General Mathematics Units 3 & 4

Practice Examinations

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Contents

Exam 1 — Multiple-choice (Easy)

Reading time: 15 minutes. Writing time: 1 hour 30 minutes. 40 multiple-choice questions, 40 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Choose the response that is ***correct*** *for the question. Unless otherwise indicated, diagrams are not drawn to scale.*

Section A — Multiple-choice

Question 1

A veterinary nurse records the following four variables for each dog that visits a clinic: its breed, its age (in years), whether it has been desexed (yes or no), and its weight (in kilograms). How many of these four variables are numerical?

A. 1

B. 2

C. 3

D. 4

Question 2

The maximum temperatures, in °C, recorded on nine days were

$$24, 19, 22, 27, 25, 20, 23, 26, 21.$$

The median maximum temperature is

A. 22 °C

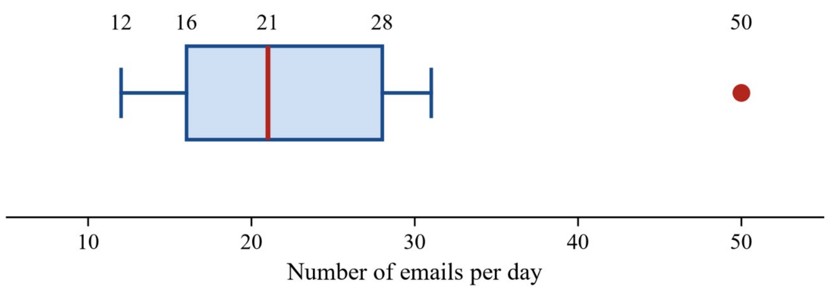
B. 24 °C

C. 25 °C

D. 23 °C

Question 3

The boxplot below summarises the number of emails received per day by an office worker over 11 days.



The interquartile range (IQR) of this data set is

A. 12

B. 28

C. 16

D. 9

Question 4

For a data set, the first quartile is $Q_1 = 34$ and the third quartile is $Q_3 = 46$. Using the $1.5 \times \text{IQR}$ rule, a value is classified as an outlier at the upper end if it is greater than

A. 58

B. 52

C. 64

D. 78

Question 5

For the data set

$$2, 4, 6, 8, 10,$$

the sample standard deviation is closest to

A. 2.83

B. 4.00

C. 3.16

D. 10.00

Question 6

The times taken by athletes to run 100 metres are normally distributed with a mean of 14.0 seconds and a standard deviation of 0.5 seconds. The standardised value (z-score) of an athlete who runs 100 metres in 15.0 seconds is

A. 2

B. 1.5

C. 1

D. -2

Question 7

The heights of seedlings in a nursery are approximately normally distributed with a mean of 30 cm and a standard deviation of 4 cm. Using the 68–95–99.7% rule, the percentage of seedlings that are taller than 34 cm is closest to

A. 32%

B. 2.5%

C. 84%

D. 16%

Question 8

The two-way frequency table below shows whether 180 people own a bicycle, classified by age group.

Owns a bicycle	Under 18	18 or over	Total
Yes	45	39	84
No	15	81	96
Total	60	120	180

The percentage of people aged under 18 who own a bicycle is

A. 60%

B. 75%

C. 25%

D. 45%

Question 9

A scatterplot of two numerical variables is approximately linear, with a correlation coefficient of $r = 0.82$. This correlation is best described as

A. strong positive

B. moderate positive

C. weak positive

D. strong negative

Question 10

For a least-squares regression of two numerical variables, the correlation coefficient is $r = 0.7$. The coefficient of determination, expressed as a percentage, is

A. 70%

B. 84%

C. 49%

D. 7%

Question 11

The least-squares line relating a worker's weekly income, y (in dollars), to the number of hours worked, x, is

$$\hat{y} = 85 + 22x.$$

The slope of this line tells us that, on average,

A. weekly income increases by $85 for each extra hour worked

B. weekly income increases by $22 for each extra hour worked

C. weekly income decreases by $22 for each extra hour worked

D. weekly income is $22 when no hours are worked

Question 12

Using the least-squares line $\hat{y} = 85 + 22x$, where y is weekly income (in dollars) and x is the number of hours worked, the predicted weekly income of a worker who works 30 hours is

A. $660

B. $715

C. $770

D. $745

Question 13

A least-squares line is $\hat{y} = 3 + 2.5x$. For a data point with $x = 8$, the actual value is $y = 25$. The residual for this point is

A. 23

B. -2

C. 48

D. 2

Question 14

A least-squares model relating two variables is

$$\log_{10}(y) = 0.6 + 0.2x.$$

Using this model, the value of y predicted when $x = 5$ is closest to

A. 1.6

B. 40

C. 16

D. 4.0

Question 15

The numbers of umbrellas sold at a shop on five consecutive days were

$$8, 14, 11, 17, 20.$$

The three-point moving mean centred on the third day (11 umbrellas) is

A. 12.5

B. 11

C. 14

D. 42

Question 16

The seasonal index for December sales at a store is 1.6. In one particular year the actual December sales were $48,000. The deseasonalised December sales figure is

A. $30,000

B. $76,800

C. $48,000

D. $28,000

Question 17

A sequence is generated by the recurrence relation

$$V_0 = 6, V_{n+1} = 2V_n + 3.$$

The value of V_2 is

A. 15

B. 30

C. 33

D. 69

Question 18

An amount of $4000 is invested in an account paying simple interest at 6% per annum. The total interest earned after 4 years is

A. $960

B. $240

C. $1050

D. $4960

Question 19

An amount of $5000 is invested at 4% per annum, compounding annually. The value of the investment after 2 years is

A. $5400.00

B. $5200.00

C. $5624.32

D. $5408.00

Question 20

A machine is purchased for $30,000. It depreciates by a flat rate of 10% of its purchase price each year. The value of the machine after 3 years is

A. $27,000

B. $21,000

C. $21,870

D. $24,000

Question 21

A laptop worth $2000 depreciates using the reducing-balance method at a rate of 20% per annum. The value of the laptop after 2 years is

A. $1200

B. $1280

C. $1600

D. $800

Question 22

An account earns interest at 6% per annum, compounding monthly. The interest rate per compounding period and the number of compounding periods in 4 years are, respectively,

A. 6% and 4

B. 0.5% and 4

C. 2% and 16

D. 0.5% and 48

Question 23

A reducing-balance loan of $10,000 is charged interest at 1% per month. A repayment of $600 is made at the end of the first month. The balance of the loan immediately after this repayment is

A. $9500

B. $9400

C. $10,100

D. $9490

Question 24

A perpetuity pays $1000 each year, forever, from an account that earns interest at 4% per annum. The principal that must be invested to fund this perpetuity is

A. $40,000

B. $4000

C. $25,000

D. $24,000

Question 25

Consider the matrix

$$A = \begin{bmatrix} 4 & -2 & 7 \\ 1 & 0 & 5 \end{bmatrix}.$$

The order of A and the value of the element a_{23} are, respectively,

A. 3×2 and 5

B. 2×3 and 7

C. 2×3 and 0

D. 2×3 and 5

Question 26

Given

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix},$$

the matrix $A - B$ is equal to

A. $\begin{bmatrix} 6 & 2 \\ 5 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$

C. $\begin{bmatrix} -4 & -2 \\ 3 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Question 27

Given

$$P = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix},$$

the product PQ is equal to

A. $\begin{bmatrix} 2 & 4 \\ 0 & 15 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 13 \\ 4 & 17 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 13 \\ 6 & 15 \end{bmatrix}$

D. $\begin{bmatrix} 4 & 9 \\ 6 & 15 \end{bmatrix}$

Question 28

A bakery sells pies and rolls. The quantities sold one morning and their prices are given by

$$Q = \begin{bmatrix} 10 & 6 \end{bmatrix}, P = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

where the entries of Q are the numbers of pies and rolls sold and the entries of P are their prices in dollars. The total revenue, given by the product QP, is

A. \$58

B. \$46

C. \$40

D. \$18

Question 29

The determinant of the matrix

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

is

A. 14

B. 26

C. 20

D. 6

Question 30

The matrix

$$\begin{bmatrix} k & 2 \\ 3 & 6 \end{bmatrix}$$

does not have an inverse (is singular) when k is equal to

A. 0

B. 2

C. 1

D. -1

Question 31

Each year, residents move between a town (T) and a nearby city (C) according to the transition matrix below, in which the columns sum to 1.

$$\begin{array}{cc} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} & \begin{array}{c} T \\ C \end{array} \end{array}$$

This year the town has 500 residents and the city has 500 residents. According to this model, the number of residents in the town next year will be

A. 450

B. 550

C. 500

D. 400

Question 32

A population of insects is modelled by the Leslie matrix L and the initial state vector S_0, where

$$L=\begin{bmatrix}0 & 2 & 1\\ 0.5 & 0 & 0\\ 0 & 0.4 & 0\end{bmatrix}, S_0=\begin{bmatrix}100\\ 80\\ 40\end{bmatrix}.$$

The number of insects in the youngest age group (the first component) after one time step, given by $L S_0$, is

A. 160

B. 90

C. 50

D. 200

Question 33

A graph has five vertices whose degrees are

$$2,3,3,4,2.$$

The number of edges in this graph is

A. 14

B. 7

C. 5

D. 8

Question 34

A connected planar graph has 6 vertices and 9 edges. Using Euler's formula, the number of faces (including the infinite outer region) is

A. 5

B. 3

C. 15

D. 4

Question 35

The adjacency matrix of a graph with vertices A, B, C and D (in that order) is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The number of walks of length 2 from vertex A to vertex D is

A. 0

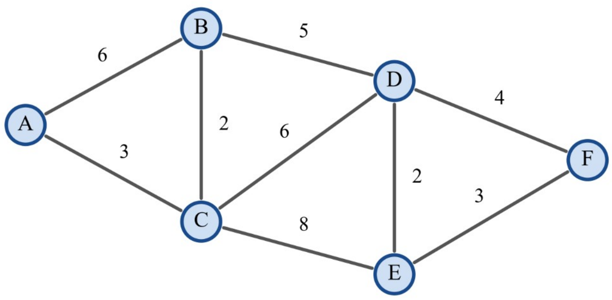
B. 1

C. 2

D. 3

Question 36

The network below shows the roads connecting six towns, A to F. The number on each edge is the length of that road, in kilometres.



The length of the shortest path from town A to town F, in kilometres, is

A. 11

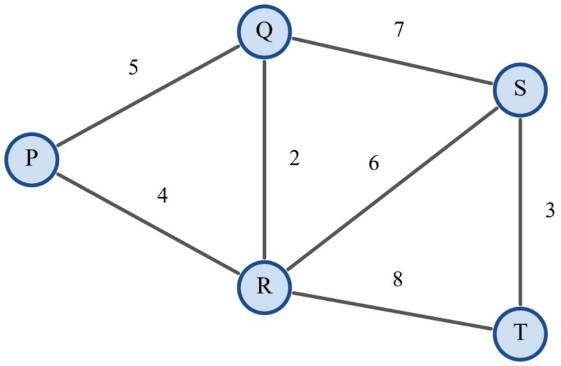
B. 12

C. 15

D. 13

Question 37

The network below shows the possible cable connections between five houses, P to T. The number on each edge is the cost, in thousands of dollars, of laying that cable.



The minimum cost of connecting all five houses (the total weight of the minimum spanning tree), in thousands of dollars, is

A. 14

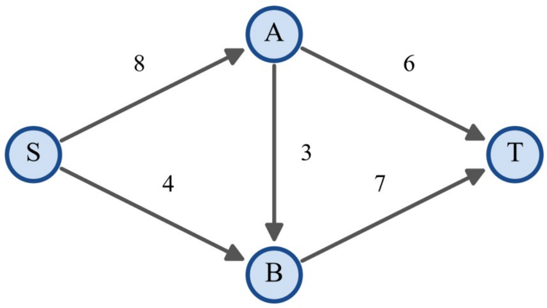
B. 16

C. 15

D. 17

Question 38

The directed network below shows a system of water pipes from a source S to a sink T. The number on each arc is the capacity of that pipe, in litres per second.



The maximum flow from S to T, in litres per second, is

A. 10

B. 11

C. 13

D. 12

Question 39

Three workers, X, Y and Z, are each to be allocated exactly one of three tasks. The table shows the time (in hours) each worker would take on each task.

	Task 1	Task 2	Task 3
Worker X	8	4	7
Worker Y	5	6	3
Worker Z	7	2	6

If each task is done by a different worker, the minimum total time to complete all three tasks, in hours, is

A. 13

B. 14

C. 15

D. 20

Question 40

A project consists of five activities. The duration of each activity and its immediate predecessor(s) are shown in the table.

Activity	Duration (hours)	Immediate predecessor(s)
A	5	—
B	4	—
C	6	A
D	3	B
E	2	C, D

The minimum time required to complete the whole project, in hours, is

A. 11

B. 13

C. 12

D. 15

End of Exam 1

Exam 1 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \bar{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + b\,x$
least squares slope	$b = \dfrac{r\,s_y}{s_x}$
least squares intercept	$a = \bar{y} - b\,\bar{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $u_0 = a\,, u_{n+1} = R\,u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100\,n}\right)^n - 1\right] \times 100\,\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\,A = a\,d - b\,c$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\,A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\,A \neq 0$

recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = T\,S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = L\,S_n$

Networks and decision mathematics

Euler's formula $v - e + f = 2$

End of Formula Sheet

Exam 1 — Answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Answer	B	D	A	C	C	A	D	B	A	C	B	D	D	B	C	A	C	A	D	B

Q	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Answer	B	D	A	C	D	B	C	A	A	C	B	D	B	A	C	D	C	D	A	B

Q1. B. Age and weight are numerical; breed and desexed status are categorical, so 2 of the four variables are numerical.

Q2. D. In order the values are $19, 20, 21, 22, 23, 24, 25, 26, 27$; with $n=9$ the median is the 5th value, 23 °C.

Q3. A. From the boxplot $Q_1=16$ and $Q_3=28$, so $\text{IQR}=Q_3-Q_1=28-16=12$.

Q4. C. $\text{IQR}=46-34=12$, so the upper fence is $Q_3+1.5\times\text{IQR}=46+1.5\times12=64$.

Q5. C. The mean is 6; $s=\sqrt{\dfrac{16+4+0+4+16}{5-1}}=\sqrt{10}\approx3.16$ (dividing by $n-1$).

Q6. A. $z=\dfrac{15.0-14.0}{0.5}=\dfrac{1.0}{0.5}=2$.

Q7. D. $34=30+1\,\text{sd}$; since 68% lie within one standard deviation, 32% lie outside, so the percentage above is $32/2=16\%$.

Q8. B. There are 60 people under 18, of whom 45 own a bicycle: $45/60=75\%$.

Q9. A. $|r|=0.82\geq0.75$ indicates a strong association, and the sign is positive.

Q10. C. $r^2=0.7^2=0.49$, i.e. 49%.

Q11. B. The slope is $+22$ dollars per hour, so on average weekly income rises by $22 for each additional hour worked.

Q12. D. $\hat{y}=85+22\times30=85+660=745$, i.e. $745.

Q13. D. Predicted $\hat{y}=3+2.5\times8=23$; residual = actual − predicted $=25-23=2$.

Q14. B. $\log_{10}(y)=0.6+0.2\times5=1.6$, so $y=10^{1.6}\approx39.8$, closest to 40.

Q15. C. The three-point moving mean is $\dfrac{14+11+17}{3}=\dfrac{42}{3}=14$.

Q16. A. Deseasonalised value = actual ÷ seasonal index $=\dfrac{48\,000}{1.6}=\$30\,000$.

Q17. C. $V_1=2\times6+3=15$ and $V_2=2\times15+3=33$.

Q18. A. Simple interest $=4000\times0.06\times4=\$960$.

Q19. D. $5000\times1.04^2=5000\times1.0816=\$5408.00$.

Q20. B. Flat-rate depreciation is $0.10\times30\,000=\$3000$ per year, so after 3 years the value is $30\,000-3\times3000=\$21\,000$.

Q21. B. Reducing-balance value $= 2000 \times (1 - 0.20)^2 = 2000 \times 0.64 = \$1280$.

Q22. D. The monthly rate is $6\%/12 = 0.5\%$ and the number of monthly periods in 4 years is $12 \times 4 = 48$.

Q23. A. Interest for the month is $0.01 \times 10\,000 = \$100$, so the balance is $10\,000 + 100 - 600 = \$9500$.

Q24. C. For a perpetuity the principal is $\dfrac{1000}{0.04} = \$25\,000$ (so that the annual interest exactly equals the \$1000 payment).

Q25. D. A has 2 rows and 3 columns, so its order is 2×3; a_{23} is the entry in row 2, column 3, which is 5.

Q26. B. $A - B = \begin{bmatrix} 5-1 & 2-0 \\ 1-4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$.

Q27. C. $PQ = \begin{bmatrix} 2(1)+1(2) & 2(4)+1(5) \\ 0(1)+3(2) & 0(4)+3(5) \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 6 & 15 \end{bmatrix}$.

