General Mathematics Units 3 & 4

Chapter 9 — Transition matrices

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Contents

Theory

Many situations involve a fixed group of things — customers, animals, voters — where each member is in exactly one of a small number of **states** at any time, and members move between those states in regular, equal steps (each week, each season, each year). A **transition matrix** records the proportion that moves between every pair of states in one step, and it is the object that lets the whole system be modelled with matrices.

States, steps and a closed system. The distinct categories a member can be in are the **states** of the system; a **step** is one time period, at the end of which each member is again in one of the states. Unless it is stated otherwise, the system is **closed**: no members enter or leave, so the total number stays the same and every member is always in exactly one state. (Later in the chapter this assumption is relaxed, and a fixed number of members is added to or removed from the system at each step.) A member may **stay** in its current state or **move** to another.

The transition matrix. For a system with n states, the transition matrix T is the **square** $n \times n$ matrix whose entry

$$t_{ij} = \text{the proportion of the members currently in state } j \text{ that move to state } i \text{ in one step.}$$

The first subscript i is the **destination** (the row); the second subscript j is the **starting** state (the column). In words, **the columns are the "from" states and the rows are the "to" states**, so reading *down* column j tells you where the members of state j go. The leading-diagonal entry t_{jj} is the proportion of state j that **stays** in state j.

Because the system is closed, every member of state j must finish the step in one of the n states, so the proportions down any column add to 1:

$$\text{each column of a transition matrix sums to } 1.$$

Every entry is a proportion between 0 and 1. Proportions may be written as decimals, fractions or percentages; a percentage is divided by 100 before it is placed in the matrix (for example 20% becomes 0.2).

As an illustration, suppose shoppers in a town buy each week from one of two supermarkets, **Apex** (A) or **Best** (B). Each week 80% of Apex's customers return to Apex and the other 20% switch to Best, while 90% of Best's customers return to Best and the other 10% switch to Apex. Taking the states in the order A, B, the "from Apex" information fills **column 1** and the "from Best" information fills **column 2**:

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{array}{l} \leftarrow \text{to } A \\ \leftarrow \text{to } B \end{array}.$$

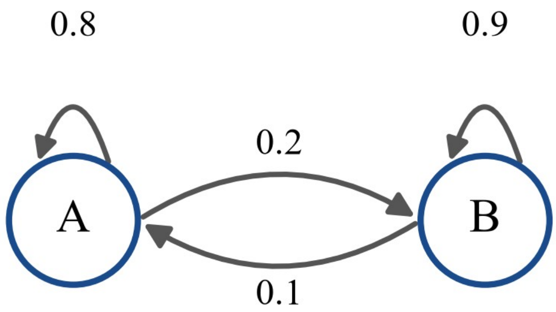
Here $t_{11} = 0.8$ is the proportion of Apex customers who stay with Apex, $t_{21} = 0.2$ is the proportion of Apex customers who move to Best, $t_{12} = 0.1$ is the proportion of Best customers who move to Apex, and $t_{22} = 0.9$ is the proportion who stay with Best. Column 1 gives $0.8 + 0.2 = 1$ and column 2 gives $0.1 + 0.9 = 1$, as required.

The initial state matrix. The starting situation is recorded in a **column matrix** called the **initial state matrix** S_0, whose entries are the amounts (or proportions) in each state at the start, listed in the **same order** as the rows and columns of T. If the town begins with 500 shoppers at Apex and 300 at Best, then

$$S_0 = \begin{bmatrix} 500 \\ 300 \end{bmatrix} \begin{array}{l} \leftarrow A \\ \leftarrow B \end{array}.$$

The order of S_0 is $n \times 1$: one row for each state. (How T and S_0 are used together to predict later states is developed later in this chapter.)

Transition diagrams. The same information is often drawn as a **transition diagram** (a state diagram): each state is a labelled node, and a directed arrow from one state to another is labelled with the proportion that moves that way in one step. An arrow from a state back to itself — a **loop** — carries the proportion that **stays**. Because everything leaving a state is accounted for, **the labels on all the arrows leaving a node add to** 1.



A = Apex, B = Best; a loop is the proportion that stays

The diagram and the matrix carry exactly the same information, and you should be able to move between them in either direction:

- the arrows **leaving** node j (its loop and its outgoing arrows) are exactly **column** j of T;
- the arrows **arriving** at node i are exactly **row** i of T.

So the loop 0.8 and the arrow $A \to B$ labelled 0.2 are column 1 of T, and the loop 0.9 and the arrow $B \to A$ labelled 0.1 are column 2.

Using CAS. General Mathematics is a technology-active course. A transition matrix and an initial state matrix are entered into the CAS matrix editor just like any other matrices, choosing the order first and then typing the entries. The essential by-hand skill in this section is **constructing them correctly** — choosing a consistent order for the states, putting each proportion in the right position so that every column sums to 1, and translating faithfully between a written description, a transition diagram and the matrix.

Worked examples

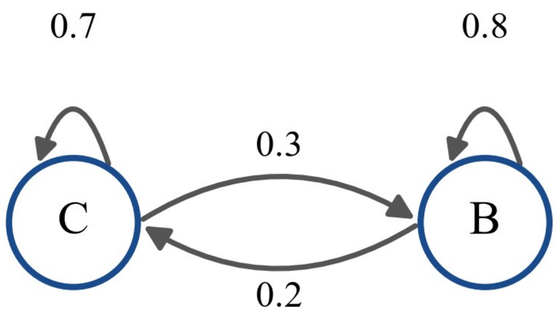
Example 1 — scaffolded

A town has two mobile-phone networks, **Zip** (Z) and **Yonder** (Y). Each month 85% of Zip's customers stay with Zip and the rest move to Yonder, while 75% of Yonder's customers stay with Yonder and the rest move to Zip. There are currently 4000 Zip customers and 6000 Yonder customers. Construct the transition matrix T (states in the order Z, Y) and the initial state matrix S_0.

