in context with units.

Q7. A ball's height is $h=-5t^2+35t$ metres after t seconds. Find when it returns to the ground and its maximum height.

Q8. A golf ball is struck from a platform 2 m above the ground; its height is $h=-5t^2+20t+2$ metres after t seconds. Find the maximum height and the time at which it occurs.

Q9. A rectangle has a perimeter of 36 m. Find its greatest possible area.

Q10. A rectangular yard is fenced on three sides (a wall forms the fourth) using 72 m of fencing. Find the maximum possible area.

Q11. Two positive numbers differ by 4 and their product is 96. Find the numbers.

Q12. The product of two consecutive positive integers is 132. Find the integers.

Q13. A shop sells $(400-10p)$ items per day at $\$p$ each. Find the price that maximises daily revenue and the maximum revenue.

Q14. A company's daily profit is $P=-2x^2+40x-150$ dollars, where x is the number of items sold. Find (a) the number of items that maximises the profit and the maximum profit, and (b) the break-even values of x (where $P=0$).

Q15. A right-angled triangle has one leg 7 cm longer than the other and a hypotenuse of 13 cm. Find the lengths of the two legs.

Q16. A rectangular photograph 10 cm by 15 cm is surrounded by a border of uniform width x cm. The total area of the photograph and border is 234 cm². Find x.

Q17. A stone is thrown from the top of a 25 m cliff; its height above the sea is $h=-5t^2+20t+25$ metres after t seconds. Find (a) the maximum height, and (b) the time it takes to reach the sea.

Q18. The sum of a positive number and its square is 56. Find the number.

Answers

Q1. Returns after 10 s; maximum height 125 m at $t=5$ s. **Q2.** $A=14x-x^2$; maximum area 49 m² (7 m × 7 m). **Q3.** $A=32x-2x^2$; width 8 m gives maximum area 128 m². **Q4.** $P=22x-x^2$; maximum product 121 (the numbers are 11 and 11). **Q5.** $R=500p-25p^2$; price $\$10$ gives maximum revenue $\$2500$. **Q6.** Initial height 15 m; hits the ground at $t=3$ s; maximum height 20 m at $t=1$ s. **Q7.** Returns after 7 s; maximum height 61.25 m at $t=3.5$ s. **Q8.** Maximum height 22 m at $t=2$ s. **Q9.** 81 m². **Q10.** 648 m². **Q11.** 8 and 12. **Q12.** 11 and 12. **Q13.** Price $\$20$ gives maximum revenue $\$4000$. **Q14.** (a) 10 items, maximum profit $\$50$; (b) break-even at $x=5$ and $x=15$. **Q15.** 5 cm and 12 cm. **Q16.** $x=1.5$ cm. **Q17.** (a) 45 m at $t=2$ s; (b) 5 s. **Q18.** 7.

Fully-worked solutions

Q1. $h=-5t^2+50t$. (a) $h=0$: $-5t(t-10)=0$, so $t=0$ or $t=10$; the stone returns after 10 s. (b) Turning point: $t=-\dfrac{50}{2(-5)}=5$; $h(5)=-5(25)+250=125$. ∴ maximum height 125 m at $t=5$ s.

Q2. Perimeter 28: $2x+2\ell=28$, so $\ell=14-x$. (a) $A=x(14-x)=14x-x^2$. (b) Maximum at $x=-\dfrac{14}{2(-1)}=7$; $A(7)=14(7)-49=49$. ∴ the maximum area is 49 m², when the rectangle is 7 m × 7 m (a square).

Q3. Fencing: $2x+\ell=32$, so $\ell=32-2x$. (a) $A=x(32-2x)=32x-2x^2$. (b) Maximum at $x=-\dfrac{32}{2(-2)}=8$; $A(8)=32(8)-2(64)=256-128=128$. ∴ width 8 m gives the maximum area 128 m².

Q4. The numbers are x and $22-x$. (a) $P=x(22-x)=22x-x^2$. (b) Maximum at $x=-\dfrac{22}{2(-1)}=11$; $P(11)=242-121=121$. ∴ the maximum product is 121, when both numbers are 11.

Q5. (a) $R=p(500-25p)=500p-25p^2$. (b) Maximum at $p=-\dfrac{500}{2(-25)}=10$; $R(10)=500(10)-25(100)=5000-2500=2500$. ∴ a price of $\$10$ gives the maximum revenue $\$2500$.