Q28. A. $QP = 10 \times 4 + 6 \times 3 = 40 + 18 = \$58$.

Q29. A. $det = 4 \times 5 - 3 \times 2 = 20 - 6 = 14$.

Q30. C. The matrix is singular when $det = 6k - 6 = 0$, i.e. $k = 1$.

Q31. B. Next year the town has $0.8 \times 500 + 0.3 \times 500 = 400 + 150 = 550$ residents.

Q32. D. The first component of LS_0 is $0 \times 100 + 2 \times 80 + 1 \times 40 = 160 + 40 = 200$.

Q33. B. The sum of the degrees is $2 + 3 + 3 + 4 + 2 = 14$; the number of edges is half of this, $14/2 = 7$.

Q34. A. Euler's formula gives $v - e + f = 2$, so $f = 2 - v + e = 2 - 6 + 9 = 5$.

Q35. C. The number of length-2 walks from A to D is the (A, D) entry of the square of the adjacency matrix, which equals 2 (via $A - B - D$ and $A - C - D$).

Q36. D. The shortest path is $A - C - D - F$ with length $3 + 6 + 4 = 13$ km.

Q37. C. The minimum spanning tree uses edges $QR(2)$, $ST(3)$, $PR(4)$ and $RS(6)$, giving a total of $2 + 3 + 4 + 6 = 15$ thousand dollars.

Q38. D. The minimum cut is across the two arcs leaving S, with capacity $8 + 4 = 12$, so the maximum flow is 12 litres per second.

Q39. A. The minimum-cost allocation is $X \rightarrow$ Task 1 (8), $Y \rightarrow$ Task 3 (3), $Z \rightarrow$ Task 2 (2), giving $8 + 3 + 2 = 13$ hours.

Q40. B. The critical path is $A - C - E$ with length $5 + 6 + 2 = 13$ hours (the other path $B - D - E$ takes only $4 + 3 + 2 = 9$ hours).

Exam 2 — Multiple-choice (Medium)

Reading time: 15 minutes. Writing time: 1 hour 30 minutes. 40 multiple-choice questions, 40 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Choose the response that is ***correct*** *for the question. Unless otherwise indicated, diagrams are not drawn to scale.*

Section A — Multiple-choice

Question 1

A veterinary clinic records the following four variables for each dog that visits: breed, weight (in kilograms), number of vaccinations received, and overall health rating (recorded as poor, fair, good or excellent). The variable that is best described as ordinal is

A. breed

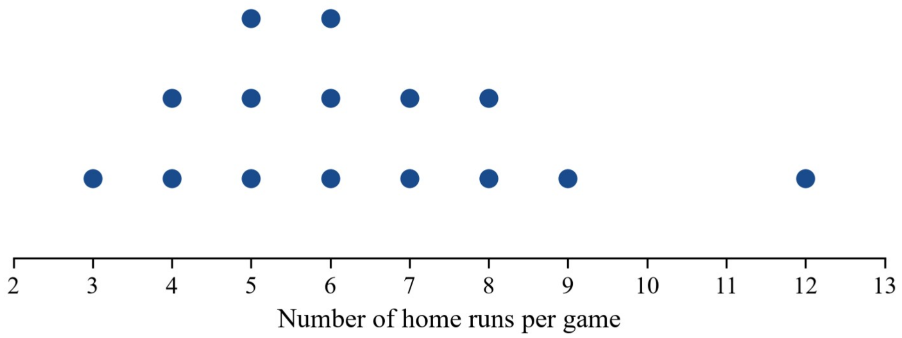
B. weight (in kilograms)

C. overall health rating

D. number of vaccinations received

Question 2

The dot plot below shows the number of home runs scored by a baseball team in each of its 15 games this season.



The percentage of games in which the team scored 7 or more home runs is

A. 40%

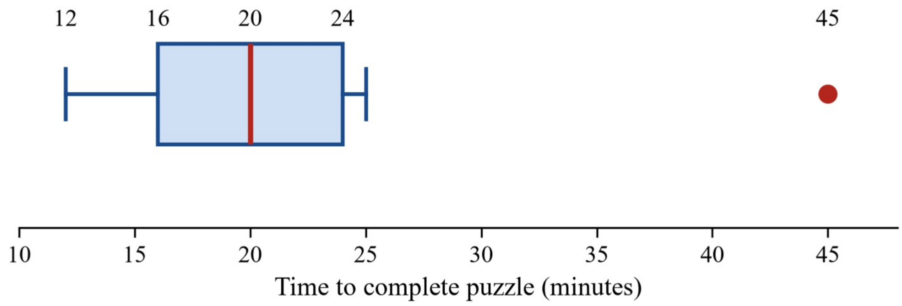
B. 20%

C. 6%

D. 60%

Question 3

The boxplot below displays the times, in minutes, taken by 11 students to complete a puzzle.



Using the $1.5 \times \text{IQR}$ rule, the value identified as an outlier is

A. 12

B. 25

C. there are no outliers

D. 45

Question 4

A data set consists of the five values

$$8, 11, 5, 14, 7.$$

The sample standard deviation of these values is closest to

A. 3.54

B. 3.16

C. 5.00

D. 12.50

Question 5

The volumes of soft drink dispensed by a machine are approximately normally distributed with a mean of 500 mL and a standard deviation of 5 mL. Using the 68–95–99.7% rule, the percentage of cups with a volume between 495 mL and 510 mL is

A. 95%

B. 81.5%

C. 68%

D. 83.85%

Question 6

In a class, the Mathematics test marks have a mean of 62 and a standard deviation of 8, and the English test marks have a mean of 68 and a standard deviation of 10. A student scored 74 in Mathematics and 80 in English. Relative to the rest of the class, this student performed

A. equally well in both subjects

B. better in English than in Mathematics

C. below average in both subjects

D. better in Mathematics than in English

Question 7

The two-way frequency table below shows the preferred hot drink of 300 people, classified by gender.

Preferred drink	Male	Female
Tea	50	60
Coffee	150	40

The percentage of males who prefer coffee is

A. 60%

B. 50%

C. 75%

D. 79%

Question 8

For two numerical variables, the coefficient of determination is 0.81, and the scatterplot shows a negative association. The value of the correlation coefficient r is

A. -0.90

B. 0.90

C. -0.81

D. -0.45

Question 9

A least-squares line relating the weight w (in kilograms) of a puppy to its age a (in weeks) is

$$w = 2.5 + 0.18 \times a.$$

Which of the following best interprets the slope of this line?

A. The puppy weighs 2.5 kg at birth.

B. On average, the weight increases by 0.18 kg each week.

C. On average, the weight increases by 18 kg each week.

D. On average, the weight increases by 2.5 kg each week.

Question 10

In a study of the relationship between the number of hours per week that students study and their test score, the correlation coefficient is $r = 0.6$. The coefficient of determination indicates that

A. 60% of the variation in test score is explained by the variation in hours studied

B. 36% of the variation in hours studied is explained by the variation in test score

C. 64% of test scores are predicted correctly

D. 36% of the variation in test score is explained by the variation in hours studied

Question 11

A least-squares line is $\hat{y} = 12.4 + 3.1\,x$. For a data point where $x = 8$, the actual value is $y = 40$. The residual for this point is

A. -2.8

B. 3.1

C. 2.8

D. 37.2

Question 12

A least-squares line was fitted to data for which the explanatory variable x ranged from 5 to 20. The line is used to predict the response when $x = 30$. This prediction

A. is an interpolation, and is reliable

B. is an extrapolation, and may be unreliable

C. is an extrapolation, and is always reliable

D. is an interpolation, and may be unreliable

Question 13

A scatterplot of the stopping distance d of a car against its speed s is curved, increasing at an increasing rate in a way consistent with a quadratic relationship. The transformation most likely to linearise the association is

A. s^2

B. $\log_{10}(s)$

C. $\frac{1}{d}$

D. $\log_{10}(d)$

Question 14

A least-squares model for a set of data is

$$\log_{10}(y) = 0.8 + 0.15x.$$

Using this model, the value of y when $x = 6$ is closest to

A. 5

B. 1.7

C. 50

D. 500

Question 15

The table below shows the quarterly sales, in thousands of dollars, of a swimwear store for one year.

Quarter	Q1	Q2	Q3	Q4
Sales ($'000)	88	92	120	100

The seasonal index for Quarter 3 is

A. 0.83

B. 1.20

C. 1.00

D. 0.88

Question 16

The number of visitors to a museum on seven consecutive days was

$$18, 22, 20, 26, 24, 30, 28.$$

The five-point moving mean centred on the fourth day (26 visitors) is

A. 26.0

B. 24.0

C. 25.0

D. 24.4

Question 17

A sequence is defined by the recurrence relation

$$V_0 = 4, V_{n+1} = 2V_n + 3.$$

The value of V_3 is

A. 53

B. 25

C. 109

D. 22

Question 18

An amount of $6000 is invested at an interest rate of 5.4% per annum, compounding quarterly. The value of the investment after 2 years is closest to

A. $6648.00

B. $6665.50

C. $6679.46

D. $6324.00

Question 19

A car purchased for $32 000 depreciates at a rate of 15% per annum using the reducing-balance method. Its value after 3 years is

A. $17 600.00

B. $19 652.00

C. $16 704.20

D. $27 200.00

Question 20

A reducing-balance loan of $15 000 is charged interest at 9% per annum, compounding monthly. The monthly repayment is $700. The balance of the loan immediately after the first repayment is

A. $14 300.00

B. $14 887.50

C. $14 475.00

D. $14 412.50

Question 21

A savings account offers a nominal interest rate of 7.2% per annum, compounding monthly. The effective annual interest rate is closest to

A. 7.44%

B. 7.20%

C. 7.50%

D. 7.42%

Question 22

A loan is to be repaid by equal monthly repayments. A financial calculator (with the future value set to $0) shows that the number of repayments required is 41.7. The loan is fully repaid after

A. 41 repayments

B. 41.7 repayments

C. 42 repayments

D. 43 repayments

Question 23

A perpetuity is to pay a scholarship of $6000 each year, funded by an investment earning 4% per annum. The amount that must be invested is

A. $24 000

B. $60 000

C. $240 000

D. $150 000

Question 24

An amount of $2000 is invested in an account earning 6% per annum, compounding monthly. At the end of each month, immediately after interest is added, a further $200 is deposited. The balance at the end of the second month is

A. $2420.00

B. $2421.05

C. $2410.00

D. $2431.05

Question 25

Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 4 & 0 \\ 2 & 5 & -2 & 6 \\ 1 & 0 & 7 & -3 \end{bmatrix}.$$

The element a_{23} is

A. 5

B. 0

C. -2

D. -3

Question 26

Given $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$, the matrix $3A - B$ is

A. $\begin{bmatrix} 1 & -10 \\ 14 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 11 & -8 \\ 10 & 6 \end{bmatrix}$

C. $\begin{bmatrix} -3 & -4 \\ 6 & -2 \end{bmatrix}$

D. $\begin{bmatrix} -1 & -7 \\ 10 & -1 \end{bmatrix}$

Question 27

A shop's sales of three products over two days are given by the quantity matrix Q, and the selling prices (in dollars) by the price matrix P:

$$Q=\begin{bmatrix}10 & 5 & 8\\ 6 & 12 & 4\end{bmatrix}, P=\begin{bmatrix}3\\ 7\\ 2\end{bmatrix}.$$

The total revenue, in dollars, on the second day is

A. 81

B. 95

C. 104

D. 110

Question 28

The matrix

$$M=\begin{bmatrix}3 & k\\ k & 12\end{bmatrix}$$

has no inverse when

A. $k=6$ only

B. $k=6$ or $k=-6$

C. $k=-6$ only

D. $k=36$

Question 29

Each week the four members W, X, Y and Z of a relay team swim in an order recorded as a column matrix. This week's order is

$$S=\begin{bmatrix}W\\ X\\ Y\\ Z\end{bmatrix},$$

and next week's order is given by PS, where P is the permutation matrix

$$P=\begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}.$$

Reading from the top of the column matrix PS, next week's swimming order is

A. Y, W, Z, X

B. X, Z, W, Y

C. Y, W, X, Z

D. W, Y, X, Z

Question 30

In a call centre, four staff members P, Q, R and S can forward calls to one another. The matrix F below records the direct forwarding links, where a 1 in row i, column j means staff member i can forward a call directly to staff member j. The staff are taken in the order P, Q, R, S.

$$F = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The number of ways a call can be passed from P to S using exactly one intermediary (a two-step link) is

A. 1

B. 3

C. 2

D. 0

Question 31

The movement of residents each year between a town (T) and a nearby city (C) is described by the transition matrix

$$M = \begin{bmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{bmatrix},$$

where the rows and columns are in the order town (T), city (C). This year there are 600 people in the town and 400 in the city. After one year, the number of people in the town is

A. 480

B. 540

C. 510

D. 560

Question 32

A population of 1200 birds moves each week between two feeding sites, X and Y, according to the transition matrix

$$T = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix},$$

where the rows and columns are in the order X, Y. In the long term, the number of birds at site X is

A. 600

B. 960

C. 400

D. 800

Question 33

A connected planar graph has 8 vertices and 13 edges. When the graph is drawn in the plane, the number of faces (regions, including the infinite outer region) is

A. 5

B. 7

C. 21

D. 6

Question 34

The adjacency matrix of a graph with vertices P, Q, R and S (in that order for both rows and columns) is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The degree of vertex R is

A. 2

B. 3

C. 5

D. 4

Question 35

In a connected graph, the degrees of the five vertices are 2, 2, 3, 3 and 4. Which of the following statements is correct?

A. An Eulerian trail exists, but an Eulerian circuit does not.

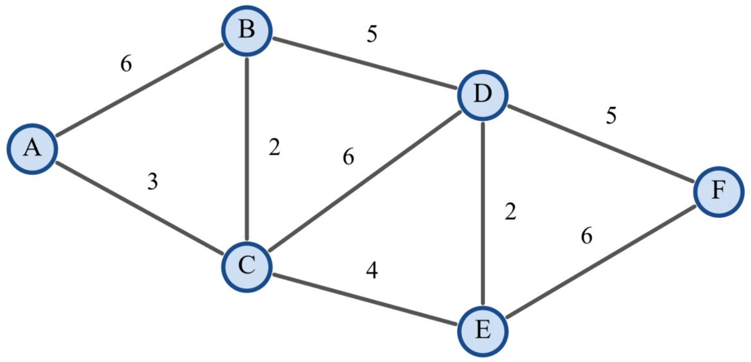
B. An Eulerian circuit exists.

C. Neither an Eulerian trail nor an Eulerian circuit exists.

D. The graph must contain a Hamiltonian cycle.

Question 36

The network below shows the distances, in kilometres, along the tracks between six checkpoints in a park.



The shortest distance from checkpoint A to checkpoint F is

A. 15

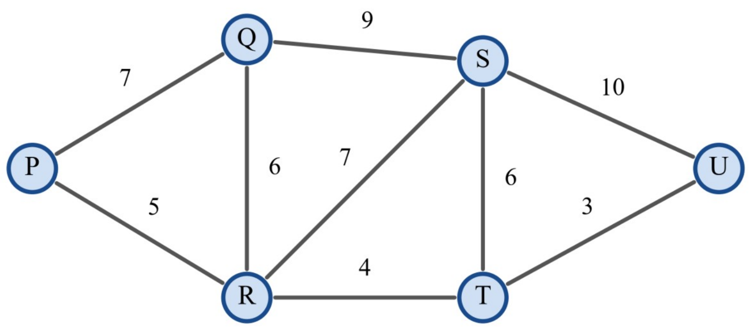
B. 14

C. 13

D. 12

Question 37

The weighted network below shows the cost, in thousands of dollars, of connecting six towns (P to U) by water pipes.



The minimum total cost, in thousands of dollars, of connecting all six towns is

A. 24

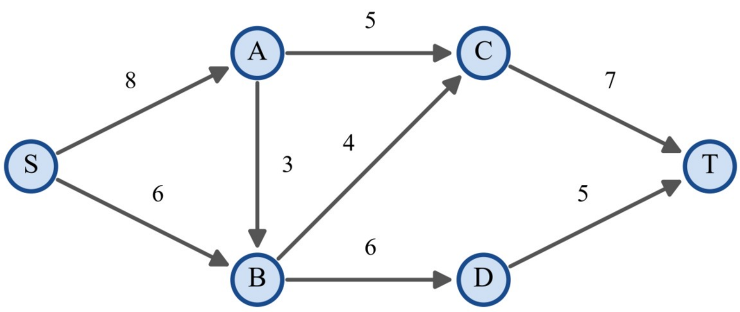
B. 25

C. 22

D. 27

Question 38

In the directed network below, the numbers on the arcs represent the maximum flow, in litres per second, that can pass through each pipe from the source S to the sink T.