Working	Comment
From Zip: stays 0.85, moves to Yonder $1-0.85=0.15$.	"The rest" of Zip's customers switch, so $Z \to Y = 0.15$.
Column 1 (from Z) $=\begin{bmatrix}0.85\\0.15\end{bmatrix}$; check $0.85+0.15=1$.	Top entry is $Z \to Z$, bottom is $Z \to Y$.
From Yonder: stays 0.75, moves to Zip $1-0.75=0.25$.	So $Y \to Z = 0.25$ and $Y \to Y = 0.75$.
Column 2 (from Y) $=\begin{bmatrix}0.25\\0.75\end{bmatrix}$; check $0.25+0.75=1$.	Top entry is $Y \to Z$, bottom is $Y \to Y$.
$T=\begin{bmatrix}0.85 & 0.25\\0.15 & 0.75\end{bmatrix}$.	Columns "from", rows "to"; each column sums to 1.
$S_0=\begin{bmatrix}4000\\6000\end{bmatrix}$.	Starting numbers in the order Z, Y.

Example 2 — scaffolded

Each day, commuters travelling into a city choose to go by **Car** (C) or by **Bus** (B). The transition diagram below shows the proportions that keep to, or change, their choice from one day to the next. Today 1200 people drive and 800 take the bus. Write down the transition matrix T (order C, B) and the initial state matrix S_0.



C = Car, B = Bus

Working	Comment
Arrows leaving C: loop 0.7 and $C \to B$ labelled 0.3.	These become column 1 (from Car); $0.7+0.3=1$.
Column 1 $=\begin{bmatrix}0.7\\0.3\end{bmatrix}$.	Top is $C \to C$, bottom is $C \to B$.
Arrows leaving B: loop 0.8 and $B \to C$ labelled 0.2.	These become column 2 (from Bus); $0.8+0.2=1$.
Column 2 $=\begin{bmatrix}0.2\\0.8\end{bmatrix}$.	Top is $B \to C$, bottom is $B \to B$.
$T=\begin{bmatrix}0.7 & 0.2\\0.3 & 0.8\end{bmatrix}$.	Each column of the diagram becomes a column of T.
$S_0=\begin{bmatrix}1200\\800\end{bmatrix}$.	Starting numbers in the order C, B.

Example 3 — unscaffolded

Each year, residents move between three neighbouring towns, **Alton** (A), **Bell** (B) and **Crest** (C). Of Alton's residents, 70 % stay, 20 % move to Bell and 10 % move to Crest. Of Bell's residents, 80 % stay, 10 % move to Alton and 10 % move to Crest. Of Crest's residents, 70 % stay, 20 % move to Alton and 10 % move to Bell. The current populations are 5000 in Alton, 8000 in Bell and 3000 in Crest. Construct the transition matrix T (order A, B, C) and the initial state matrix S_0.

Each town supplies one **column** — the proportions of that town's residents going to Alton, then Bell, then Crest (the row order).

From Alton: $A \to A=0.7$, $A \to B=0.2$, $A \to C=0.1$, giving column 1 $\begin{bmatrix}0.7\\0.2\\0.1\end{bmatrix}$. From Bell: $B \to A=0.1$, $B \to B=0.8$, $B \to C=0.1$, giving column 2 $\begin{bmatrix}0.1\\0.8\\0.1\end{bmatrix}$. From Crest: $C \to A=0.2$, $C \to B=0.1$, $C \to C=0.7$, giving column 3 $\begin{bmatrix}0.2\\0.1\\0.7\end{bmatrix}$. Each column sums to 1.

$$T=\begin{bmatrix}0.7 & 0.1 & 0.2\\0.2 & 0.8 & 0.1\\0.1 & 0.1 & 0.7\end{bmatrix}, S_0=\begin{bmatrix}5000\\8000\\3000\end{bmatrix}.$$

$\therefore$ the transition matrix and initial state matrix are as shown, with every column of T summing to 1.

Example 4 — unscaffolded

A market has two brands of toothpaste, **Gleam** (G) and **Shine** (S). Part of the transition matrix (order G, S) has been recorded, but two entries are missing:

$$T=\begin{bmatrix}0.65 & 0.1\\a & b\end{bmatrix}.$$

Find a and b, and state what each of t_{21} and t_{12} represents.

Every column of a transition matrix must sum to 1, because each brand's customers must all end the step using one of the two brands. Applying this to each column gives the missing entries.

Column 1 (from Gleam): $0.65+a=1$, so $a=0.35$. Column 2 (from Shine): $0.1+b=1$, so $b=0.9$. Hence

$$T=\begin{bmatrix}0.65 & 0.1\\0.35 & 0.9\end{bmatrix}.$$

The entry $t_{21}=a=0.35$ lies in column 1 (from G), row 2 (to S), so it is the proportion of Gleam users who switch to Shine in one step. The entry $t_{12}=0.1$ lies in column 2 (from S), row 1 (to G), so it is the proportion of Shine users who switch to Gleam.

$\therefore a=0.35$ and $b=0.9$; t_{21} is the proportion of Gleam users who move to Shine and t_{12} is the proportion of Shine users who move to Gleam.

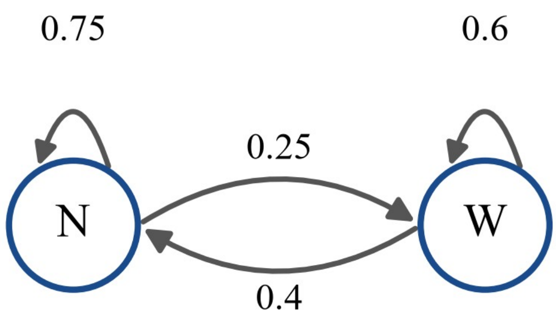
Practice questions

Scaffolded

Q1. Two cafes, **Roast** (R) and **Bean** (B), compete for the same customers. Each week 90% of Roast's customers return to Roast and the rest go to Bean; 85% of Bean's customers return to Bean and the rest go to Roast. Take the states in the order R, B. (a) State the proportions $R \to R$ and $R \to B$. (b) State the proportions $B \to B$ and $B \to R$. (c) Write column 1 (from Roast) and column 2 (from Bean) of the transition matrix. (d) Check that each column sums to 1. (e) Write the transition matrix T.