Q6. $h=-5t^2+10t+15$. (a) Initial height $h(0)=15$ m. (b) $h=0$: $-5t^2+10t+15=0 \Rightarrow t^2-2t-3=0 \Rightarrow (t-3)(t+1)=0$, so $t=3$ or $t=-1$. Time cannot be negative, so the ball hits the ground at $t=3$ s. (c) Turning point: $t=-\dfrac{10}{2(-5)}=1$; $h(1)=-5+10+15=20$. ∴ maximum height 20 m at $t=1$ s.

Q7. $h=-5t^2+35t=-5t(t-7)$, so $h=0$ at $t=0$ or $t=7$; it returns after 7 s. Turning point: $t=-\dfrac{35}{2(-5)}=3.5$; $h(3.5)=-5(12.25)+35(3.5)=-61.25+122.5=61.25$. ∴ maximum height 61.25 m at $t=3.5$ s.

Q8. $h=-5t^2+20t+2$. Turning point: $t=-\dfrac{20}{2(-5)}=2$; $h(2)=-5(4)+40+2=-20+42=22$. ∴ maximum height 22 m at $t=2$ s.

Q9. Perimeter 36: $\ell=18-x$, so $A=x(18-x)=18x-x^2$. Maximum at $x=-\dfrac{18}{2(-1)}=9$; $A(9)=18(9)-81=81$. ∴ the greatest area is 81 m² (a 9 m × 9 m square).

Q10. Fencing: $\ell=72-2x$, so $A=x(72-2x)=72x-2x^2$. Maximum at $x=-\dfrac{72}{2(-2)}=18$; $A(18)=72(18)-2(324)=1296-648=648$. ∴ the maximum area is 648 m².

Q11. Let the numbers be x and $x+4$. Then $x(x+4)=96 \Rightarrow x^2+4x-96=0 \Rightarrow (x-8)(x+12)=0$, so $x=8$ (rejecting $x=-12$, as the numbers are positive). ∴ the numbers are 8 and 12.

Q12. Let the integers be x and $x+1$. Then $x(x+1)=132 \Rightarrow x^2+x-132=0 \Rightarrow (x-11)(x+12)=0$, so $x=11$ (rejecting $x=-12$). ∴ the integers are 11 and 12.

Q13. $R=p(400-10p)=400p-10p^2$. Maximum at $p=-\dfrac{400}{2(-10)}=20$;
$R(20)=400(20)-10(400)=8000-4000=4000$. ∴ a price of $\$20$ gives the maximum revenue $\$4000$.

Q14. $P=-2x^2+40x-150$. (a) Maximum at $x=-\dfrac{40}{2(-2)}=10$; $P(10)=-2(100)+400-150=-200+250=50$. ∴ selling 10 items gives the maximum profit $\$50$. (b) $P=0$:
$-2x^2+40x-150=0 \Rightarrow x^2-20x+75=0 \Rightarrow (x-5)(x-15)=0$, so $x=5$ or $x=15$. ∴ the company breaks even at 5 and 15 items.

Q15. Let the shorter leg be x cm; the longer leg is $x+7$. By Pythagoras' theorem, $x^2+(x+7)^2=13^2$:

$$x^2+x^2+14x+49=169 \Rightarrow 2x^2+14x-120=0 \Rightarrow x^2+7x-60=0 \Rightarrow (x-5)(x+12)=0.$$

So $x=5$ (rejecting $x=-12$). ∴ the legs are 5 cm and 12 cm (check: $5^2+12^2=25+144=169=13^2$).

Q16. With a border of width x, the outer rectangle is $(10+2x)$ by $(15+2x)$:

$$(10+2x)(15+2x)=234 \Rightarrow 4x^2+50x+150=234 \Rightarrow 4x^2+50x-84=0 \Rightarrow 2x^2+25x-42=0.$$

Factorising, $(2x-3)(x+14)=0$, so $x=3/2$ or $x=-14$. A width cannot be negative, so $x=1.5$ cm (check: $13\times 18=234$).

Q17. $h=-5t^2+20t+25$. (a) Turning point: $t=-\dfrac{20}{2(-5)}=2$; $h(2)=-5(4)+40+25=45$, so the maximum height is 45 m at $t=2$ s. (b) At the sea $h=0$: $-5t^2+20t+25=0 \Rightarrow t^2-4t-5=0 \Rightarrow (t-5)(t+1)=0$, so $t=5$ (rejecting $t=-1$). ∴ it reaches the sea after 5 s.

Q18. Let the number be x. Then $x+x^2=56 \Rightarrow x^2+x-56=0 \Rightarrow (x-7)(x+8)=0$, so $x=7$ (rejecting $x=-8$, as the number is positive). ∴ the number is 7.