The maximum flow from S to T is

A. 14

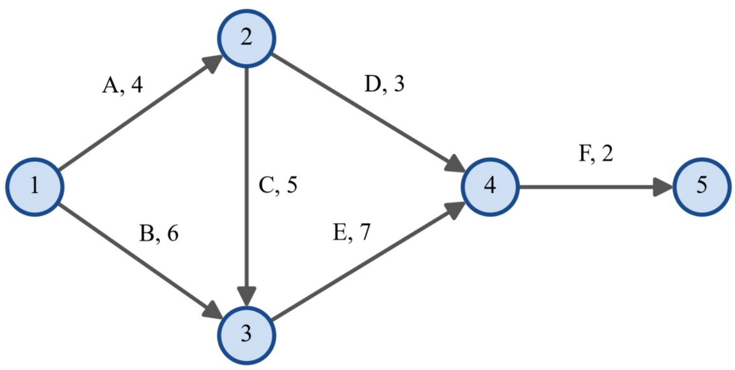
B. 11

C. 13

D. 12

Question 39

The activity network below shows the activities required to complete a project, with each arc labelled by its activity name and duration (in hours).



The minimum time required to complete the project is

A. 16

B. 18

C. 17

D. 20

Question 40

Three workers are each to be assigned to exactly one of three tasks. The time (in minutes) each worker takes on each task is shown in the table.

Worker	Task 1	Task 2	Task 3
W_1	9	6	12
W_2	7	5	8
W_3	10	9	7

The minimum total time to complete all three tasks is

A. 21

B. 18

C. 20

D. 24

End of Exam 2

Exam 2 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \bar{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + b x$
least squares slope	$b = \dfrac{r s_y}{s_x}$
least squares intercept	$a = \bar{y} - b \bar{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $u_0 = a, u_{n+1} = R u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100 n}\right)^n - 1\right] \times 100\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\ A = a d - b c$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\ A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\ A \neq 0$

recurrence relation	$S_0 = \text{initial state}, S_{n+1} = T S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}, S_{n+1} = L S_n$

Networks and decision mathematics

Euler's formula $v - e + f = 2$

End of Formula Sheet

Exam 2 — Answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Answer	C	A	D	A	B	D	C	A	B	D	C	B	A	C	B	D	A	C	B	D

Q	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Answer	A	C	D	B	C	A	D	B	A	C	B	D	B	D	A	C	A	D	B	C

Q1. C. A health rating recorded as poor/fair/good/excellent has a natural order, so it is an ordinal categorical variable; breed is nominal, and weight and the number of vaccinations are numerical.

Q2. A. Six of the 15 games had 7 or more home runs (values 7, 7, 8, 8, 9, 12), and $6/15=0.40=40\%$.

Q3. D. Here $Q_1=16$ and $Q_3=24$, so IQR $=8$ and the upper fence is $24+1.5\times 8=36$; since $45>36$, the value 45 is an outlier.

Q4. A. The mean is $8+11+5+14+7/5=9$; the sum of squared deviations is $1+4+16+25+4=50$, so $s=\sqrt{50/4}=3.54$ (dividing by $n-1$).

Q5. B. $495=500-1\,\text{sd}$ and $510=500+2\,\text{sd}$, so the percentage between is $68/2+95/2=34+47.5=81.5\%$.

Q6. D. The standardised scores are $z_{\text{M}}=74-62/8=1.5$ and $z_{\text{E}}=80-68/10=1.2$; the larger z-score (Mathematics) indicates the stronger relative performance.

Q7. C. There are $50+150=200$ males, of whom 150 prefer coffee: $150/200=75\%$.

Q8. A. $r=\pm\sqrt{0.81}=\pm 0.9$; because the association is negative, $r=-0.90$.

Q9. B. The slope 0.18 is measured in kilograms per week, so on average the weight increases by 0.18 kg for each additional week of age.

Q10. D. $r^2=0.6^2=0.36$; 36% of the variation in the response (test score) is explained by the variation in the explanatory variable (hours studied).

Q11. C. Predicted $\hat{y}=12.4+3.1\times 8=37.2$; residual = actual − predicted $=40-37.2=2.8$.

Q12. B. $x=30$ lies outside the data range 5 to 20, so the prediction is an extrapolation and may be unreliable.

Q13. A. A curve that increases at an increasing rate like a quadratic is straightened by squaring the explanatory variable ($s\to s^2$).

Q14. C. $\log_{10}(y)=0.8+0.15\times 6=1.7$, so $y=10^{1.7}\approx 50$.

Q15. B. The mean quarterly sales are $88+92+120+100/4=100$, so the Quarter 3 index is $120/100=1.20$.

Q16. D. The five-point moving mean centred on the fourth day is $22+20+26+24+30/5=122/5=24.4$.

Q17. A. $V_1=2(4)+3=11$, $V_2=2(11)+3=25$, $V_3=2(25)+3=53$.

Q18. C. The quarterly rate is $5.4\%/4=1.35\%$ over $4\times 2=8$ periods: $6000\times 1.0135^8=\$6679.46$.

Q19. B. Reducing-balance depreciation gives $32\,000\times 0.85^3=\$19\,652.00$.

Q20. D. The monthly rate is $9\%/12=0.75\%$, so the first month's interest is $0.0075\times 15\,000=\$112.50$; the principal reduction is $700-112.50=\$587.50$, leaving $15\,000-587.50=\$14\,412.50$.

Q21. A. The effective annual rate is $(1+0.072/12)^{12}-1=0.0744=7.44\%$.

Q22. C. After 41 full repayments the loan is not yet cleared, so a 42nd (smaller) repayment is required; the loan is fully repaid after 42 repayments.

Q23. D. For a perpetuity, principal $=\frac{\text{payment}}{\text{rate}}=\frac{6000}{0.04}=\$150\,000$.

Q24. B. Month 1: $2000+0.005\times 2000+200=\$2210.00$. Month 2: $2210+0.005\times 2210+200=2210+11.05+200=\$2421.05$.

Q25. C. The element a_{23} is in row 2, column 3, which is -2.

Q26. A. $3A=\begin{bmatrix} 6 & -9 \\ 12 & 3 \end{bmatrix}$, so $3A-B=\begin{bmatrix} 6-5 & -9-1 \\ 12-(-2) & 3-3 \end{bmatrix}=\begin{bmatrix} 1 & -10 \\ 14 & 0 \end{bmatrix}$.

Q27. D. The second day's revenue is $6\times 3+12\times 7+4\times 2=18+84+8=110$.

Q28. B. $det(M)=3\times 12-k\times k=36-k^2$, which is 0 when $k^2=36$, that is, when $k=6$ or $k=-6$.

Q29. A. Row 1 of P has its 1 in column 3, so the first entry of PS is Y; row 2 has its 1 in column 1, giving W; row 3 has its 1 in column 4, giving Z; and row 4 has its 1 in column 2, giving X. Hence PS lists the order Y, W, Z, X.

Q30. C. The two-step links from P to S are $P\rightarrow Q\rightarrow S$ and $P\rightarrow R\rightarrow S$; equivalently $(F^2)_{PS}=2$ (the direct link $P\rightarrow S$ is not a two-step link and is not counted).

Q31. B. After one year the town has $0.8\times 600+0.15\times 400=480+60=540$ people.

Q32. D. The steady-state proportions satisfy $0.2x=0.4y$ with $x+y=1$, giving $x=2/3$; so $2/3\times 1200=800$ birds at site X.

Q33. B. By Euler's formula $v-e+f=2$, so $f=2-v+e=2-8+13=7$.

Q34. D. The off-diagonal entries in row R sum to $1+1+0=2$, and the diagonal 1 is a loop that adds 2 to the degree, so the degree of R is $2+2=4$.

Q35. A. Exactly two vertices have odd degree (the two of degree 3), so an Eulerian trail exists but an Eulerian circuit (which needs all even degrees) does not.

Q36. C. The shortest route is $A\rightarrow C\rightarrow E\rightarrow F=3+4+6=13$ km (shorter than $A\rightarrow C\rightarrow D\rightarrow F=14$).

Q37. A. The minimum spanning tree uses edges $TU(3)$, $RT(4)$, $PR(5)$, $QR(6)$ and $ST(6)$, giving a total of $3+4+5+6+6=24$.

Q38. D. The minimum cut is the two arcs into the sink, $C\rightarrow T$ and $D\rightarrow T$, with capacity $7+5=12$, so the maximum flow is 12.

Q39. B. The critical (longest) path is $A\rightarrow C\rightarrow E\rightarrow F=4+5+7+2=18$ hours.

Q40. C. The optimal allocation is $W_1\rightarrow$ Task 2 (6), $W_2\rightarrow$ Task 1 (7) and $W_3\rightarrow$ Task 3 (7), giving a minimum total of $6+7+7=20$ minutes (checking all six allocations, the next best is 21).

Exam 3 — Multiple-choice (Hard)

*Reading time: 15 minutes. Writing time: 1 hour 30 minutes. 40 multiple-choice questions, 40 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Choose the response that is **correct** for the question. Unless otherwise indicated, diagrams are not drawn to scale.*

Section A — Multiple-choice

Question 1

In a Physics test the marks are approximately normally distributed with a mean of 62 and a standard deviation of 5; Ann scored 70. In a Chemistry test the marks are approximately normally distributed with a mean of 70 and a standard deviation of 8; Ben scored 82. Comparing their standardised scores, which statement is correct?

A. Ben performed better relative to his class, because his mark of 82 was higher than Ann's mark of 70.

B. They performed equally well relative to their classes.

C. Ann performed better relative to her class, because her standardised score of 1.6 exceeds Ben's standardised score of 1.5.

D. Ben performed better relative to his class, because his standardised score of 1.6 exceeds Ann's standardised score of 1.5.

Question 2

The masses of bags of flour are approximately normally distributed with a mean of 500 grams and a standard deviation of 25 grams. In a batch of 2000 bags, the number of bags with a mass between 475 grams and 550 grams is closest to

A. 1360

B. 1630

C. 1900

D. 1980

Question 3

The populations of a group of country towns range from a few hundred to well over a million, so they are displayed on a $\log_{10}$ scale. On this scale the population of Ashford is recorded as 4.7 and the population of Brenton as 2.7. The population of Ashford is

A. 2 times the population of Brenton

B. 1.74 times the population of Brenton

C. 20 times the population of Brenton

D. 100 times the population of Brenton

Question 4

The grouped frequency table below shows the daily rainfall, in millimetres, recorded at a weather station on 40 days.

Rainfall (mm)	Frequency
$0\text{–}<10$	4
$10\text{–}<20$	9
$20\text{–}<30$	12

Rainfall (mm)	Frequency
30–< 40	8
40–< 50	7

Using the interval midpoints, the estimated mean daily rainfall, in millimetres, is

A. 26.25

B. 25.50

C. 27.50

D. 30.00

Question 5

The two-way frequency table below classifies 250 people by age group and whether or not they own a smartwatch.

	Under 40	40 or over	Total
Owns a smartwatch	132	60	192
Does not own one	18	40	58
Total	150	100	250

Of the people who own a smartwatch, the percentage who are aged under 40 is closest to

A. 52.8 %

B. 88.0 %

C. 68.75 %

D. 31.25 %

Question 6

Across 40 suburbs, a strong positive association ($r = 0.86$) is found between the number of public library branches in a suburb and the number of bicycle thefts reported in that suburb. Which one of the following is the most appropriate conclusion?

A. The association is likely to be explained by a common cause, such as the size of the suburb's population, so it does not show that opening library branches causes bicycle thefts.

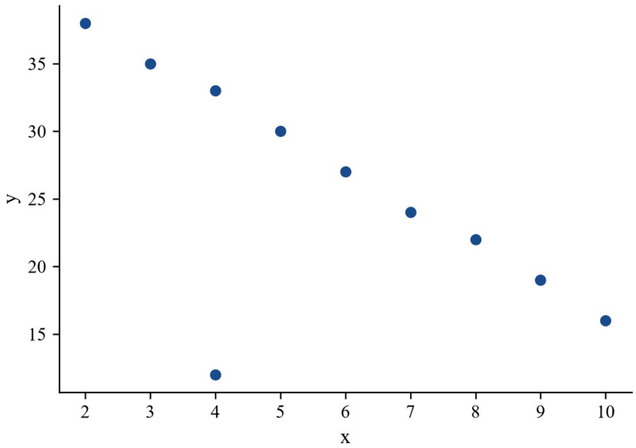
B. Because r is close to 1, opening more library branches causes more bicycle thefts.

C. Because the data were collected by observation rather than by experiment, no association between the two variables can be claimed.

D. Because the association is positive, the bicycle thefts must be causing the suburbs to open more library branches.

Question 7

The scatterplot below shows the values of two numerical variables x and y recorded for ten observations.



Which one of the following statements best describes the association shown?

A. There is a strong positive linear association with no outliers.

B. There is a moderate positive linear association.

C. There is no association between x and y.

D. There is a negative, approximately linear association, with one outlier.

Question 8

Over 30 days, the correlation coefficient between the number of hours of sunlight (the explanatory variable) and the energy generated by a solar panel (the response variable) is $r = 0.92$. Which one of the following is a correct interpretation of the coefficient of determination?

A. 92% of the variation in the energy generated is explained by the variation in the hours of sunlight.

B. 84.6% of the variation in the energy generated is explained by the variation in the hours of sunlight.

C. 84.6% of the variation in the hours of sunlight is explained by the variation in the energy generated.

D. 84.6% of the days had their energy generation predicted correctly.

Question 9

For a set of bivariate data, the summary statistics are $\bar{x} = 20$, $\bar{y} = 50$, $s_x = 5$, $s_y = 15$ and $r = 0.8$. The equation of the least-squares regression line $\hat{y} = a + b\,x$ is

A. $\hat{y} = 2 + 2.4\,x$

B. $\hat{y} = 44.7 + 0.27\,x$

C. $\hat{y} = 2 - 2.4\,x$

D. $\hat{y} = 98 + 2.4\,x$

Question 10

A least-squares regression line is $\hat{y}=3.5+0.8x$. For the data point at $x=15$, the observed value is $y=14$. The residual for this point is

A. 1.5

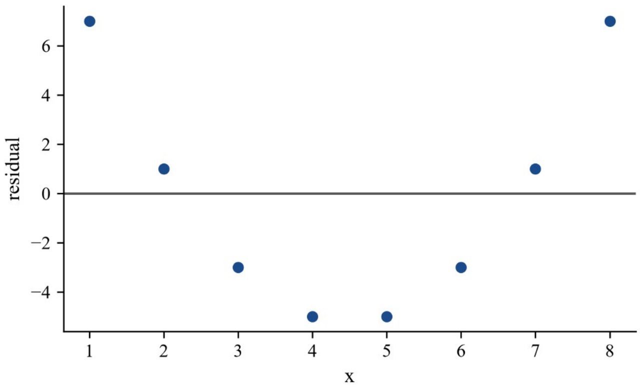
B. 15.5

C. -1.5

D. -0.5

Question 11

A least-squares line is fitted to a set of bivariate data and has coefficient of determination $r^2=0.94$. The residual plot for the fit is shown below.



Which one of the following conclusions is best supported?

A. The residuals are randomly scattered, so the linear model is appropriate.

B. The high value of r^2 confirms that the linear model fits the data well.

C. The residuals show a linear trend, so a second least-squares line should be added.

D. The residuals show a clear curved (U-shaped) pattern, so a linear model is not appropriate and a transformation should be applied.

Question 12

A $\log_{10}$ transformation applied to the response variable gives the least-squares line

$$\log_{10}(y)=0.6+0.18x.$$

Using this model, the value of y predicted when $x=8$ is closest to

A. 2.04

B. 110

C. 55

D. 1096

Question 13

A reciprocal transformation applied to the response variable gives the least-squares line

$$\frac{1}{y}=0.04+0.012x.$$

Using this model, the value of y predicted when $x=5$ is

A. 10

B. 0.10

C. 100

D. 8.3

Question 14

A scatterplot of y against x is curved. The table below gives the coefficient of determination r^2 obtained when a least-squares line is fitted after each of four possible transformations.