Q2. At a school canteen each student chooses a **Hot** (H) or a **Cold** (C) lunch, and the day-to-day changes are modelled by $T=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$ (order H, C). Today 150 students chose Hot and 90 chose Cold. (a) Write the initial state matrix S_0. (b) State the order of S_0 and the order of T. (c) State what the entry $t_{21}=0.2$ represents. (d) Verify that each column of T sums to 1.

Q3. A shopping centre has two car parks, the **North lot** (N) and the **West lot** (W). The transition diagram below shows, for drivers who visit on consecutive days, the proportions that return to the same lot or change lots.



N = North lot, W = West lot

(a) State the proportion of North-lot drivers who stay at the North lot, and the proportion who move to the West lot.
(b) State the proportion of West-lot drivers who stay, and the proportion who move to the North lot.
(c) Write the transition matrix T (order N, W).
(d) On one day 320 cars use the North lot and 180 use the West lot. Write the initial state matrix S_0.

Q4. A transition matrix for two states X and Y (order X, Y) has two entries missing:

$$T = \begin{bmatrix} 0.6 & 0.45 \\ p & q \end{bmatrix}.$$

(a) Use the fact that column 1 sums to 1 to find p. (b) Use the fact that column 2 sums to 1 to find q. (c) Write the completed transition matrix. (d) Explain why each column of a transition matrix must sum to 1.

Q5. Deer are tracked as they move each season between three reserves, **P**, **Q** and **R**. Of the deer in P, 60 % stay, 30 % move to Q and 10 % move to R. Of the deer in Q, 70 % stay, 20 % move to P and 10 % move to R. Of the deer in R, 70 % stay, 10 % move to P and 20 % move to Q. Take the states in the order P, Q, R. (a) Write column 1 (from P). (b) Write column 2 (from Q). (c) Write column 3 (from R). (d) Write the transition matrix T. (e) Check that each column sums to 1.

Q6. Two mobile plans, **Basic** (B) and **Premium** (P), have transition matrix $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$ (order B, P). (a) State what t_{11} represents. (b) State what t_{12} represents. (c) State what t_{21} represents. (d) Which column describes customers who start on the Premium plan? (e) Verify that each column sums to 1.

Unscaffolded

Q7. Two rideshare apps, **Dash** (D) and **Glide** (G), compete in a city. Each month 88 % of Dash's users stay with Dash and the rest switch to Glide, while 95 % of Glide's users stay with Glide and the rest switch to Dash. There are currently 7000 Dash users and 3000 Glide users. Write the transition matrix T (order D, G) and the initial state matrix S_0.

Q8. Movement between the **city** (C) and the **suburbs** (U) each year is modelled by $T = \begin{bmatrix} 0.92 & 0.06 \\ 0.08 & 0.94 \end{bmatrix}$ (order C, U).

(a) State the proportion of city residents who move to the suburbs in a year. (b) State the proportion of suburb residents who remain in the suburbs. (c) Describe in words what the entry $t_{12} = 0.06$ represents.

Q9. A transition matrix for three states A, B and C (order A, B, C) has three entries missing:

$$T=\begin{bmatrix}0.5 & 0.2 & 0.3\\0.3 & r & 0.1\\s & 0.3 & t\end{bmatrix}.$$

Find r, s and t.

Q10. In a certain town every day is classified as **Dry** (D), **Showery** (S) or **Wet** (W). After a dry day, the next day is dry, showery or wet with proportions 0.7, 0.2, 0.1. After a showery day the proportions are 0.4, 0.3, 0.3. After a wet day they are 0.2, 0.3, 0.5. Construct the transition matrix T (order D, S, W). Given that today is dry, write the initial state matrix S_0.

Q11. In a survey of hot-drink habits, each year 15 % of tea drinkers switch to coffee and the rest keep drinking tea, while 8 % of coffee drinkers switch to tea and the rest keep drinking coffee. Write the transition matrix (order Tea, Coffee) with the entries as decimals.

Q12. State, with a reason in each case, which of the following could be a transition matrix.

$$P=\begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}, Q=\begin{bmatrix}0.3 & 0.7\\0.7 & 0.2\end{bmatrix}, Rn=\begin{bmatrix}0.9 & 0.1 & 0.5\\0.1 & 0.9 & 0.5\end{bmatrix}.$$

Q13. Employment status in a region is modelled with two states, **Employed** (E) and **Unemployed** (U), by $T=\begin{bmatrix}0.95 & 0.4\\0.05 & 0.6\end{bmatrix}$ (order E, U). Describe the transition diagram for this model by listing every arrow — including the loops — together with its label.

Q14. Residents move each decade between three towns, **Highton** (H), **Lowdale** (L) and **Midbury** (M). Of Highton's residents 80 % stay, 15 % move to Lowdale and 5 % move to Midbury. Of Lowdale's residents 85 % stay, 10 % move to Highton and 5 % move to Midbury. Of Midbury's residents 85 % stay, 5 % move to Highton and 10 % move to Lowdale. The current populations are 12 000, 9000 and 6000 respectively. Write the transition matrix T (order H, L, M) and the initial state matrix S_0.

Q15. Each week 30 % of the members of **Gym A** cancel and join **Gym B**, while 25 % of the members of Gym B move to Gym A; no one else leaves the two gyms. Write the transition matrix (order A, B).

Q16. A shop's customers are in one of two states, buying **Online** (N) or **In-store** (I). Its transition diagram has a loop at Online labelled 0.6, an arrow Online → In-store labelled 0.4, a loop at In-store labelled 0.85, and an arrow In-store → Online labelled 0.15. Write the transition matrix (order N, I).

Q17. In a town of 40 000 people, 55 % support party X, 30 % support party Y and 15 % are undecided (U). Write the initial state matrix S_0 (order X, Y, U) giving the **number** of people in each state, and check that the entries add to the total population.

Q18. A magazine finds that each year it keeps 70 % of its subscribers, with the rest not renewing (becoming non-subscribers), while 10 % of non-subscribers become subscribers. Using the two states **Subscriber** (S) and **Non-subscriber** (N) in the order S, N, write the transition matrix and state what the entry t_{12} represents.

Q19. A student writes $T=\begin{bmatrix}0.8 & 0.3\\0.3 & 0.7\end{bmatrix}$ as the transition matrix for a two-state system. Explain why this cannot be correct. Given that the "stay" proportions 0.8 and 0.7 on the leading diagonal are right, write the correct transition matrix.

Q20. A car-hire firm has depots in three cities, **East** (E), **West** (W) and **North** (N). Of the cars hired from East, 50 % are returned to East, 40 % to West and 10 % to North. Of those hired from West, 60 % are returned to West, 30 % to East and 10 % to North. Of those hired from North, 60 % are returned to North, 20 % to East and 20 % to West. Currently there are 50 cars at East, 40 at West and 30 at North. (a) Write the transition matrix T (order E, W, N). (b) Write the initial state matrix S_0. (c) State what the entry t_{23} represents. (d) Explain why each column of T sums to 1.

Answers

Q1. (a) $R \to R = 0.9$, $R \to B = 0.1$; (b) $B \to B = 0.85$, $B \to R = 0.15$; (c) $\begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$, $\begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$; (d) $0.9 + 0.1 = 1$, $0.15 + 0.85 = 1$; (e) $T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$. **Q2.** (a) $S_0 = \begin{bmatrix} 150 \\ 90 \end{bmatrix}$; (b) S_0 is 2×1, T is 2×2; (c) the proportion of Hot-lunch students who choose Cold the next day; (d) $0.8 + 0.2 = 1$, $0.3 + 0.7 = 1$. **Q3.** (a) stay 0.75, move 0.25; (b) stay 0.6, move 0.4; (c) $T = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$; (d) $S_0 = \begin{bmatrix} 320 \\ 180 \end{bmatrix}$. **Q4.** (a) $p = 0.4$; (b) $q = 0.55$; (c) $T = \begin{bmatrix} 0.6 & 0.45 \\ 0.4 & 0.55 \end{bmatrix}$; (d) every member of a state must move to exactly one state, so the proportions in each column account for the whole of that state and add to 1. **Q5.** (a) $\begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$; (b) $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.1 \end{bmatrix}$; (c) $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$; (d) $T = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$; (e) each column: 1. **Q6.** (a) the proportion of Basic customers who stay on Basic; (b) the proportion of Premium customers who move to Basic; (c) the proportion of Basic customers who move to Premium; (d) column 2; (e) $0.9 + 0.1 = 1$, $0.2 + 0.8 = 1$. **Q7.** $T = \begin{bmatrix} 0.88 & 0.05 \\ 0.12 & 0.95 \end{bmatrix}$, $S_0 = \begin{bmatrix} 7000 \\ 3000 \end{bmatrix}$. **Q8.** (a) 0.08; (b) 0.94; (c) the proportion of suburb residents who move to the city in a year. **Q9.** $r = 0.5$, $s = 0.2$, $t = 0.6$. **Q10.** $T = \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}$, $S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. **Q11.** $T = \begin{bmatrix} 0.85 & 0.08 \\ 0.15 & 0.92 \end{bmatrix}$. **Q12.** Only P. Q fails because column 2 gives $0.7 + 0.2 = 0.9 \neq 1$; Rn is 2×3, not square, so it cannot be a transition matrix. **Q13.** Loop at E: 0.95; arrow $E \to U$: 0.05; arrow $U \to E$: 0.4; loop at U: 0.6. **Q14.** $T = \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.15 & 0.85 & 0.1 \\ 0.05 & 0.05 & 0.85 \end{bmatrix}$, $S_0 = \begin{bmatrix} 12\,000 \\ 9000 \\ 6000 \end{bmatrix}$. **Q15.** $T = \begin{bmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{bmatrix}$. **Q16.** $T = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix}$. **Q17.** $S_0 = \begin{bmatrix} 22\,000 \\ 12\,000 \\ 6000 \end{bmatrix}$; $22\,000 + 12\,000 + 6000 = 40\,000$. **Q18.** $T = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$; $t_{12} = 0.1$ is the proportion of non-subscribers who become subscribers. **Q19.** Column 1 gives $0.8 + 0.3 = 1.1 \neq 1$, so it is not a valid transition matrix. Correct: $T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. **Q20.** (a) $T = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}$; (b) $S_0 = \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}$; (c) the proportion of cars hired from North that are returned to West; (d) every hired car is returned to exactly one depot, so each column accounts for all of that depot's cars.

Fully-worked solutions

Q1. The state order is R, B. From Roast, 90% return, so $R \to R = 0.9$ and the rest $R \to B = 1 - 0.9 = 0.1$; this is column 1, $\begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$, and $0.9 + 0.1 = 1$. From Bean, 85% return, so $B \to B = 0.85$ and $B \to R = 1 - 0.85 = 0.15$; this is column 2, $\begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$, and $0.15 + 0.85 = 1$. Placing each "from" column in order, $T = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix}$.

Q2. (a) The initial state matrix lists the current numbers as a column in the order H, C: $S_0 = \begin{bmatrix} 150 \\ 90 \end{bmatrix}$. (b) S_0 has 2 rows and 1 column, so it is 2×1; T has 2 rows and 2 columns, so it is 2×2. (c) The entry t_{21} is in column 1 (from Hot) and row 2 (to Cold), so 0.2 is the proportion of Hot-lunch students who switch to a Cold lunch the next day. (d) Column 1 gives $0.8+0.2=1$ and column 2 gives $0.3+0.7=1$.