Transformation	$x \to x^2$	$x \to \frac{1}{x}$	$x \to \log_{10}(x)$	none
r^2	0.76	0.92	1.00	0.91

The transformation that best linearises the association is

A. $x \to x^2$

B. $x \to \frac{1}{x}$

C. $x \to \log_{10}(x)$

D. no transformation

Question 15

A store's quarterly sales have the four seasonal indices 0.80, 1.10, ? and 1.30 for Quarters 1 to 4 respectively. The actual sales in Quarter 3 were $\$46\,000$. The deseasonalised Quarter 3 sales figure is

A. $\$36\,800$

B. $\$46\,000$

C. $\$50\,000$

D. $\$57\,500$

Question 16

The quarterly sales (in thousands of dollars) of a business over six consecutive quarters were

$$42, 48, 60, 50, 46, 54.$$

The four-point centred moving average for Quarter 3 is

A. 50.0

B. 50.5

C. 51.0

D. 54.5

Question 17

A sequence is generated by the first-order recurrence relation

$$u_0 = 8, u_{n+1} = 1.25u_n - 3.$$

The value of u_3 is

A. 4.1875

B. 5.75

C. 7

D. 3.19

Question 18

A machine is purchased for $\$24\,000$ and depreciates by 15% of its value each year (reducing-balance depreciation). The number of complete years after which its value first falls below half of its purchase price is

A. 3

B. 4

C. 5

D. 6

Question 19

An account pays interest at 6% per annum, compounding monthly. The effective annual interest rate is closest to

A. 6.00%

B. 6.09%

C. 6.14%

D. 6.17%

Question 20

A loan of $\$ 20\,000$ is charged interest at $8.4\,\%$ per annum, compounding monthly (that is, $0.7\,\%$ per month), and is repaid with monthly instalments of $\$ 850$. Interest is rounded to the nearest cent each month. The balance owing immediately after the third repayment is

A. $\$ 19\,290.00$

B. $\$ 17\,855.06$

C. $\$ 18\,575.03$

D. $\$ 17\,450.00$

Question 21

$\$ 5000$ is invested in an account earning $0.5\,\%$ interest per month, and a deposit of $\$ 300$ is added at the end of each month. Interest is rounded to the nearest cent each month. The balance in the account immediately after the third deposit is

A. $\$ 5900.00$

B. $\$ 5651.63$

C. $\$ 5979.89$

D. $\$ 6900.00$

Question 22

A loan of $\$ 30\,000$ is charged interest at $7.2\,\%$ per annum, compounding monthly, and is to be repaid in equal monthly instalments over 5 years (60 payments). The monthly repayment, rounded up to the nearest cent, is

A. $\$ 596.88$

B. $\$ 596.87$

C. $\$ 500.00$

D. $\$ 600.00$

Question 23

A scholarship pays $\$ 600$ at the end of each quarter, indefinitely, from a fund that earns $4.8\,\%$ per annum, compounding quarterly. The amount that must be invested in the fund is

A. $\$ 12\,500$

B. $\$ 50\,000$

C. $\$ 60\,000$

D. $\$ 500\,000$

Question 24

$\$10\,000$ is invested at 5% per annum, compounding annually. The least number of complete years for the investment to first exceed $\$13\,000$ is

A. 4

B. 5

C. 7

D. 6

Question 25

Consider the matrices

$$A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}, B=\begin{bmatrix}5 & 2\\1 & -3\end{bmatrix}.$$

The element in row 2, column 1 of the product AB is

A. 9

B. 13

C. 19

D. -6

Question 26

A drinks stall operates at two outlets. The matrix Q gives the number of units of three items sold at each outlet, and the matrix P gives the selling price, in dollars, of each item.

$$Q=\begin{bmatrix}20 & 15 & 30\\25 & 10 & 18\end{bmatrix}, P=\begin{bmatrix}6\\8\\4\end{bmatrix}.$$

The total takings at the second outlet are

A. $\$360$

B. $\$302$

C. $\$662$

D. $\$306$

Question 27

The inverse of the matrix

$$A=\begin{bmatrix}4 & 3\\ 3 & 2\end{bmatrix}$$

is

A. $\begin{bmatrix}2 & -3\\ -3 & 4\end{bmatrix}$

B. $\begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$

C. $\begin{bmatrix}-2 & -3\\ -3 & -4\end{bmatrix}$

D. $\begin{bmatrix}-2 & 3\\ 3 & -4\end{bmatrix}$

Question 28

The matrix equation

$$\begin{bmatrix}3 & 2\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12\\ 5\end{bmatrix}$$

has solution

A. $x=2, y=3$

B. $x=3, y=2$

C. $x=-22, y=-33$

D. $x=2, y=-3$

Question 29

Four players W, X, Y, Z each play every other player once, with no draws. The dominance matrix below records a 1 where the row player defeats the column player, with the players taken in the order W, X, Y, Z.

$$D=\begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}$$

Ranking the players using the sum of one-step and two-step dominances (that is, the row sums of $D+D^2$), from strongest to weakest, gives

A. W, X, Y, Z

B. W, Y, X, Z

C. Y, W, X, Z

D. Y, X, W, Z

Question 30

Commuters choose between train (T) and car (C) each week according to the transition matrix

$$\begin{array}{cc} T & C \\ \left[\begin{matrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{matrix}\right. & \left.\right] \end{array}\begin{matrix} T \\ C \end{matrix}$$

If this week 600 commuters travel by train and 400 by car, the numbers travelling by train and by car two weeks later are

A. 700 by train, 300 by car

B. 750 by train, 250 by car

C. 800 by train, 200 by car

D. 775 by train, 225 by car

Question 31

For the transition matrix in Question 30, with a total of 1000 commuters, the long-run (steady-state) numbers travelling by train and by car are

A. 600 by train, 400 by car

B. 750 by train, 250 by car

C. 700 by train, 300 by car

D. 800 by train, 200 by car

Question 32

A population of animals is divided into three age classes. Its Leslie matrix and initial state are

$$L=\begin{bmatrix} 0 & 3 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}, S_0=\begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}.$$

The number of animals in the youngest age class after two years (that is, in S_2) is

A. 150

B. 260

C. 130

D. 290

Question 33

The walking tracks joining five huts J, K, L, M and N are represented by the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

with the huts taken in the order J, K, L, M, N. The number of walks of length 3 from J to N is

A. 1

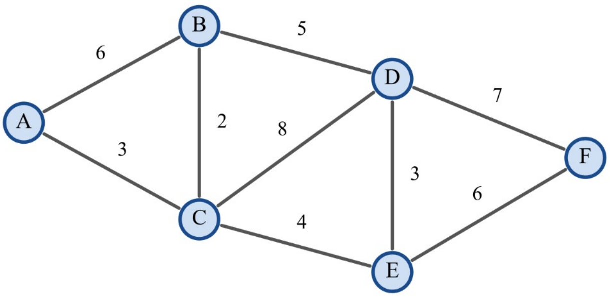
B. 3

C. 2

D. 6

Question 34

For the weighted graph shown below, the length of the shortest path from A to F is



A. 11

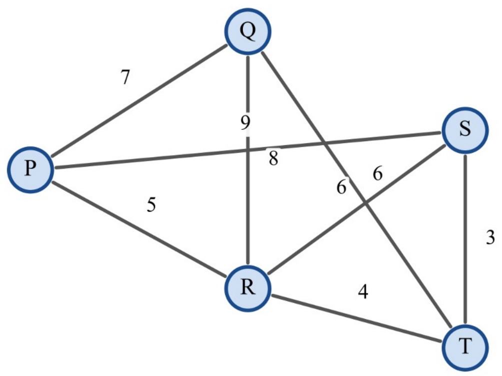
B. 12

C. 13

D. 16

Question 35

The network below shows the possible connections between five sites, with the number on each edge its weight. The total weight of the minimum spanning tree of this network is



A. 18

B. 20

C. 22

D. 15

Question 36

A connected graph has the degree sequence $2, 2, 2, 4, 4, 4$. Which one of the following statements is true?

A. The graph has an Eulerian trail but not an Eulerian circuit.

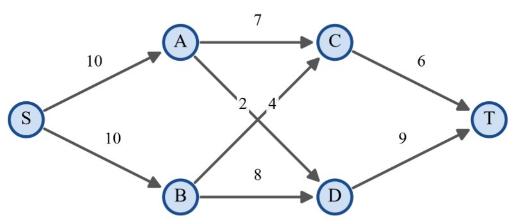
B. The graph has 18 edges.

C. The graph must contain a bridge.

D. The graph has an Eulerian circuit.

Question 37

For the directed network below, the number on each arc is the capacity of that arc. The maximum flow from the source S to the sink T is



A. 13

B. 15

C. 20

D. 21

Question 38

The table shows the cost, in dollars, for each of four workers to complete each of four tasks. Each worker is assigned exactly one task, and each task is done by exactly one worker.

Worker	Task 1	Task 2	Task 3	Task 4
A	14	9	12	11
B	8	13	10	14
C	12	10	9	13
D	11	8	12	10

The minimum total cost of completing all four tasks is

A. $\$34$

B. $\$35$

C. $\$36$

D. $\$38$

Question 39

The precedence table below shows the activities in a project, their durations and their immediate predecessors.

Activity	Duration (hours)	Immediate predecessor(s)
A	4	—
B	6	—
C	5	A
D	3	A
E	4	B, C
F	2	D
G	5	E, F

The float (slack) of activity D, in hours, is

A. 4

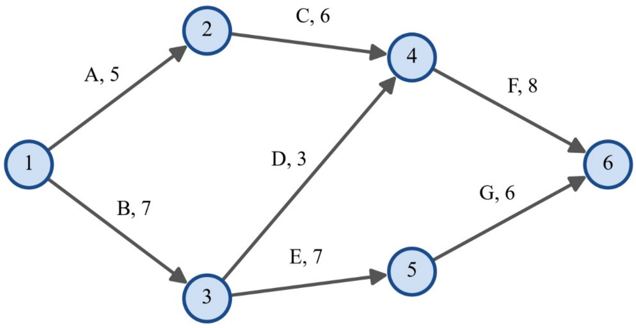
B. 0

C. 2

D. 3

Question 40

The activity network below shows a project; each arc is an activity labelled with its letter and its duration in hours. Every activity except D can be crashed (shortened) by at most 2 hours. The cost of crashing, per hour, is A $\$30$, B $\$60$, C $\$45$, E $\$55$, F $\$70$ and G $\$40$.



The minimum extra cost of reducing the completion time of the project by 2 hours is

A. $\$80$

B. $\$130$

C. $\$70$

D. $\$110$

End of Exam 3

Exam 3 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \overline{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + b\,x$
least squares slope	$b = \dfrac{r\,s_y}{s_x}$
least squares intercept	$a = \overline{y} - b\,\overline{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $\quad u_0 = a\,, u_{n+1} = R\,u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100\,n}\right)^n - 1\right] \times 100\,\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\ A = a\,d - b\,c$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\ A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\ A \neq 0$

recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = T\,S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = L\,S_n$

Networks and decision mathematics

Euler's formula $\quad v - e + f = 2$

End of Formula Sheet

Exam 3 — Answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans	C	B	D	A	C	A	D	B	A	C	D	B	A	C	D	B	A	C	D	B

Q	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans	C	A	B	D	C	B	D	A	C	B	D	A	B	C	A	D	B	C	A	D

Q1. C — the standardised scores are Ann $z=\frac{70-62}{5}=1.6$ and Ben $z=\frac{82-70}{8}=1.5$; Ann's higher z means she did better relative to her class.

Q2. B — $475=500-1\,\text{sd}$ and $550=500+2\,\text{sd}$, so the percentage is $34+47.5=81.5\,\%$; $0.815\times 2000=1630$ bags.

Q3. D — a reading of 4.7 means a population of $10^{4.7}$ and a reading of 2.7 means $10^{2.7}$, so the ratio is $\frac{10^{4.7}}{10^{2.7}}=10^{2}=100$; on a $\log_{10}$ scale a difference of 2 is a factor of 100, not a difference or ratio of the readings themselves.

Q4. A — using midpoints $5,15,25,35,45$: mean $=\frac{4(5)+9(15)+12(25)+8(35)+7(45)}{40}=\frac{1050}{40}=26.25$.

Q5. C — there are 192 smartwatch owners, of whom 132 are under 40: $\frac{132}{192}=68.75\,\%$ (not $\frac{132}{150}$, which is the reverse conditional).

Q6. A — a strong correlation never establishes cause and effect on its own; both counts rise with the population of the suburb, which is a common cause (a confounding variable). Observational data can still establish an association, so C is wrong, and there is no reason to prefer the reversed claim in D.

Q7. D — nine points fall on a clear downward (negative) line, while the point $(4,12)$ sits far below that trend, making it an outlier.

Q8. B — $r^2=0.92^2=0.8464$; $84.6\,\%$ of the variation in the response (energy) is explained by the variation in the explanatory variable (sunlight).

Q9. A — $b=r\frac{s_y}{s_x}=0.8\times\frac{15}{5}=2.4$ and $a=\acute{y}-b\acute{x}=50-2.4\times 20=2$, giving $\hat{y}=2+2.4x$.

Q10. C — predicted $\hat{y}=3.5+0.8\times 15=15.5$; residual = actual − predicted $=14-15.5=-1.5$.

Q11. D — a high r^2 does not guarantee a good fit; the clear U-shaped pattern in the residuals shows the association is non-linear, so a transformation is needed.

Q12. B — $\log_{10}(y)=0.6+0.18\times 8=2.04$, so $y=10^{2.04}\approx 110$.

Q13. A — $\frac{1}{y}=0.04+0.012\times 5=0.10$, so $y=\frac{1}{0.10}=10$.

Q14. C — the $\log_{10}(x)$ transformation gives $r^2=1.00$, the closest to 1 of the four options, so it best straightens the association.

Q15. D — the four indices sum to 4, so the missing index is $4-0.80-1.10-1.30=0.80$; deseasonalised $=\frac{46\,000}{0.80}=\$57\,500$.

Q16. B — the four-point centred moving average is $\frac{1/2(42)+48+60+50+1/2(46)}{4}=\frac{202}{4}=50.5$.

Q17. A — $u_1=1.25(8)-3=7$, $u_2=1.25(7)-3=5.75$, $u_3=1.25(5.75)-3=4.1875$.

Q18. C — half of $\$24\,000$ is $\$12\,000$; the reducing-balance values are $\$12\,528.15$ after 4 years and $\$10\,648.93$ after 5 years, so the value first drops below $\$12\,000$ after 5 years.

Q19. D — the effective annual rate is $\left(1+\frac{0.06}{12}\right)^{12}-1=0.0617\ldots$, i.e. about 6.17%.

Q20. B — with 0.7% monthly interest and $\$850$ repayments the balances are $\$19\,290.00$, $\$18\,575.03$ then $\$17\,855.06$ after the third repayment (option D is what is left if the interest is ignored altogether).

Q21. C — each month the interest is 0.5% of the balance and $\$300$ is then added:
$\$5000+\$25.00+\$300=\$5325.00$, then $\$5325.00+\$26.63+\$300=\$5651.63$, then
$\$5651.63+\$28.26+\$300=\$5979.89$ after the third deposit.

Q22. A — the annuity payment is $\text{PMT}=\frac{PV\cdot r\cdot R^N}{R^N-1}=\$596.8708\ldots$, which rounds up to $\$596.88$.

Q23. B — the quarterly rate is $\frac{4.8\%}{4}=1.2\%$; a perpetuity needs principal $=\frac{600}{0.012}=\$50\,000$.

Q24. D — $1.05^5=1.2763<1.3$ but $1.05^6=1.3401>1.3$, so $\$10\,000$ first exceeds $\$13\,000$ after 6 years.

Q25. C — the $(2,1)$ entry of AB is (row 2 of A) $\cdot$ (column 1 of B) $=3\times5+4\times1=19$.

Q26. B — the second-outlet takings are $25\times6+10\times8+18\times4=150+80+72=\$302$.

Q27. D — $\det A=4\times2-3\times3=-1$, so $A^{-1}=\frac{1}{-1}\begin{bmatrix}2&-3\\-3&4\end{bmatrix}=\begin{bmatrix}-2&3\\3&-4\end{bmatrix}$; option A omits the factor $\frac{1}{\det A}$.

Q28. A — $\det=3(-1)-2(4)=-11\neq0$, so the inverse exists and gives $x=2$, $y=3$ (check $3(2)+2(3)=12$ and $4(2)-3=5$); option C is the result of omitting the factor $\frac{1}{\det}$.

Q29. C — the row sums of $D+D^2$ are $W\,4, X\,3, Y\,5, Z\,2$; ranking from highest gives Y, W, X, Z (the one-step tie between W and Y is broken by two-step dominances).

Q30. B — $S_1=\begin{bmatrix}0.9(600)+0.4(400)\\0.1(600)+0.6(400)\end{bmatrix}=\begin{bmatrix}700\\300\end{bmatrix}$ and $S_2=\begin{bmatrix}0.9(700)+0.4(300)\\0.1(700)+0.6(300)\end{bmatrix}=\begin{bmatrix}750\\250\end{bmatrix}$ (option A stops after one week, option D goes on to a third).

Q31. D — solving $\begin{bmatrix}0.9&0.4\\0.1&0.6\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$ with $a+b=1000$ gives $0.1a=0.4b$, so $a=800$ train, $b=200$ car.

Q32. A — $S_1=\begin{bmatrix}260\\40\\15\end{bmatrix}$ and $S_2=LS_1=\begin{bmatrix}3(40)+2(15)\\0.5(260)\\0.25(40)\end{bmatrix}=\begin{bmatrix}150\\130\\10\end{bmatrix}$, so the youngest class has 150.

Q33. B — the number of walks of length 3 from J to N is the (J,N) entry of A^3, which is 3; the walks are $J-K-L-N$, $J-K-M-N$ and $J-L-M-N$ (option A is the single walk of length 2, $J-L-N$).

Q34. C — Dijkstra from A gives distances $A\,0, C\,3, B\,5, E\,7, D\,10, F\,13$; the shortest path is $A-C-E-F$ of length $3+4+6=13$.

Q35. A — the minimum spanning tree uses edges $S-T(3), R-T(4), P-R(5), Q-T(6)$, total $3+4+5+6=18$.