Q3. The arrows leaving each node give that node's column. (a) At the North lot the loop is 0.75 (stays) and the arrow $N \to W$ is 0.25 (moves). (b) At the West lot the loop is 0.6 (stays) and the arrow $W \to N$ is 0.4 (moves). (c) Column 1 (from N) is $\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$ and column 2 (from W) is $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$, so $T=\begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$. (d) The starting numbers in the order N, W give $S_0=\begin{bmatrix} 320 \\ 180 \end{bmatrix}$.

Q4. Each column of a transition matrix sums to 1. (a) Column 1: $0.6+p=1$, so $p=0.4$. (b) Column 2: $0.45+q=1$, so $q=0.55$. (c) Hence $T=\begin{bmatrix} 0.6 & 0.45 \\ 0.4 & 0.55 \end{bmatrix}$. (d) In a closed system every member of a given state must move to exactly one of the states, so the proportions in that state's column account for the whole of it and therefore add to 1.

Q5. Each reserve provides a "from" column, with the rows in the order P, Q, R. From P: 0.6 stay, 0.3 to Q, 0.1 to R, giving $\begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$. From Q: 0.2 to P, 0.7 stay, 0.1 to R, giving $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.1 \end{bmatrix}$. From R: 0.1 to P, 0.2 to Q, 0.7 stay, giving $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$. Thus $T=\begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$, and each column sums to 1 ($0.6+0.3+0.1$, $0.2+0.7+0.1$, $0.1+0.2+0.7$).

Q6. The order is B, P, so column 1 is "from Basic" and column 2 is "from Premium". (a) $t_{11}=0.9$ is in column 1 (from Basic), row 1 (to Basic): the proportion of Basic customers who stay on Basic. (b) $t_{12}=0.2$ is in column 2 (from Premium), row 1 (to Basic): the proportion of Premium customers who move to Basic. (c) $t_{21}=0.1$ is in column 1 (from Basic), row 2 (to Premium): the proportion of Basic customers who move to Premium. (d) Customers who start on Premium are described by column 2. (e) $0.9+0.1=1$ and $0.2+0.8=1$.

Q7. Order D, G. From Dash, 88% stay, so $D \to D=0.88$ and $D \to G=0.12$; from Glide, 95% stay, so $G \to G=0.95$ and $G \to D=0.05$. Placing the "from" columns in order, $T=\begin{bmatrix} 0.88 & 0.05 \\ 0.12 & 0.95 \end{bmatrix}$, and the current numbers give $S_0=\begin{bmatrix} 7000 \\ 3000 \end{bmatrix}$.

Q8. The order is C, U, so column 1 is "from city" and column 2 is "from suburbs". (a) The proportion of city residents who move to the suburbs is $t_{21}=0.08$. (b) The proportion of suburb residents who remain is $t_{22}=0.94$. (c) The entry $t_{12}=0.06$ is in column 2 (from suburbs), row 1 (to city), so it is the proportion of suburb residents who move to the city in a year.

Q9. Each column must sum to 1. Column 1: $0.5+0.3+s=1$, so $s=0.2$. Column 2: $0.2+r+0.3=1$, so $r=0.5$. Column 3: $0.3+0.1+t=1$, so $t=0.6$. Hence $r=0.5$, $s=0.2$, $t=0.6$ and $T=\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.6 \end{bmatrix}$.

Q10. Each "starting day" gives a column, with the rows in the order D, S, W. After a dry day: $\begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix}$; after a showery day: $\begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}$; after a wet day: $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$. Thus $T = \begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}$, and each column sums to 1. Today is dry, so the initial state matrix has all of the "weight" in the Dry state: $S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Q11. Order Tea, Coffee. Of tea drinkers, 15% switch to coffee, so 85% keep drinking tea: column 1 is $\begin{bmatrix} 0.85 \\ 0.15 \end{bmatrix}$. Of coffee drinkers, 8% switch to tea, so 92% keep drinking coffee: column 2 is $\begin{bmatrix} 0.08 \\ 0.92 \end{bmatrix}$. Hence $T = \begin{bmatrix} 0.85 & 0.08 \\ 0.15 & 0.92 \end{bmatrix}$.

Q12. A transition matrix must be square with every column summing to 1. For P, the columns give $0.6 + 0.4 = 1$ and $0.5 + 0.5 = 1$, and it is 2×2, so P is a valid transition matrix. For Q, column 1 gives $0.3 + 0.7 = 1$ but column 2 gives $0.7 + 0.2 = 0.9$, which is not 1, so Q is not valid. Rn is 2×3, so it is not square and cannot be a transition matrix. Only P could be a transition matrix.

Q13. The order is E, U. The leading-diagonal entries are the loops: $t_{11} = 0.95$ is a loop at E and $t_{22} = 0.6$ is a loop at U. The off-diagonal entries are the connecting arrows: $t_{21} = 0.05$ is the arrow $E \to U$ (column E, row U) and $t_{12} = 0.4$ is the arrow $U \to E$ (column U, row E). So the diagram has a loop at E labelled 0.95, an arrow $E \to U$ labelled 0.05, an arrow $U \to E$ labelled 0.4, and a loop at U labelled 0.6.

Q14. Order H, L, M; each town gives a "from" column. From Highton: 0.8 stay, 0.15 to Lowdale, 0.05 to Midbury $\Rightarrow \begin{bmatrix} 0.8 \\ 0.15 \\ 0.05 \end{bmatrix}$. From Lowdale: 0.1 to Highton, 0.85 stay, 0.05 to Midbury $\Rightarrow \begin{bmatrix} 0.1 \\ 0.85 \\ 0.05 \end{bmatrix}$. From Midbury: 0.05 to Highton, 0.1 to Lowdale, 0.85 stay $\Rightarrow \begin{bmatrix} 0.05 \\ 0.1 \\ 0.85 \end{bmatrix}$. Hence $T = \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.15 & 0.85 & 0.1 \\ 0.05 & 0.05 & 0.85 \end{bmatrix}$ (each column sums to 1) and $S_0 = \begin{bmatrix} 12\,000 \\ 9000 \\ 6000 \end{bmatrix}$.

Q15. Order A, B. From Gym A, 30% move to B, so $A \to B = 0.3$ and the rest stay, $A \to A = 1 - 0.3 = 0.7$. From Gym B, 25% move to A, so $B \to A = 0.25$ and $B \to B = 1 - 0.25 = 0.75$. Placing the "from" columns in order, $T = \begin{bmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{bmatrix}$.

Q16. Order N, I. The arrows leaving Online are its column: loop 0.6 (stays) and $N \to I = 0.4$, giving column 1 $\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$. The arrows leaving In-store are its column: $I \to N = 0.15$ and loop 0.85 (stays), giving column 2 $\begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$. Hence $T = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix}$.

Q17. The initial state matrix lists the number in each state, found by taking each percentage of $40\,000$. Party X: $0.55 \times 40\,000 = 22\,000$; party Y: $0.30 \times 40\,000 = 12\,000$; undecided: $0.15 \times 40\,000 = 6000$. In the order X, Y, U, $S_0 = \begin{bmatrix} 22\,000 \\ 12\,000 \\ 6000 \end{bmatrix}$, and the entries add to $22\,000 + 12\,000 + 6000 = 40\,000$, the whole population.

Q18. Order S, N. The magazine keeps 70% of subscribers, so $S \to S = 0.7$ and the rest lapse, $S \to N = 0.3$. Of non-subscribers, 10% become subscribers, so $N \to S = 0.1$ and $N \to N = 0.9$. Hence $T = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$. The entry t_{12} is in column 2 (from non-subscribers), row 1 (to subscribers), so $t_{12} = 0.1$ is the proportion of non-subscribers who become subscribers.

Q19. In a transition matrix each column must sum to 1. Column 1 of the student's matrix gives $0.8 + 0.3 = 1.1$, which is greater than 1, so the matrix cannot be correct. Keeping the "stay" proportions 0.8 and 0.7 on the leading diagonal, the "move" entries must be the remainders of each column: $1 - 0.8 = 0.2$ in column 1 and $1 - 0.7 = 0.3$ in column 2. The correct matrix is $T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

Q20. Order E, W, N; each depot gives a "from" column. From East: 0.5 to East, 0.4 to West, 0.1 to North $\Rightarrow \begin{bmatrix} 0.5 \\ 0.4 \\ 0.1 \end{bmatrix}$. From West: 0.3 to East, 0.6 to West, 0.1 to North $\Rightarrow \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}$. From North: 0.2 to East, 0.2 to West, 0.6 to North $\Rightarrow \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$. (a) Hence $T = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{bmatrix}$. (b) $S_0 = \begin{bmatrix} 50 \\ 40 \\ 30 \end{bmatrix}$. (c) t_{23} is in column 3 (from North) and row 2 (to West), so it is the proportion of cars hired from North that are returned to West, namely 0.2. (d) Every car hired from a depot is returned to exactly one depot, so the proportions in each column account for all of that depot's cars and therefore sum to 1.

Theory

An earlier section showed how to **set up** a transition matrix from a description or a transition diagram. This section works the other way: given a transition matrix T, we read what its entries mean in context, check that it is a valid transition matrix, turn it back into a transition diagram, and carry the population forward **one step** to see how many members move where. Repeating that step many times — the long-run behaviour — is the subject of a later section.

The convention.

A transition matrix T records the proportions of a population that move between a fixed set of **states** from one step to the next. The rows and columns are labelled by the states in the **same order**, and throughout this course:

- each **column** is a *current* (this-step) state — the state a member is moving **from**;
- each **row** is a *next* (next-step) state — the state a member is moving **to**.

So the entry t_{ij} in **row** i**, column** j is the proportion of the members that are in state j now and will be in state i next step. The state of the population is held in a **column matrix** S_n, and one step of the model is the product

$$S_{n+1} = T\,S_n .$$

Reading the entries.

Consider a town whose shoppers each buy their weekly groceries at one of two supermarkets, **Metro (**M**)** or **Nova (**N**)**. Research finds that from one week to the next, $80\,\%$ of Metro's shoppers return to Metro (so $20\,\%$ switch to Nova), while $30\,\%$ of Nova's shoppers switch to Metro (so $70\,\%$ stay at Nova). Listing the states in the order M, N,

$$T = \begin{array}{cl} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} & \begin{array}{l} \leftarrow \text{to } M \\ \leftarrow \text{to } N \end{array} \end{array} .$$

The left column is "from Metro", the right column is "from Nova". Each entry is a proportion of the group named by its **column**:

- $t_{MM} = 0.8$ is on the **leading diagonal**: it is the proportion of Metro shoppers who **stay** at Metro. Diagonal entries are **retention** proportions.
- $t_{NM} = 0.2$ is **off-diagonal**: it is the proportion of Metro shoppers who **move** to Nova. Off-diagonal entries are **movement** proportions.
- $t_{MN} = 0.3$ is the proportion of Nova shoppers who move to Metro, and $t_{NN} = 0.7$ is Nova's retention proportion.

The table below summarises where to look.

Where you read	What it tells you
diagonal entry t_{ii}	proportion of state i that **stays** in state i (the retention proportion)
off-diagonal t_{ij} ($i \neq j$)	proportion of state j that **moves** to state i
column j	how this step's state-j group **splits** next step
row i	all the groups that **feed into** state i next step

The column-sums-to-1 check.

Every member currently in a state must go **somewhere** next step — either it stays or it moves to one of the other states — and these are all the possibilities. So the proportions down any one column must account for the whole of that group:

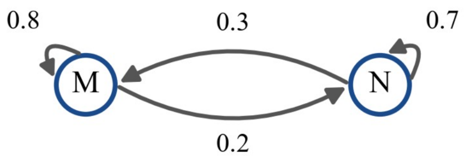
each column of a transition matrix adds to 1.

For T above, column M gives $0.8+0.2=1$ and column N gives $0.3+0.7=1$, so T is a valid transition matrix. If a column does **not** add to 1, the array cannot be a transition matrix. This check is also handy for **finding a missing entry**: if one entry of a column is unknown, it must be 1 minus the sum of the others.