Q36. D — every vertex has even degree and the graph is connected, so an Eulerian circuit exists; A is wrong because that circuit is itself a closed Eulerian trail. The degree sum $2+2+2+4+4+4=18$ is twice the number of edges, so the graph has 9 edges, not 18, and every edge of a graph with an Eulerian circuit lies on a cycle, so there is no bridge.

Q37. B — every unit of flow into T uses $C-T(6)$ or $D-T(9)$, capping flow at $6+9=15$; this flow is achievable, so the maximum flow is 15 (not the source-cut sum of 20).

Q38. C — the optimal allocation is $A \to 2, B \to 1, C \to 3, D \to 4$ with total $9+8+9+10=36$ (the greedy sum of row minima, 34, is infeasible as A and D both want Task 2).

Q39. A — the critical path is $A-C-E-G$ of length 18; activity D has earliest start 4 and latest start 8, so its float is $8-4=4$ hours.

Q40. D — the critical path $B-E-G(20)$ must lose 2 hours (crash G by 2: $2 \times \$40 = \$80$) and $A-C-F(19)$ must lose 1 hour (crash A by 1: $\$30$), for a minimum extra cost of $\$110$.

Exam 4 — Short-answer (Easy)

Reading time: 15 minutes. Writing time: 1 hour 30 minutes. Answer all questions. Total 60 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Data analysis

Question 1 (6 marks)

On one morning at a suburban medical clinic, the waiting time, in minutes, was recorded for each of the 11 patients seen. The values, in order, are

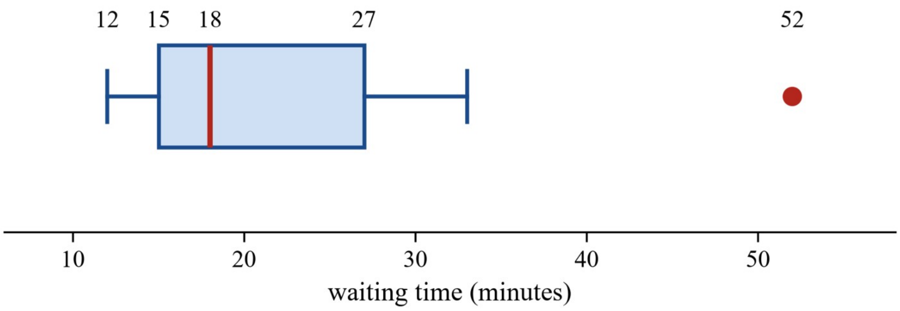
$$12, 14, 15, 16, 17, 18, 20, 23, 27, 33, 52.$$

a. Write down the five-number summary for this data. 2 marks

b. Determine the interquartile range (IQR) and the upper fence, and hence show that the waiting time of 52 minutes is an outlier.

2 marks

The data is displayed in the boxplot below, with the outlier shown separately.



c. Describe the shape of the distribution, referring to any outliers. 2 marks

Question 2 (6 marks)

The time taken by junior swimmers to complete one lap of a pool is approximately normally distributed with a mean of 42 seconds and a standard deviation of 3 seconds.

a. Find the standardised value (z-score) of a lap time of 48 seconds. 1 mark

b. A lap time has a standardised value of $z = -1$. Find this lap time, in seconds. 1 mark

c. Using the 68–95–99.7 % rule, find the percentage of lap times between 36 seconds and 48 seconds. 2 marks

d. Using the 68–95–99.7 % rule, find the percentage of lap times greater than 45 seconds. 2 marks

Question 3 (6 marks)

A survey of 200 commuters recorded their preferred type of transport (car or public transport) and where they live (inner city or suburbs). The results are shown in the two-way frequency table below.

Preferred transport	Inner city	Suburbs	Total
Car	30	90	120
Public transport	50	30	80
Total	80	120	200

Preferred transport	Inner city	Suburbs	Total

a. What percentage of the commuters surveyed live in the suburbs? 1 mark

b. Of the commuters who live in the inner city, what percentage prefer public transport? 2 marks

c. By comparing the percentage of inner-city commuters who prefer public transport with the percentage of suburban commuters who prefer public transport, determine whether there is an association between preferred transport and where a commuter lives. Justify your answer. 3 marks

Question 4 (6 marks)

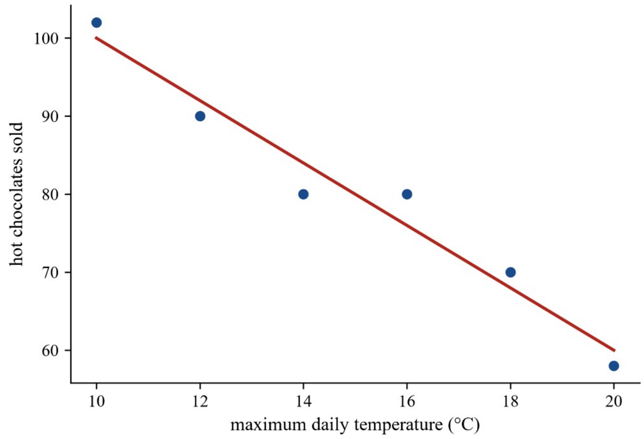
A school canteen recorded the maximum daily temperature, in °C, and the number of hot chocolates sold, on each of six winter days.

Maximum temperature (°C)	10	12	14	16	18	20
Hot chocolates sold	102	90	80	80	70	58

The equation of the least-squares line, found using technology, is

$$\text{sold} = 140 - 4 \times \text{temperature}.$$

The data and this line are shown in the scatterplot below.



a. Interpret the slope of the least-squares line in terms of maximum temperature and the number of hot chocolates sold. 2 marks

b. Use the equation to predict the number of hot chocolates sold on a day with a maximum temperature of 16 °C. 2 marks

c. On the day in the data set with a maximum temperature of 16 °C, 80 hot chocolates were actually sold. Calculate the residual for this day. 2 marks

Recursion and financial modelling

Question 5 (6 marks)

Priya invests $4,000 in an account earning compound interest at a rate of 0.5 % per month. The value of this investment after n months, V_n dollars, is modelled by the recurrence relation

$$V_0 = 4000, V_{n+1} = 1.005V_n.$$

Interest is rounded to the nearest cent each month.

a. Use the recurrence relation to find the value of the investment after 1 month and after 2 months. 2 marks

In a second account, Priya invests $4,000 at the same interest rate of 0.5% per month, but she also adds $150 to the account at the end of each month.

b. Write down a recurrence relation, in terms of W_{n+1} and W_n, that models the value of the second investment after n months. 2 marks

c. Find the value of the second investment after 2 months. 2 marks

Question 6 (6 marks)

Sam borrows $8,000, to be repaid with interest charged at 1% per month on the reducing balance. Sam makes a repayment of $700 at the end of each month. The first row of the amortisation table is shown below.

Payment number	Payment ($)	Interest ($)	Principal reduction ($)	Balance ($)
0	0.00	0.00	0.00	8000.00
1	700.00			
2	700.00			

a. Show that the interest charged in the first month is $80.00, and find the balance owing after the first payment. 2 marks

b. Calculate the interest, the principal reduction and the balance owing for payment number 2. 3 marks

c. State the total amount paid over the first two months, and the total reduction in the balance owing over those two months. 1 mark

Matrices

Question 7 (6 marks)

Consider the matrices

$$A=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}5 & 0\\ 1 & 2\end{bmatrix}.$$

a. Write down the order of matrix A. 1 mark

b. Find $A+B$. 1 mark

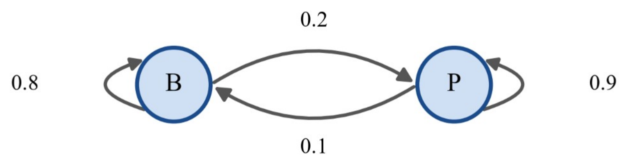
c. Find $3A$. 1 mark

d. Find the matrix product AB. 2 marks

e. Find $det(A)$, the determinant of A. 1 mark

Question 8 (6 marks)

A music streaming app has 1200 subscribers. Each month, every subscriber is on either the Basic plan (B) or the Premium plan (P). The transition diagram below shows how subscribers change plans from one month to the next.



This information is summarised by the transition matrix

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix},$$

where the columns and rows are taken in the order B, P. In the first month there are 900 subscribers on the Basic plan and 300 on the Premium plan, so the initial state matrix is $S_0 = \begin{bmatrix} 900 \\ 300 \end{bmatrix}$.

a. State what the value 0.2 in the transition matrix represents. 1 mark

b. Find the state matrix $S_1 = T\,S_0$, and hence state the number of subscribers on each plan after one month. 2 marks

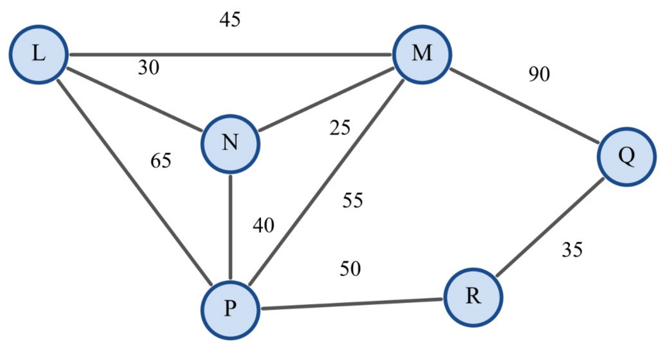
c. Find the state matrix S_2. 2 marks

d. In the long run, the number of subscribers on each plan settles to a steady state. State the number of subscribers on the Premium plan in the steady state. 1 mark

Networks and decision mathematics

Question 9 (6 marks)

A university campus has six buildings, L, M, N, P, Q and R, joined by paved walkways. The weighted graph below shows these walkways. Each number gives the length of that walkway, in metres.



a. Write down the degree of vertex N and the total number of edges in the graph. 2 marks

b. A security guard starts at building L and walks along every walkway exactly once. Explain why the guard cannot finish back at building L, and state the building at which this walk must finish. 2 marks

c. Lighting cable is to be laid along some of the walkways so that every building is connected, using the least possible total length of cable. State the walkways used and find the total length of cable required. 2 marks

Question 10 (6 marks)

A council is upgrading a local playground. The six activities involved are labelled A to F. The table below shows what each activity is, how long it takes, in days, and the activities that must be finished immediately before it can start.

Activity	Description	Duration (days)	Immediate predecessor(s)
A	remove the old equipment	6	—
B	excavate the soft-fall pit	4	—
C	pour the concrete footings	3	A
D	lay the soft-fall surface	7	B
E	install the new equipment	5	C, D
F	build the shade sail	2	C

a. Find the earliest finish time of each of the six activities, and hence write down the minimum completion time for the project and the activities that make up the critical path. 3 marks

b. Determine the float (slack time) of activity A, and the number of days by which activity F could be delayed without delaying the whole project. 2 marks

c. Write down all the activities that have zero float. 1 mark

End of Exam 4

Exam 4 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \bar{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + bx$
least squares slope	$b = \dfrac{r s_y}{s_x}$
least squares intercept	$a = \bar{y} - b\bar{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $u_0 = a, u_{n+1} = R u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100n}\right)^n - 1\right] \times 100\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\ A = ad - bc$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\ A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\ A \neq 0$

recurrence relation	$S_0 = \text{initial state}, S_{n+1} = T S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}, S_{n+1} = L S_n$

Networks and decision mathematics

Euler's formula $v - e + f = 2$

End of Formula Sheet

Exam 4 — Answers and solutions

Data analysis

Q1. (a) The data is already in order ($n=11$). The minimum is 12 and the maximum is 52. The median is the 6th value, $M=18$. The lower half (excluding the median) is $12, 14, 15, 16, 17$, whose median is $Q_1=15$; the upper half is $20, 23, 27, 33, 52$, whose median is $Q_3=27$. The five-number summary is

$$\min=12, Q_1=15, M=18, Q_3=27, \max=52.$$

(b) The interquartile range is $\text{IQR}=Q_3-Q_1=27-15=12$. The upper fence is

$$Q_3+1.5\times\text{IQR}=27+1.5\times12=27+18=45.$$

Since $52>45$, a waiting time of 52 minutes lies beyond the upper fence and is therefore an outlier. (The lower fence is $15-18=-3$, so there are no outliers at the low end.)

(c) Ignoring the outlier, both the box and the whisker are longer on the right of the median than on the left: $Q_3-M=27-18=9$ compared with $M-Q_1=18-15=3$, and the upper whisker runs from 27 to 33 (6 minutes) compared with the lower whisker from 15 to 12 (3 minutes). The distribution is therefore **positively skewed** (skewed to the right), with one outlier at 52 minutes.

Q2. (a) $z=\dfrac{x-\bar{x}}{s}=\dfrac{48-42}{3}=\dfrac{6}{3}=2.$

(b) A standardised value of $z=-1$ corresponds to a time one standard deviation below the mean:

$$x=\bar{x}+z\,s=42+(-1)(3)=39\,\text{seconds}.$$

(c) $36=42-2\times3$ is 2 standard deviations below the mean, and $48=42+2\times3$ is 2 standard deviations above the mean. By the 68–95–99.7 % rule, about 95 % of the lap times lie between 36 and 48 seconds.

(d) $45=42+3$ is 1 standard deviation above the mean. About 68 % of times lie within 1 standard deviation of the mean, leaving $100\%-68\%=32\%$ in the two tails, or 16 % in each tail. Therefore about 16 % of lap times are greater than 45 seconds.

Q3. (a) The number who live in the suburbs is 120, out of 200 surveyed:

$$\frac{120}{200}\times100\%=60\%.$$

(b) Of the 80 inner-city commuters, 50 prefer public transport:

$$\frac{50}{80}\times100\%=62.5\%.$$

(c) The percentage of inner-city commuters who prefer public transport is $\dfrac{50}{80}\times100\%=62.5\%$, while the percentage of suburban commuters who prefer public transport is $\dfrac{30}{120}\times100\%=25\%$. Because these two percentages are very different (62.5 % compared with 25 %), the preferred type of transport depends on where a commuter lives, so there **is an association** between preferred transport and location.

Q4. (a) The slope is -4. This means that, on average, the number of hot chocolates sold **decreases by 4 for each increase of 1 °C in the maximum daily temperature**.

(b) Substituting $\text{temperature}=16$:

$$\text{sold}=140-4\times16=140-64=76,$$

so the predicted number sold is 76 hot chocolates.

(c) The residual is the actual value minus the predicted value. The actual number sold was 80 and the predicted number was 76:

$$\text{residual} = 80 - 76 = 4.$$

(The point lies 4 units above the least-squares line.)

Recursion and financial modelling

Q5. (a) The monthly multiplier is $R = 1.005$.

$$V_1 = 1.005 \times 4000 = 4020.00, V_2 = 1.005 \times 4020 = 4040.10.$$

(Month 2 interest is $4020 \times 0.005 = 20.10$, so $V_2 = 4020 + 20.10 = \$4\,040.10$.) After 1 month the investment is worth \$4,020.00 and after 2 months it is worth \$4,040.10.

(b) Each month the balance grows by 0.5% and then \$150 is added, so

$$W_0 = 4000, W_{n+1} = 1.005\,W_n + 150.$$

(c) Iterating:

$$W_1 = 1.005 \times 4000 + 150 = 4020 + 150 = 4170.00,$$

$$W_2 = 1.005 \times 4170 + 150 = 4170 + 20.85 + 150 = 4340.85.$$

(Month 2 interest is $4170 \times 0.005 = 20.85$.) The second investment is worth \$4,340.85 after 2 months.

Q6. (a) The interest for month 1 is 1% of the opening balance \$8,000:

$$0.01 \times 8000 = \$80.00.$$

The principal reduction is $700 - 80 = \$620.00$, so the balance owing after the first payment is

$$8000 - 620 = \$7\,380.00.$$

(b) The interest for month 2 is 1% of \$7,380.00:

$$0.01 \times 7380 = \$73.80.$$

The principal reduction is $700 - 73.80 = \$626.20$, and the new balance is

$$7380 - 626.20 = \$6\,753.80.$$

The completed rows are:

Payment number	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance (\$)
1	700.00	80.00	620.00	7380.00
2	700.00	73.80	626.20	6753.80

(c) The total amount paid over the two months is $2 \times 700 = \$1\,400.00$. The total reduction in the balance is $8000 - 6753.80 = \$1\,246.20$.

Matrices

Q7. (a) A has 2 rows and 2 columns, so its order is 2×2.

(b) $A + B = \begin{bmatrix} 2+5 & 1+0 \\ 3+1 & 4+2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 6 \end{bmatrix}$.

(c) $3A = \begin{bmatrix} 3\times 2 & 3\times 1 \\ 3\times 3 & 3\times 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix}$.

(d) $AB = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(5)+1(1) & 2(0)+1(2) \\ 3(5)+4(1) & 3(0)+4(2) \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ 19 & 8 \end{bmatrix}$.

(e) $det(A) = 2 \times 4 - 1 \times 3 = 8 - 3 = 5$.

Q8. (a) The value 0.2 is in the column headed B and the row headed P. It represents the proportion of subscribers who are on the Basic plan in one month and on the Premium plan the next month — that is, 20% of Basic subscribers move to Premium each month.