Converting a matrix back to a transition diagram.

A transition diagram draws each state as a labelled circle. For every entry, draw an arrow **from column-state to row-state** and label it with the proportion — a diagonal entry becomes a **self-loop** (an arrow leaving a state and returning to it). Reading T column by column:

- from M: a self-loop 0.8 and an arrow $M \to N$ labelled 0.2;
- from N: a self-loop 0.7 and an arrow $N \to M$ labelled 0.3.



an arrow from state j to state i is labelled t_ij

The two arrows leaving each state carry labels that add to 1 — the same column-sum check, seen on the diagram.

Interpreting a single step.

Suppose that this week 500 shoppers use Metro and 500 use Nova, so the state matrix is $S_0=\begin{bmatrix}500\\500\end{bmatrix}$ (order M, N).

One step of the model gives

$$S_1=T\,S_0=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}\begin{bmatrix}500\\500\end{bmatrix}=\begin{bmatrix}0.8(500)+0.3(500)\\0.2(500)+0.7(500)\end{bmatrix}=\begin{bmatrix}400+150\\100+350\end{bmatrix}=\begin{bmatrix}550\\450\end{bmatrix}.$$

Reading the product **element by element** tells us exactly how many people move where:

- Next week's Metro total, 550, is made of the 400 Metro shoppers who **stayed** (0.8×500) **plus** the 150 Nova shoppers who **switched in** (0.3×500).
- Next week's Nova total, 450, is the 100 who left Metro plus the 350 who stayed at Nova.

Comparing S_1 with S_0, Metro has **gained** 50 shoppers ($500\to550$) while Nova has **lost** 50 ($500\to450$). The two changes are equal and opposite because no shopper is created or destroyed: the total, 1000, is unchanged. That conservation of the total is a direct consequence of every column summing to 1.

Worked examples

Example 1 — scaffolded

Each month a person belongs to one of two gyms, **Pulse (P)** or **Rush (R)**. From month to month $90\,\%$ of Pulse members renew at Pulse and $10\,\%$ switch to Rush, while $15\,\%$ of Rush members switch to Pulse and $85\,\%$ stay at

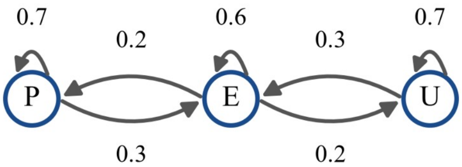
Rush. The transition matrix (order P, R) is $T=\begin{bmatrix}0.9 & 0.15\\0.1 & 0.85\end{bmatrix}$. This month 200 people belong to Pulse and 400 to Rush. Interpret the diagonal entries, check T is valid, and find next month's membership.

Working	Comment
$t_{PP}=0.9$: the proportion of Pulse members who **stay** at Pulse.	Diagonal = retention.
$t_{RR}=0.85$: the proportion of Rush members who stay at Rush.	The other diagonal entry.
Column P: $0.9+0.1=1$; column R: $0.15+0.85=1$.	Each column adds to 1, so T is valid.
$S_0=\begin{bmatrix}200\\400\end{bmatrix}$ (order P, R).	This month's membership as a column matrix.
$S_1=T\,S_0=\begin{bmatrix}0.9(200)+0.15(400)\\0.1(200)+0.85(400)\end{bmatrix}=\begin{bmatrix}180+60\\20+340\end{bmatrix}=\begin{bmatrix}240\\360\end{bmatrix}$.	One step $S_1=T\,S_0$.
Pulse: $240=180$ stayed $+60$ joined from Rush.	Element-by-element reading of row P.
Pulse $200\rightarrow 240$ (gains 40); Rush $400\rightarrow 360$ (loses 40).	Pulse is the state that gains.

So next month Pulse has 240 members and Rush has 360; Pulse gains 40 members from Rush.

Example 2 — scaffolded

A tagged animal is tracked between three zones — **Park** (P), **Edge** (E) and **Urban** (U) — with the transition diagram below (proportions per season, order P, E, U).



Write down the transition matrix, state what the entry 0.3 in the "from U" column means, and find the next state matrix if this season 300 animals are in the Park, 400 at the Edge and 300 in Urban zones.

Working	Comment
From P: loop 0.7, arrow $P\rightarrow E$ of 0.3. Column $P=\begin{bmatrix}0.7\\0.3\\0\end{bmatrix}$.	No arrow $P\rightarrow U$, so that entry is 0.
From E: arrows $E\rightarrow P$ of 0.2, loop 0.6, $E\rightarrow U$ of 0.2. Column $E=\begin{bmatrix}0.2\\0.6\\0.2\end{bmatrix}$.	Reads down the "from E" arrows.
From U: arrow $U\rightarrow E$ of 0.3, loop 0.7. Column	No arrow $U\rightarrow P$, so that entry is 0.

Working	Comment
$U=\begin{bmatrix}0\\0.3\\0.7\end{bmatrix}$.	
$T=\begin{bmatrix}0.7&0.2&0\\0.3&0.6&0.3\\0&0.2&0.7\end{bmatrix}$.	Columns sum $1,1,1$: valid.
The 0.3 in the "from U" column (row E) is t_{EU}: the proportion of Urban animals that **move to the Edge** next season.	An off-diagonal = movement.
$S_1=T\begin{bmatrix}300\\400\\300\end{bmatrix}=\begin{bmatrix}0.7(300)+0.2(400)+0(300)\\0.3(300)+0.6(400)+0.3(300)\\0(300)+0.2(400)+0.7(300)\end{bmatrix}=\begin{bmatrix}29\\42\\29\end{bmatrix}$.	One step $S_1=T\,S_0$.
Edge: $420=90$ from Park $+240$ stayed $+90$ from Urban.	The Edge is the only zone that gains (400 → 420).

The next state matrix is $\begin{bmatrix}290\\420\\290\end{bmatrix}$: the Edge gains 20 animals while the Park and Urban zones each lose 10.