(b) $S_1 = T S_0 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 900 \\ 300 \end{bmatrix} = \begin{bmatrix} 0.8(900)+0.1(300) \\ 0.2(900)+0.9(300) \end{bmatrix} = \begin{bmatrix} 750 \\ 450 \end{bmatrix}$.

After one month there are 750 subscribers on the Basic plan and 450 on the Premium plan.

(c) $S_2 = T S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 750 \\ 450 \end{bmatrix} = \begin{bmatrix} 0.8(750)+0.1(450) \\ 0.2(750)+0.9(450) \end{bmatrix} = \begin{bmatrix} 645 \\ 555 \end{bmatrix}$.

(d) Keep generating state matrices with $S_{n+1} = T S_n$ until there is no noticeable change from one state matrix to the next. Using technology,

$$S_5 \approx \begin{bmatrix} 484.0 \\ 716.0 \end{bmatrix}, S_{10} \approx \begin{bmatrix} 414.1 \\ 785.9 \end{bmatrix}, S_{20} \approx \begin{bmatrix} 400.4 \\ 799.6 \end{bmatrix}, S_{30} \approx \begin{bmatrix} 400.0 \\ 800.0 \end{bmatrix},$$

so the state matrix settles at 400 subscribers on Basic and 800 on Premium. In the long run there are 800 subscribers on the Premium plan.

Networks and decision mathematics

Q9. (a) Vertex N is joined to L, M and P, so its degree is 3. The degrees of L, M, N, P, Q, R are $3, 4, 3, 4, 2, 2$; their sum is 18, which is twice the number of edges, so the graph has $18 \div 2 = 9$ edges.

(b) A walk that uses every edge exactly once is an **Eulerian trail**. Here L and N have degree 3 (odd), while M, P, Q, R have degrees $4, 4, 2, 2$ (all even), so exactly two vertices have odd degree.

A walk that finished back where it started would be an Eulerian *circuit*, and a graph has an Eulerian circuit only when **every** vertex has even degree. Since L (and N) have odd degree, no Eulerian circuit exists, so the guard cannot finish back at building L. An Eulerian trail can only begin at one odd-degree vertex and finish at the other, so a guard who starts at L must finish at building N. (One such walk is $L - M - Q - R - P - M - N - P - L - N$, which uses each of the 9 walkways exactly once.)

(c) Applying Prim's algorithm from L: the shortest edge from L is $L - N(30)$; from $\{L, N\}$ the shortest edge to a new building is $M - N(25)$; then $N - P(40)$; then $P - R(50)$; and finally $Q - R(35)$. The minimum spanning tree uses the walkways

$$M - N(25), L - N(30), Q - R(35), N - P(40), P - R(50),$$

so the total length of cable required is

$$25 + 30 + 35 + 40 + 50 = 180 \text{ metres.}$$

Q10. (a) A forward scan gives the earliest finish times

$$A\,6, B\,4, C\,9, D\,11, E\,16, F\,11$$

(activity E cannot start until both C and D are finished, that is, at time $max(9, 11) = 11$). The largest earliest finish time is 16, so the minimum completion time is 16 days, along the critical path $B - D - E$ ($4 + 7 + 5 = 16$).

(b) The full schedule of earliest/latest times gives:

Activity	Dur	EST	EFT	LST	LFT	Float
A	6	0	6	2	8	2

Activity	Dur	EST	EFT	LST	LFT	Float
B	4	0	4	0	4	0
C	3	6	9	8	11	2
D	7	4	11	4	11	0
E	5	11	16	11	16	0
F	2	9	11	14	16	5

The float of activity A is $\text{LST} - \text{EST} = 2 - 0 = 2$ days. For activity F the float is $14 - 9 = 5$ days, so activity F could be delayed by up to 5 days — starting as late as day 14 and still finishing by day 16 — without delaying the whole project.

(c) The activities with zero float are those on the critical path: B, D and E.

Exam 5 — Short-answer (Medium)

Reading time: 15 minutes. Writing time: 1 hour 30 minutes. Answer all questions. Total 60 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Data analysis

Question 1 (6 marks)

The resting heart rate, in beats per minute (bpm), was recorded for each of 15 members of a fitness club. The values, arranged in ascending order, are

$$52\ 55\ 58\ 60\ 62\ 64\ 65\ 66\ 68\ 70\ 72\ 74\ 76\ 80\ 96$$

a. Determine the five-number summary for this data set. 2 marks

b. Using the $1.5 \times \text{IQR}$ rule, show that 96 is an outlier, and state whether the data set has any other outliers. 2 marks

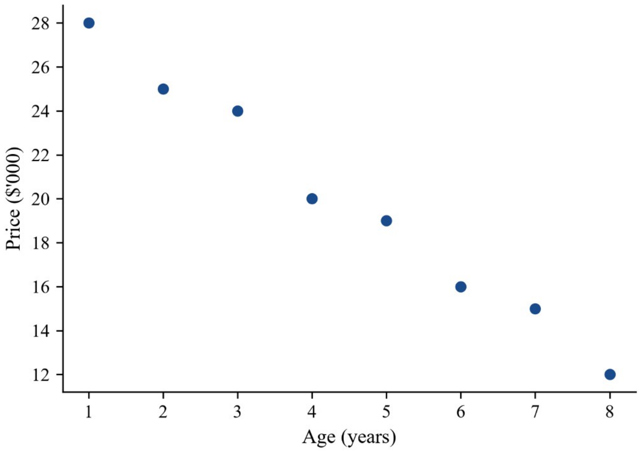
c. The mean of the data set is 67.87 bpm (to two decimal places). By comparing the mean with the median, describe the shape of the distribution. 1 mark

d. Across the whole club population, resting heart rate is approximately normally distributed with a mean of 68 bpm and a standard deviation of 8 bpm. Determine the percentage of members with a resting heart rate greater than 92 bpm. 1 mark

Question 2 (7 marks)

A used-car dealer records the age (in years) and the price (in thousands of dollars) of eight cars of the same model. The data are shown in the table and scatterplot below.

Age (years)	1	2	3	4	5	6	7	8
Price ($'000)	28	25	24	20	19	16	15	12



Age is the explanatory variable.

a. Determine the equation of the least-squares regression line that allows the price to be predicted from the age. Write the coefficients correct to two decimal places. 2 marks

b. Interpret the slope of the least-squares line in terms of the age and price of a car. 1 mark

c. Determine the coefficient of determination, correct to four decimal places, and interpret it in this context. 2 marks

d. Use the least-squares line to predict the price of a car that is 5 years old, and state whether this prediction is an interpolation or an extrapolation. 2 marks

Question 3 (5 marks)

The table shows the number of downloads N, in thousands, of a new app, t days after its release.

t (days)	0	1	2	3	4
N ('000)	8	20	50	125	312

A plot of N against t is clearly non-linear, so a $\log_{10}$ transformation is applied to the response variable N.

a. With reference to the values in the table, explain why a $\log_{10}$ transformation of N is appropriate here. 1 mark

b. Determine the equation of the least-squares line for $\log_{10}(N)$ against t, giving the coefficients correct to three significant figures. 2 marks

c. Use this model to predict the number of downloads (in thousands) after 6 days, correct to the nearest 10. 2 marks

Question 4 (6 marks)

A surf shop's quarterly sales, in thousands of units, for its first year of trading are shown below.

Quarter	Q1	Q2	Q3	Q4
Sales ('000)	48	56	64	32

a. Show that the seasonal index for Quarter 1 is 0.96, and determine the seasonal indices for the other three quarters. 2 marks

b. Interpret the Quarter 3 seasonal index in the context of the shop's sales. 1 mark

c. In Year 2, the actual sales in Quarter 3 were $76\,800$ units. Calculate the deseasonalised sales figure for that quarter. 1 mark

d. The deseasonalised sales (in thousands) follow the trend line $\text{sales} = 45 + 1.5t$, where t is the quarter number, with $t = 1$ for Year 1, Quarter 1. Use this trend line together with the appropriate seasonal index to forecast the actual sales for Year 3, Quarter 2. 2 marks

Recursion and financial modelling

Question 5 (5 marks)

A landscaping business buys a ride-on mower for \$45,000. The mower will be depreciated over time.

a. Under reducing-balance depreciation at $20\,\%$ per annum, write down a recurrence relation, including the value of V_0, that gives the value V_n (in dollars) of the mower after n years. 1 mark

b. Find the value of the mower after 3 years under this reducing-balance depreciation. 2 marks

c. The business instead considers flat-rate (straight-line) depreciation of \$7500 per year. Find the value of the mower after 3 years under this method, and state which of the two methods gives the greater value at that time. 2 marks

Question 6 (7 marks)

Rohan takes out a reducing-balance loan of \$18,000 to renovate his kitchen. Interest is charged at $9\,\%$ per annum, compounding monthly, and he repays \$1200 at the end of each month.

a. Show that the monthly interest rate is $0.75\,\%$, and complete the first two rows of the amortisation table below. 3 marks

Month	Repayment ($)	Interest ($)	Principal reduction ($)	Balance ($)
0	—	—	—	18 000.00
1	1200.00			
2	1200.00			

b. Calculate the total interest charged over the first two months. 1 mark

c. Determine the effective annual interest rate for this loan, correct to two decimal places. 2 marks

d. The loan is fully repaid by a final repayment that is smaller than $1200. Explain why this final repayment is less than $1200. 1 mark

Matrices

Question 7 (6 marks)

Two stores of a homewares chain, Store 1 and Store 2, each sell mugs, plates and bowls. In one hour, the number of each item sold is given by the matrix

$$Q=\begin{bmatrix} 8 & 5 & 12 \\ 6 & 10 & 4 \end{bmatrix},$$

where the columns give mugs, plates and bowls (in that order) and row 1 is Store 1, row 2 is Store 2. The selling prices, in dollars, are given by

$$P=\begin{bmatrix} 6 \\ 9 \\ 4 \end{bmatrix}.$$

a. Evaluate the matrix product QP and explain what its entries represent. 2 marks

A stocktake at a third store leads to the matrix equation

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 16 \\ 30 \end{bmatrix}.$$

b. Find the determinant and the inverse of $A=\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$. 2 marks

c. Hence solve the matrix equation for x and y. 2 marks

Question 8 (6 marks)

Each week, 2000 commuters in a town choose between two coffee brands, Aroma (A) and Bean (B). From one week to the next:

- 95 % of Aroma's customers stay with Aroma and 5 % switch to Bean;
- 15 % of Bean's customers switch to Aroma and 85 % stay with Bean.

This is modelled by the transition matrix T acting on the state vector $\begin{bmatrix} A \\ B \end{bmatrix}$, where

$$T=\begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}.$$

Initially, 500 commuters choose Aroma and 1500 choose Bean, so $S_0=\begin{bmatrix} 500 \\ 1500 \end{bmatrix}$.

a. Interpret the entry 0.05 of the transition matrix T in this context. 1 mark

b. Find the number of customers choosing each brand after one week and after two weeks. 2 marks

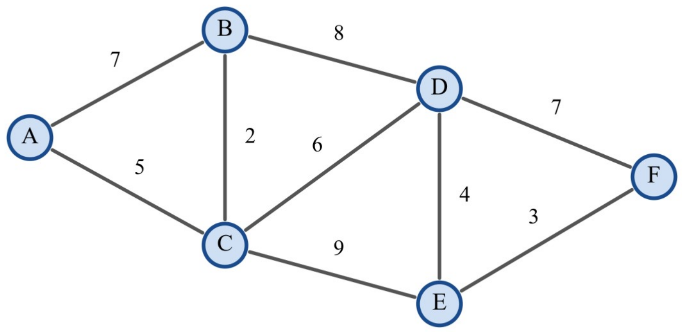
c. Determine the long-term (steady-state) number of customers choosing each brand. 2 marks

d. Hence state the long-term percentage of customers who choose Aroma. 1 mark

Networks and decision mathematics

Question 9 (6 marks)

The network below shows the roads connecting six towns, A to F. The number on each edge is the length of that road, in kilometres.



a. Write down the degree of town D, and hence, by considering the sum of the degrees of all the towns, determine the total number of roads in the network. 2 marks

b. Find the length of the shortest path from town A to town F, and state the route taken. 2 marks

c. A cable company will connect all six towns, laying cable along the roads, using the least possible total length of cable (a minimum spanning tree). Determine the total length of cable required. 2 marks

Question 10 (6 marks)

A project consists of seven activities, A to G. The duration of each activity, in days, and its immediate predecessor(s) are shown in the table below.

Activity	Duration (days)	Immediate predecessor(s)
A	4	—
B	3	—
C	5	A
D	6	B
E	2	C,D
F	4	D
G	3	E,F

a. Determine the minimum completion time for the project, and write down the critical path. 3 marks

b. Write down the earliest start time of activity G. 1 mark

c. Determine the float (slack time) of activity C, and explain what this value means for the scheduling of the project. 2 marks

End of Exam 5

Exam 5 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \overline{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + b\,x$
least squares slope	$b = \dfrac{r\,s_y}{s_x}$
least squares intercept	$a = \overline{y} - b\,\overline{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $\quad u_0 = a\,,\, u_{n+1} = R\,u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100\,n}\right)^n - 1\right] \times 100\,\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\,A = a\,d - b\,c$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\,A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\,A \neq 0$

recurrence relation	$S_0 = \text{initial state}\,,\, S_{n+1} = T\,S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}\,,\, S_{n+1} = L\,S_n$

Networks and decision mathematics

Euler's formula $\quad v - e + f = 2$

End of Formula Sheet

Exam 5 — Answers and solutions

Data analysis

Question 1

(a) With $n = 15$, the median is the 8th value, $Q_2 = 66$. The lower half (the first 7 values) has median $Q_1 = 60$; the upper half (the last 7 values) has median $Q_3 = 74$. The five-number summary is

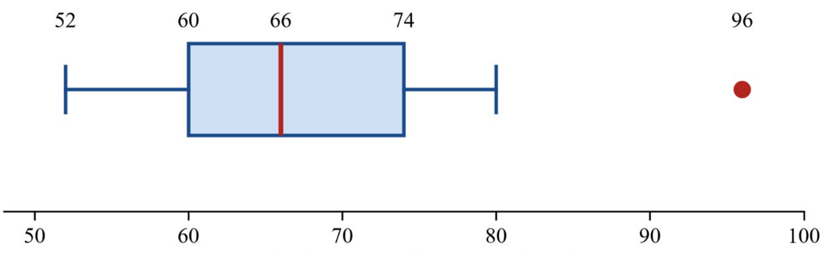
$$\min = 52, Q_1 = 60, Q_2 = 66, Q_3 = 74, \max = 96.$$

(b) $\text{IQR} = Q_3 - Q_1 = 74 - 60 = 14$. The upper fence is

$$Q_3 + 1.5 \times \text{IQR} = 74 + 1.5 \times 14 = 74 + 21 = 95.$$

Since $96 > 95$, the value 96 is an outlier. The lower fence is $Q_1 - 1.5 \times \text{IQR} = 60 - 21 = 39$, and the smallest value (52) is greater than 39, so 96 is the **only** outlier.

(c) The median is 66 bpm and the mean is 67.87 bpm, so the mean is greater than the median. The mean is pulled up by the single high outlier (96), so the distribution is **positively skewed** (skewed to the right). A boxplot confirms this: the upper whisker extends only to the largest non-outlier (80), with 96 shown as a separate outlier.



(d) A resting heart rate of 92 bpm is $92 = 68 + 3 \times 8$, that is, three standard deviations above the mean. By the 68–95–99.7 rule, 99.7 % of values lie within three standard deviations of the mean, leaving $100 - 99.7 = 0.3\,\%$ in the two tails combined; by symmetry the upper tail is half of this. Hence $\dfrac{100 - 99.7}{2} = 0.15\,\%$, that is 0.15 % of members have a resting heart rate greater than 92 bpm.

Question 2

(a) Using technology, the least-squares regression line of price on age is

$$\text{price} = 29.89 - 2.23 \times \text{age},$$

with the intercept and slope each given to two decimal places.

(b) The slope is -2.23 (with price in thousands of dollars). On average, for each additional year of age, the price of a car of this model **decreases by about \$2230**.

(c) The coefficient of determination is $r^2 = 0.9871$. This means that about 98.71 % of the variation in the price of these cars can be explained by the variation in their age. (Equivalently, the linear model fits the data very well.)

(d) Substituting $\text{age} = 5$ into the least-squares line,

$$\text{price} = 29.89 - 2.23 \times 5 = 29.89 - 11.15 = 18.74,$$

that is, about \$18,740. Because 5 lies within the range of the data (1 to 8 years), this prediction is an **interpolation** and is reliable.

Question 3

(a) From one day to the next the number of downloads is multiplied by roughly the same factor ($20 \div 8 = 2.5$, $50 \div 20 = 2.5$, $125 \div 50 = 2.5$, $312 \div 125 \approx 2.5$). Repeated multiplication by a constant factor describes exponential growth, so the graph of N against t curves upward. Taking $\log_{10}(N)$ straightens this exponential relationship, making a linear model appropriate.

(b) The transformed values $\log_{10}(N)$ are

t	0	1	2	3	4
$\log_{10}(N)$	0.90	1.30	1.70	2.10	2.49

The least-squares line for $\log_{10}(N)$ against t is

$$\log_{10}(N) = 0.903 + 0.398\,t\,,$$

with coefficients to three significant figures (the fit is almost perfect, $r^2 \approx 1$).

(c) At $t = 6$,

$$\log_{10}(N) = 0.903 + 0.398 \times 6 = 0.903 + 2.388 = 3.291\,,$$

so $N = 10^{3.291} \approx 1954$. To the nearest 10, the model predicts about 1950 **thousand** downloads (that is, about $1\,950\,000$ downloads).

Question 4

(a) The mean of the four quarterly figures is

$$\frac{48 + 56 + 64 + 32}{4} = \frac{200}{4} = 50\,.$$

The seasonal index for a quarter is (quarterly value) ÷ (mean). For Quarter 1, $48 \div 50 = 0.96$, as required. The remaining indices are

$$\text{Q2:}\,\frac{56}{50} = 1.12\,,\text{Q3:}\,\frac{64}{50} = 1.28\,,\text{Q4:}\,\frac{32}{50} = 0.64\,.$$

(As a check, $0.96 + 1.12 + 1.28 + 0.64 = 4.00$.)

(b) The Quarter 3 seasonal index is 1.28. This means that, on average, Quarter 3 sales are $28\,\%$ **above** the average quarterly sales for the year.

(c) Deseasonalised value $= \dfrac{\text{actual value}}{\text{seasonal index}} = \dfrac{76\,800}{1.28} = 60\,000$ units.

(d) Year 3, Quarter 2 is quarter number $t = 10$ (Year 1: $t = 1$–4, Year 2: $t = 5$–8, Year 3: $t = 9$–12). The trend value is

$$\text{sales} = 45 + 1.5 \times 10 = 60\,(\text{thousand})\,.$$

Reseasonalising with the Quarter 2 index (1.12),

$$\text{forecast actual sales} = 60 \times 1.12 = 67.2\,\text{thousand} = 67\,200\,\text{units}.$$

Recursion and financial modelling

Question 5

(a) Reducing-balance depreciation at $20\,\%$ per annum multiplies the value by $1 - 0.20 = 0.8$ each year. With an initial value of \$45,000,

$$V_0 = 45\,000\,,\,V_{n+1} = 0.8\,V_n\,.$$

(b) After 3 years,

$$V_3 = 45\,000 \times 0.8^3 = 45\,000 \times 0.512 = 23\,040.$$

The mower is worth **$23,040** after 3 years.

(c) Flat-rate depreciation of $7500 per year over 3 years reduces the value by $3 \times 7500 = 22\,500$, so

$$V_3 = 45\,000 - 22\,500 = 22\,500.$$

Under flat-rate depreciation the mower is worth **$22,500**. Since $23\,040 > 22\,500$, the **reducing-balance** method gives the greater value after 3 years.

Question 6

(a) The annual rate is $9\,\%$ compounding monthly, so the monthly rate is $\frac{9\,\%}{12} = 0.75\,\%$ (that is, 0.0075 per month).

- Month 1: interest $= 0.0075 \times 18\,000 = \$135.00$; principal reduction $= 1200 - 135 = \$1065.00$; balance $= 18\,000 - 1065 = \$16\,935.00$.
- Month 2: interest $= 0.0075 \times 16\,935 = \$127.01$ (to the nearest cent); principal reduction $= 1200 - 127.01 = \$1072.99$; balance $= 16\,935 - 1072.99 = \$15\,862.01$.

Month	Repayment ($)	Interest ($)	Principal reduction ($)	Balance ($)
0	—	—	—	18 000.00
1	1200.00	135.00	1065.00	16 935.00
2	1200.00	127.01	1072.99	15 862.01

(b) Total interest over the first two months $= 135.00 + 127.01 = \$262.01$.

(c) The effective annual interest rate is

$$\left(\left(1 + \frac{0.09}{12}\right)^{12} - 1\right) \times 100\,\% = \left(1.0075^{12} - 1\right) \times 100\,\% = 9.38\,\% \text{ (to two decimal places).}$$

(d) With each repayment the balance owing falls. When the outstanding balance (plus that month's interest) has become less than $1200, a full $1200 repayment would more than clear the debt (it would leave a negative balance). Only the exact amount owing is repaid, so the final repayment is less than $1200. (Here the loan clears with a 16th repayment of $1167.14.)

Matrices

Question 7

(a) $QP = \begin{bmatrix} 8 & 5 & 12 \\ 6 & 10 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 8(6) + 5(9) + 12(4) \\ 6(6) + 10(9) + 4(4) \end{bmatrix} = \begin{bmatrix} 141 \\ 142 \end{bmatrix}.$

The entries give the total takings, in dollars, in one hour: **$141 at Store 1** and **$142 at Store 2**.

(b) $det(A) = 3 \times 4 - 2 \times 5 = 12 - 10 = 2$. Since $det(A) = 2 \neq 0$, the inverse exists:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}.$$

(c) Multiplying both sides of $AX = B$ on the left by A^{-1},

$$X = A^{-1}B = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 16 \\ 30 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 64-60 \\ -80+90 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Hence $x=2$ and $y=5$. (Check: $3(2)+2(5)=16$ and $5(2)+4(5)=30$.)

Question 8

(a) The entry 0.05 is in row B (Bean), column A (Aroma). It means that each week 5% of Aroma's customers switch to Bean.

(b) After one week,

$$S_1 = T\,S_0 = \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}\begin{bmatrix} 500 \\ 1500 \end{bmatrix} = \begin{bmatrix} 475+225 \\ 25+1275 \end{bmatrix} = \begin{bmatrix} 700 \\ 1300 \end{bmatrix}.$$

After two weeks,

$$S_2 = T\,S_1 = \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}\begin{bmatrix} 700 \\ 1300 \end{bmatrix} = \begin{bmatrix} 665+195 \\ 35+1105 \end{bmatrix} = \begin{bmatrix} 860 \\ 1140 \end{bmatrix}.$$

So after one week 700 choose Aroma and 1300 choose Bean; after two weeks 860 choose Aroma and 1140 choose Bean.

(c) In the steady state $\begin{bmatrix} a \\ b \end{bmatrix}$ the distribution is unchanged, so $0.95a+0.15b=a$ with $a+b=2000$. This gives $0.15b=0.05a$, i.e. $b=1/3a$, so $a+1/3a=2000$, giving $a=1500$ and $b=500$. The long-term distribution is 1500 **customers with Aroma and** 500 **with Bean**. (This can also be obtained by computing $T^n S_0$ for large n.)

(d) In the long run the fraction choosing Aroma is $\frac{1500}{2000}=75\%$.

Networks and decision mathematics

Question 9

(a) Town D is joined to B, C, E and F, so its degree is 4. The degrees of the towns are $A2, B3, C4, D4, E3, F2$, with sum $2+3+4+4+3+2=18$. Since the sum of the degrees equals twice the number of edges, the number of roads is $18 \div 2 = 9$.

(b) Applying Dijkstra's algorithm from A gives shortest distances $A0, C5, B7, D11, E14, F17$. The shortest path from A to F has length 17 **km**, along the route $A-C-E-F$ ($5+9+3=17$).

(c) Using Prim's (or Kruskal's) algorithm, the minimum spanning tree uses the roads

$$B-C(2), E-F(3), D-E(4), A-C(5), C-D(6),$$

which connect all six towns with 5 edges. The least total length of cable required is

$$2+3+4+5+6=20\,\text{km}.$$

Question 10

(a) A forward scan gives the earliest finish times $A4, B3, C9, D9, E11, F13, G16$ (activity E waits for both C and D; activity G waits for both E and F). The minimum completion time is 16 **days**. Tracing the longest path gives the critical path $B-D-F-G$ ($3+6+4+3=16$).

(b) Activity G cannot start until both E (earliest finish 11) and F (earliest finish 13) are complete, so the earliest start time of G is 13.

(c) For activity C: earliest start time $= 4$ and latest start time $= 6$, so the float is $6 - 4 = 2$ days. (Equivalently, EST $= 4$, LFT $= 11$, duration 5, float $= 11 - 4 - 5 = 2$.) This means activity C can be delayed by up to 2 days without delaying the completion of the whole project.

Exam 6 — Short-answer (Hard)

Reading time: 15 minutes. Writing time: 1 hour 30 minutes. Answer all questions. Total 60 marks. One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown. Unless otherwise indicated, diagrams are not drawn to scale.

Data analysis

Question 1 (7 marks)

A laboratory tests a batch of 16 rechargeable batteries and records the time, in hours, that each battery runs under a fixed load. The results, arranged in order, are

$$14.2, 15.8, 16.1, 16.5, 16.9, 17.2, 17.4, 17.6, 17.8, 18.0, 18.3, 18.5, 18.8, 19.1, 19.6, 23.5.$$

a. Determine the five-number summary for this data set and hence the interquartile range. 2 marks

b. Using the $1.5 \times \text{IQR}$ rule, determine whether the data set contains any outliers. Show the working that supports your decision. 2 marks

c. Construct a boxplot for the data, showing any outliers separately. 2 marks

d. A different production line makes batteries whose running times are approximately normally distributed with a mean of 18.0 hours and a standard deviation of 1.5 hours. Using the 68–95–$99.7\,\%$ rule, find the percentage of these batteries expected to run for more than 21 hours. 1 mark

Question 2 (5 marks)

A company surveyed 240 of its employees. Each employee was classified by whether they usually work from home or are office-based, and by whether or not they reported high job satisfaction. The results are shown below.

	Works from home	Office-based	Total
Reports high job satisfaction	72	45	117
Does not report high job satisfaction	48	75	123
Total	120	120	240

a. What percentage of employees who work from home report high job satisfaction, and what percentage of office-based employees report high job satisfaction? 2 marks

b. Do these data support the contention that there is an association between work location and job satisfaction? Justify your answer by quoting the percentages from part **a.** 2 marks

c. Identify the explanatory variable and the response variable in this investigation, and state the type (categorical or numerical) of each. 1 mark

Question 3 (7 marks)

Ecologists introduce a small population of rabbits to a fenced reserve and record the population size, P, at the time of release and again at the end of each of the next five years. The results are shown below, where t is the number of years since the rabbits were introduced.

Time t (years)	0	1	2	3	4	5
Populatio	15	23	36	58	91	145

Time t (years)	0	1	2	3	4	5
n P						

A scatterplot of P against t increases at an increasing rate, so a $\log_{10}$ transformation is applied to the response variable P.

a. Explain why fitting a least-squares line directly to P against t would not be appropriate. 1 mark

b. Complete a table of $\log_{10}(P)$ values, correct to two decimal places. 2 marks

c. Determine the equation of the least-squares line for $\log_{10}(P)$ against t, giving the coefficients correct to three decimal places, and state the value of the coefficient of determination. 2 marks

d. Use your model to predict the rabbit population at the end of year 8, correct to the nearest ten. State whether this prediction is an interpolation or an extrapolation, and comment on its reliability. 2 marks

Question 4 (5 marks)

The quarterly sales of a firewood merchant for one year, in thousands of dollars, are shown below. Quarter 1 covers January to March, Quarter 2 April to June, and so on.

Quarter	Q1	Q2	Q3	Q4
Sales ($’000)	75	110	130	85

a. Calculate the four seasonal indices for this year. 2 marks

b. Interpret the seasonal index for Quarter 3. 1 mark

c. In the following year, the actual Quarter 3 sales were $143,000. Calculate the deseasonalised Quarter 3 sales figure. 1 mark

d. The deseasonalised quarterly sales are modelled by the trend line $\hat{s}=96+2t$, where $\hat{s}$ is in thousands of dollars and t is the quarter number ($t=1$ for Quarter 1 of the first year). Forecast the actual sales for Quarter 2 of the third year ($t=10$). 1 mark

Recursion and financial modelling

Question 5 (7 marks)

A café owner takes out a loan of $9,000 to buy an espresso machine, charged interest at 15% per annum compounding monthly. The loan is repaid by monthly repayments of $1,600 at the end of each month. Throughout, round the interest to the nearest cent each month.

a. State the interest rate per month and write a recurrence relation for the balance owing V_n after n months. 1 mark

b. Construct the full amortisation table until the loan is repaid, showing the payment, interest, principal reduction and balance for each month. 4 marks

c. State the size of the final repayment and find the total interest charged over the life of the loan. 2 marks

Question 6 (5 marks)

On the first day of a month, $5,000 is deposited into an account paying 6% per annum compounding monthly. At the end of each month, a further $200 is deposited. Throughout, round the interest to the nearest cent each month.

a. Write a recurrence relation for the balance V_n after n months. 1 mark

b. Construct a table showing the interest earned, the deposit and the balance at the end of each of the first four months. 2 marks

c. Find the effective annual interest rate for this account, correct to two decimal places. 1 mark

d. Find the first month at the end of which the balance exceeds $6,000. 1 mark

Matrices

Question 7 (6 marks)

A workshop runs four machines, numbered 1 to 4. In the current month the operators of machines $1, 2, 3$ and 4 are J, K, L and M respectively. This roster is recorded as the column matrix

$$R_0 = \begin{bmatrix} J \\ K \\ L \\ M \end{bmatrix}.$$

At the start of every month the operators are rotated according to the following rule.

- The operator of machine 2 moves to machine 1.
- The operator of machine 4 moves to machine 2.
- The operator of machine 1 moves to machine 3.
- The operator of machine 3 moves to machine 4.

The roster after n rotations is R_n, where $R_{n+1} = P R_n$ and P is a permutation matrix.

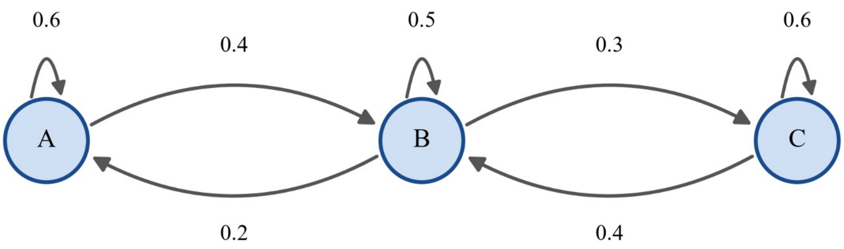
a. Write down the permutation matrix P. 2 marks

b. Find $R_2 = P^2 R_0$ and hence state the machine that operator M works on after two rotations. 2 marks

c. Find the smallest number of rotations after which every operator is back on the machine they started on. Justify your answer. 2 marks

Question 8 (6 marks)

A bike-share scheme owns 900 bicycles. Each bicycle is parked overnight at one of three stations, A, B or C. The transition diagram below shows the proportion of bicycles that move between the stations from one night to the next. A bicycle parked at A is never parked at C the following night, and a bicycle parked at C is never parked at A the following night.



On Monday night, 400 bicycles are parked at station A, 300 at station B and 200 at station C.

a. Write down the transition matrix T for this scheme, with the rows and columns in the order A, B, C, and explain why the entries in each column of T must add to 1. 2 marks

b. Write down the matrix recurrence relation, including the initial state matrix S_0, that generates the sequence of state matrices for this scheme.

1 mark

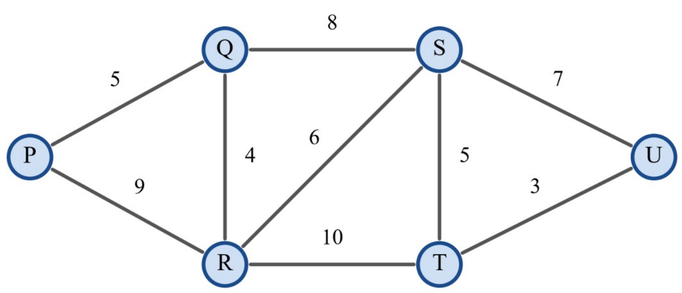
c. Find the state matrices S_1 and S_2, and hence state the number of bicycles parked at station A on Wednesday night. 2 marks

d. Determine the number of bicycles parked at each station in the long term. 1 mark

Networks and decision mathematics

Question 9 (6 marks)

A regional water authority plans to lay pipelines between six pumping stations P, Q, R, S, T and U. The network below shows the possible pipelines; each number is the cost of that pipeline, in hundreds of thousands of dollars.



a. The authority will connect all six pumping stations as cheaply as possible using a minimum spanning tree. State how many pipelines the minimum spanning tree will contain. 1 mark

b. Determine the minimum spanning tree. List the pipelines it uses and state its total cost, in hundreds of thousands of dollars. 3 marks

c. Separately, the authority wishes to pump water from P to U through the network at the least total cost. Find the shortest path from P to U and state its total cost, in hundreds of thousands of dollars. 2 marks

Question 10 (6 marks)

A project consists of seven activities. The duration of each activity, in hours, and its immediate predecessors are shown in the table below.

Activity	Duration (hours)	Immediate predecessor(s)
A	5	—
B	7	—
C	6	A
D	6	A
E	4	B, C
F	5	D
G	4	E, F

a. Determine the minimum completion time for the project and write down the critical path. 2 marks

b. Find the float (slack) of activity B and of activity C. 2 marks

Each of activities A, D, F and G can be crashed (shortened), by at most the number of hours shown, at the cost per hour shown below. No other activity can be crashed.

Activity	A	D	F	G
Maximum reduction (hours)	2	2	1	2

Activity	A	D	F	G
Cost per hour ($)	60	40	55	50

c. The project manager wants to reduce the completion time by 2 hours. Determine which activities should be crashed, the new completion time, and the minimum extra cost. 2 marks

End of Exam 6

Exam 6 Formula Sheet

This Formula Sheet is provided with every examination in this booklet.

Data analysis

standardised score	$z = \dfrac{x - \bar{x}}{s_x}$
lower fence of a boxplot	$Q_1 - 1.5 \times \text{IQR}$
upper fence of a boxplot	$Q_3 + 1.5 \times \text{IQR}$
least squares line of best fit	$y = a + b\,x$
least squares slope	$b = \dfrac{r\,s_y}{s_x}$
least squares intercept	$a = \bar{y} - b\,\bar{x}$
residual value	actual value − predicted value
seasonal index	$\dfrac{\text{actual figure}}{\text{deseasonalised figure}}$

Recursion and financial modelling

first-order linear recurrence relation $\quad u_0 = a\,, u_{n+1} = R\,u_n + d$

effective rate of interest for a compound interest loan or investment: $r_{\text{effective}} = \left[\left(1 + \dfrac{r}{100\,n}\right)^n - 1\right] \times 100\,\%$

Matrices

determinant of a 2×2 matrix: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det\ A = a\,d - b\,c$

inverse of a 2×2 matrix: $A^{-1} = \dfrac{1}{det\ A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $det\ A \neq 0$

recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = T\,S_n + B$
Leslie matrix recurrence relation	$S_0 = \text{initial state}\,, S_{n+1} = L\,S_n$

Networks and decision mathematics

Euler's formula $\quad v - e + f = 2$

End of Formula Sheet

Exam 6 — Answers and solutions

Data analysis

Question 1.

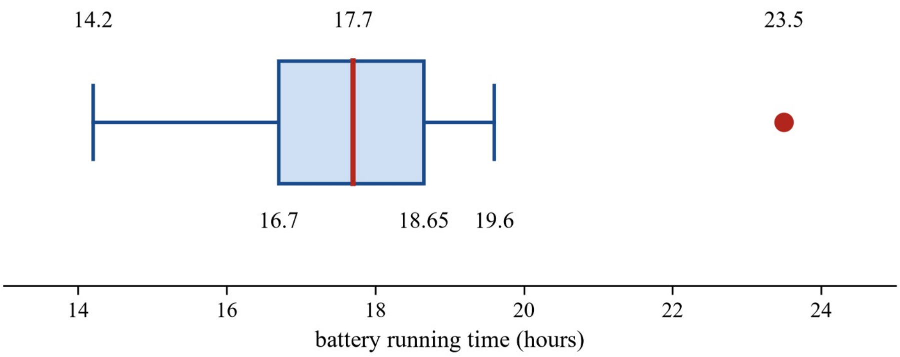
a. There are $n=16$ values, so the median lies between the 8th and 9th values: $M=\frac{17.6+17.8}{2}=17.7$. The lower half is the first eight values, whose median is $Q_1=\frac{16.5+16.9}{2}=16.7$; the upper half is the last eight values, whose median is $Q_3=\frac{18.5+18.8}{2}=18.65$. With $min=14.2$ and $max=23.5$, the five-number summary is

$$14.2, 16.7, 17.7, 18.65, 23.5,$$

and the interquartile range is $\text{IQR}=Q_3-Q_1=18.65-16.7=1.95$ hours.

b. The fences are at $Q_1-1.5\times\text{IQR}$ and $Q_3+1.5\times\text{IQR}$. Here $1.5\times1.95=2.925$, so the upper fence is $18.65+2.925=21.575$ and the lower fence is $16.7-2.925=13.775$. Since $23.5>21.575$, the value 23.5 is an outlier; since the smallest value $14.2>13.775$, there is no low outlier. The data set contains exactly one outlier, 23.5 hours.

c. The box spans $Q_1=16.7$ to $Q_3=18.65$ with the median at 17.7; the outlier 23.5 is plotted separately and the upper whisker stops at the largest non-outlier, 19.6.



d. For the second production line, $21=18.0+2\times1.5$, so 21 hours is two standard deviations above the mean. By the 68–95–99.7% rule, 95% of values lie within two standard deviations of the mean, leaving 5% in the two tails combined; by symmetry the upper tail is half of this. The percentage running for more than 21 hours is $\frac{100\%-95\%}{2}=2.5\%$.

Question 2.

a. Of the 120 employees who work from home, 72 report high job satisfaction: $\frac{72}{120}=60\%$. Of the 120 office-based employees, 45 report high job satisfaction: $\frac{45}{120}=37.5\%$.

b. Yes. The percentage reporting high job satisfaction is much higher among those who work from home (60%) than among those who are office-based (37.5%). Because these percentages differ markedly, the data support an association between work location and job satisfaction.

c. The explanatory variable is work location (works from home or office-based) and the response variable is whether or not the employee reports high job satisfaction. Both variables are categorical.

Question 3.

a. The scatterplot of P against t is curved — it increases at an increasing rate — so a straight line does not describe the pattern. A least-squares line fitted to the raw data would leave a clear curved pattern in the residuals rather than random scatter, so it would not be an appropriate model.

b. Taking $\log_{10}$ of each population value, correct to two decimal places:

t	0	1	2	3	4	5
$\log_{10}(P)$	1.18	1.36	1.56	1.76	1.96	2.16

c. Fitting the least-squares line to $\log_{10}(P)$ against t using technology gives

$$\log_{10}(P)=1.168+0.198\,t\,,r^2=0.9998\,.$$

The very high coefficient of determination confirms that the transformed data are close to linear.

d. At $t=8$, $\log_{10}(P)=1.168+0.198\times 8=2.752$, so $P=10^{2.752}\approx 565$, i.e. about 560 rabbits to the nearest ten. Since $t=8$ lies beyond the range of the data ($t=0$ to $t=5$), this is an extrapolation. It is therefore less reliable, because the exponential pattern is assumed to continue outside the observed range — in practice the growth of a caged population would eventually be limited by the reserve's capacity.

Question 4.

a. The mean quarterly sales are $\dfrac{75+110+130+85}{4}=\dfrac{400}{4}=100$. Each seasonal index is that quarter's sales divided by the mean:

$$\text{Q1}:75/100=0.75\,,\text{Q2}:110/100=1.10\,,\text{Q3}:130/100=1.30\,,\text{Q4}:85/100=0.85\,.$$

(The four indices sum to 4, as required.)

b. The Quarter 3 index of 1.30 means that Quarter 3 (July to September) sales are typically $30\,\%$ above the average quarterly sales for the year — as expected for a firewood merchant in mid-winter.

c. Deseasonalised value = actual ÷ seasonal index $=\dfrac{143\,000}{1.30}=\$\,110\,000$.

d. The trend value at $t=10$ is $\hat{s}=96+2\times 10=116$ (thousand dollars, deseasonalised). Reseasonalising with the Quarter 2 index gives $116\times 1.10=127.6$, i.e. a forecast of $\$\,127\,600$.

Recursion and financial modelling

Question 5.

a. The monthly interest rate is $r=\dfrac{15\,\%}{12}=1.25\,\%$ per month, so $R=1.0125$ and the recurrence relation is

$$V_{n+1}=1.0125\,V_n-1\,600\,,V_0=9\,000\,.$$

b. Each month the interest is $1.25\,\%$ of the previous balance (rounded to the nearest cent), the principal reduction is $1\,600-\text{interest}$, and the new balance is the previous balance less the principal reduction. In the final month the balance owing plus its interest is less than $\$\,1\,600$, so a smaller final repayment clears the loan: the balance after five months is $\$\,1\,374.23$, whose interest is $0.0125\times 1\,374.23=17.1779\to\$\,17.18$, giving a final repayment of $1\,374.23+17.18=\$\,1\,391.41$.

Payment number	Payment	Interest	Principal reduction	Balance
0	—	—	—	$\$\,9\,000.00$
1	$\$\,1\,600.00$	$\$\,112.50$	$\$\,1\,487.50$	$\$\,7\,512.50$

Payment number	Payment	Interest	Principal reduction	Balance
2	$\$1600.00$	$\$93.91$	$\$1506.09$	$\$6006.41$
3	$\$1600.00$	$\$75.08$	$\$1524.92$	$\$4481.49$
4	$\$1600.00$	$\$56.02$	$\$1543.98$	$\$2937.51$
5	$\$1600.00$	$\$36.72$	$\$1563.28$	$\$1374.23$
6	$\$1391.41$	$\$17.18$	$\$1374.23$	$\$0.00$

c. The final repayment is $\$1391.41$. The total interest is the sum of the interest column,

$$112.50+93.91+75.08+56.02+36.72+17.18=\$391.41.$$

(Check: total repaid − amount borrowed $=5\times1600+1391.41-9000=\$391.41$.)

Question 6.

a. The monthly interest rate is $r=\frac{6\%}{12}=0.5\%$ per month, so $R=1.005$ and the recurrence relation is

$$V_{n+1}=1.005V_n+200,\, V_0=5000.$$

b. Each month the interest is 0.5% of the previous balance (rounded to the nearest cent), and then $\$200$ is deposited.

Month	Interest earned	Deposit	Balance
0	—	—	$\$5000.00$
1	$\$25.00$	$\$200.00$	$\$5225.00$
2	$\$26.13$	$\$200.00$	$\$5451.13$
3	$\$27.26$	$\$200.00$	$\$5678.39$
4	$\$28.39$	$\$200.00$	$\$5906.78$

c. One year is 12 months, so the effective annual rate is

$$r_{\text{eff}}=\left(1.005^{12}-1\right)\times100\%=6.1678\ldots\%\approx6.17\%.$$

d. Continuing the iteration, the interest in month 5 is $0.005\times5906.78=29.5339\rightarrow\$29.53$, giving a balance of $5906.78+29.53+200=\$6136.31$. At the end of month 4 the balance is $\$5906.78$ (below $\$6000$) and at the end of month 5 it is $\$6136.31$ (above $\$6000$), so the balance first exceeds $\$6000$ at the end of **month** 5.

Matrices

Question 7.

a. Row i of P records where the operator of machine i comes from, so row i has a 1 in the column of the machine whose operator moves to machine i, and 0 elsewhere. Machine 1 takes the operator of machine 2, machine 2 takes the operator of machine 4, machine 3 takes the operator of machine 1 and machine 4 takes the operator of machine 3, so

$$P=\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix}.$$

(Each row and each column contains exactly one 1, as it must for a permutation matrix.)

b. After one rotation,

$$R_1 = P\,R_0 = \begin{bmatrix} K \\ M \\ J \\ L \end{bmatrix}, \text{and so } R_2 = P\,R_1 = P^2\,R_0 = \begin{bmatrix} M \\ L \\ K \\ J \end{bmatrix}.$$

The first entry of R_2 is M, so after two rotations operator M works on machine 1.

c. Continuing the rotation,

$$R_3 = P^3\,R_0 = \begin{bmatrix} L \\ J \\ M \\ K \end{bmatrix}, R_4 = P^4\,R_0 = \begin{bmatrix} J \\ K \\ L \\ M \end{bmatrix} = R_0.$$

Since R_1, R_2 and R_3 all differ from R_0 but $R_4 = R_0$ (equivalently, $P^4 = I_4$ while P, P^2 and P^3 are not the identity matrix), the smallest number of rotations after which every operator is back on their original machine is 4. This is because the rotation moves the operators around a single cycle of the four machines, $1 \to 3 \to 4 \to 2 \to 1$.

Question 8.

a. Column j of T lists where the bicycles that start the night at station j are parked the next night, so reading the arrows leaving each station in the order A, B, C,

$$T = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.4 & 0.5 & 0.4 \\ 0 & 0.3 & 0.6 \end{bmatrix}.$$

The zeros are the missing arrows: no bicycle moves directly between A and C. Each column must add to 1 because every bicycle parked at that station must be parked at exactly one of the three stations the following night, so the proportions leaving a station account for all of its bicycles.

b. With the stations in the order A, B, C, Monday night's distribution is $S_0 = \begin{bmatrix} 400 \\ 300 \\ 200 \end{bmatrix}$ and the recurrence relation is

$$S_0 = \begin{bmatrix} 400 \\ 300 \\ 200 \end{bmatrix}, S_{n+1} = T\,S_n.$$

c.

$$S_1 = T\,S_0 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.4 & 0.5 & 0.4 \\ 0 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 400 \\ 300 \\ 200 \end{bmatrix} = \begin{bmatrix} 240+60+0 \\ 160+150+80 \\ 0+90+120 \end{bmatrix} = \begin{bmatrix} 300 \\ 390 \\ 210 \end{bmatrix},$$

$$S_2 = T\,S_1 = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.4 & 0.5 & 0.4 \\ 0 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 300 \\ 390 \\ 210 \end{bmatrix} = \begin{bmatrix} 180+78+0 \\ 120+195+84 \\ 0+117+126 \end{bmatrix} = \begin{bmatrix} 258 \\ 399 \\ 243 \end{bmatrix}.$$

Monday night is S_0, so Wednesday night is S_2: 258 bicycles are parked at station A. (Each state matrix still totals 900 bicycles.)

d. Continuing the sequence $S_n = T^n\,S_0$ with technology, the state matrices settle down to

$$\begin{bmatrix} 200 \\ 400 \\ 300 \end{bmatrix},$$

so in the long term 200 bicycles are parked at station A, 400 at station B and 300 at station C. (Check: this matrix is unchanged by T, since $0.6 \times 200 + 0.2 \times 400 = 200$, $0.4 \times 200 + 0.5 \times 400 + 0.4 \times 300 = 400$ and $0.3 \times 400 + 0.6 \times 300 = 300$.)

Networks and decision mathematics

Question 9.

a. A spanning tree of a network with 6 pumping stations has $6 - 1 = 5$ edges, so the minimum spanning tree contains 5 pipelines.

b. Applying Prim's algorithm, start at P and repeatedly add the cheapest pipeline that joins a station already in the tree to one that is not:

- from $\{P\}$: choose $P - Q(5)$ in preference to $P - R(9)$;
- from $\{P, Q\}$: choose $Q - R(4)$ in preference to $Q - S(8)$ and $P - R(9)$;
- from $\{P, Q, R\}$: choose $R - S(6)$ in preference to $Q - S(8)$ and $R - T(10)$;
- from $\{P, Q, R, S\}$: choose $S - T(5)$ in preference to $S - U(7)$ and $R - T(10)$;
- from $\{P, Q, R, S, T\}$: choose $T - U(3)$ in preference to $S - U(7)$.

All six stations are now joined, so the minimum spanning tree uses

$$P - Q, Q - R, R - S, S - T, T - U,$$

with minimum total cost

$$5 + 4 + 6 + 5 + 3 = 23,$$

that is, $\$2.3$ million. (Starting Prim's algorithm at any other station produces the same five pipelines.)

c. Applying Dijkstra's algorithm from P gives the shortest distances

Vertex	P	Q	R	S	T	U
Shortest distance from P	0	5	9	13	18	20

The shortest path from P to U is $P - Q - S - U$, with total cost $5 + 8 + 7 = 20$ hundred thousand dollars, that is $\$2$ million. (The alternative route through T costs $13 + 5 + 3 = 21$.)

Question 10.

a. A forward scan through the network gives the earliest finish times $A\,5, B\,7, C\,11, D\,11, E\,15, F\,16, G\,20$ (activity E waits for both B and C; activity G waits for both E and F). The minimum completion time is 20 hours, along the critical path $A - D - F - G$ ($5 + 6 + 5 + 4 = 20$).

b. The three paths through the network have lengths $A - D - F - G = 20$, $A - C - E - G = 5 + 6 + 4 + 4 = 19$ and $B - E - G = 7 + 4 + 4 = 15$. Activity B lies only on the path of length 15, so its float is $20 - 15 = 5$ hours. Activity C lies on the path of length 19, so its float is $20 - 19 = 1$ hour. (Formally, float = LST − EST: for B, $5 - 0 = 5$; for C, $6 - 5 = 1$.)

c. To finish in 18 hours, the critical path $A - D - F - G$ (length 20) must lose 2 hours **and** the near-critical path $A - C - E - G$ (length 19) must lose at least 1 hour, otherwise it would still take 19 hours. Activities A and G lie on **both** long paths, so crashing them shortens both at once.

- Crash G by 1 hour (cost $\$50$): both paths fall by 1 hour, to $A-D-F-G=19$ and $A-C-E-G=18$.
- Then crash D by 1 hour (cost $\$40$): only $A-D-F-G$ falls, to 18.

Now every path is at most 18 hours, so the new completion time is 18 hours. The minimum extra cost is $\$50+\$40=\$90$. (Crashing D alone by 2 hours would leave $A-C-E-G$ at 19 hours and fail; crashing G by 2 hours would cost $\$100$, so $\$90$ is the least.)