Example 3 — unscaffolded

At the start of each week a taxi is working in one of three zones — **North (N)**, **Central (C)** or **South (S)**. The transition matrix (order N,C,S) is

$$T=\begin{bmatrix}0.5&0.25&0.2\\0.3&0.5&0.2\\0.2&0.25&0.6\end{bmatrix}.$$

This week 400 taxis work North, 400 Central and 200 South. Find next week's distribution and state which zone's fleet grows the most.

Each column sums to 1 ($0.5+0.3+0.2$, $0.25+0.5+0.25$, $0.2+0.2+0.6$), so T is a valid transition matrix. With $S_0=\begin{bmatrix}400\\400\\200\end{bmatrix}$,

$$S_1=T\,S_0=\begin{bmatrix}0.5(400)+0.25(400)+0.2(200)\\0.3(400)+0.5(400)+0.2(200)\\0.2(400)+0.25(400)+0.6(200)\end{bmatrix}=\begin{bmatrix}200+100+40\\120+200+40\\80+100+120\end{bmatrix}=\begin{bmatrix}340\\360\\300\end{bmatrix}.$$

Comparing with S_0: North 400 → 340 (down 60), Central 400 → 360 (down 40) and South 200 → 300 (up 100). The total is 1000 both weeks, as it must be. The South total 300 is made up of 80 taxis that came from North, 100 from Central and 120 that stayed in the South.

∴ next week the fleet is 340 North, 360 Central and 300 South; the **South** zone grows the most, gaining 100 taxis.

Example 4 — unscaffolded

Readers of a monthly magazine choose between the **Herald (H)** and the **Gazette (G)**. Each month 65 % of Herald readers stay with the Herald and 40 % of Gazette readers switch to the Herald. Part of the transition matrix has been filled in (order H,G):

$$T = \begin{bmatrix} 0.65 & 0.40 \\ \square & 0.60 \end{bmatrix}.$$

Complete T, describe its transition diagram, and find next month's readership if 300 people currently read the Herald and 200 read the Gazette.

The missing entry t_{GH} is the proportion of Herald readers who switch to the Gazette. Because column H must sum to 1, $t_{GH} = 1 - 0.65 = 0.35$ (equivalently, the 35% of Herald readers who do not stay). Checking column G: $0.40 + 0.60 = 1$. So

$$T = \begin{bmatrix} 0.65 & 0.40 \\ 0.35 & 0.60 \end{bmatrix}.$$

The transition diagram has two states H and G, a self-loop of 0.65 on H and 0.60 on G, an arrow $H \to G$ labelled 0.35 and an arrow $G \to H$ labelled 0.40. With $S_0 = \begin{bmatrix} 300 \\ 200 \end{bmatrix}$,

$$S_1 = \begin{bmatrix} 0.65(300) + 0.40(200) \\ 0.35(300) + 0.60(200) \end{bmatrix} = \begin{bmatrix} 195 + 80 \\ 105 + 120 \end{bmatrix} = \begin{bmatrix} 275 \\ 225 \end{bmatrix}.$$

The Herald falls from 300 to 275 (down 25) while the Gazette rises from 200 to 225 (up 25).

$\therefore$ the completed matrix is $\begin{bmatrix} 0.65 & 0.40 \\ 0.35 & 0.60 \end{bmatrix}$, and next month the **Gazette** gains 25 readers to reach 225, with the Herald on 275.

Practice questions

Scaffolded

Q1. Shoppers use one of two supermarkets, Apex (A) or Bright (B), with transition matrix $T = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}$ (order A, B). (a) Show that each column of T sums to 1. (b) What proportion of Apex shoppers stay at Apex? (c) What proportion of Apex shoppers switch to Bright? (d) What proportion of Bright shoppers switch to Apex? (e) If $S_0 = \begin{bmatrix} 400 \\ 600 \end{bmatrix}$, find $S_1 = T\,S_0$. (f) Which store gains shoppers, and how many?

Q2. Each day a commuter either drives (D) or catches the train (R), with $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ (order D, R). (a) Interpret the entries 0.6 and 0.4 in the "from D" column. (b) If today 500 people drive and 500 catch the train, find tomorrow's state matrix. (c) Show how the "drive" total of 450 splits into those who kept driving and those who switched from the train. (d) Which mode gains commuters, and by how many?

Q3. Customers bank with Coastal (C) or Delta (D). The transition matrix (order C, D) is $T = \begin{bmatrix} 0.85 & \square \\ 0.15 & 0.70 \end{bmatrix}$. (a) Use the column-sum rule to find the missing entry. (b) State what that entry means in context. (c) Write down the retention proportion of each bank. (d) If $S_0 = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$, find S_1 and state which bank gains customers.

Q4. A brand of drink is either the market leader Zesty (Z) or Other (O). The transition matrix (order Z, O) is $T = \begin{bmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{bmatrix}$. (a) Draw the transition diagram, labelling every arrow. (b) Interpret the entry 0.9. (c) Verify that T is a valid transition matrix.

Q5. A bike-share scheme has three docks — North (N), Mid (M) and South (S) — with

$$T = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} (\text{order } N, M, S).$$

(a) Check that every column sums to 1. (b) What proportion of Mid bikes stay at Mid? (c) What proportion of North bikes move to Mid? (d) What proportion of South bikes move to North? (e) Which dock has the highest retention proportion?

Q6. Using the matrix T from **Q5**, suppose this morning there are 300 bikes at North, 400 at Mid and 300 at South, so $S_0 = \begin{bmatrix} 300 \\ 400 \\ 300 \end{bmatrix}$. (a) Find $S_1 = T\,S_0$. (b) Show how the North total splits into bikes that stayed and bikes that arrived from each other dock. (c) Which dock gains bikes, and by how many?

Unscaffolded

Q7. Filmgoers attend the Palace (P) or the Odeon (O), with $T = \begin{bmatrix} 0.8 & 0.35 \\ 0.2 & 0.65 \end{bmatrix}$ (order P, O) and $S_0 = \begin{bmatrix} 300 \\ 200 \end{bmatrix}$. Interpret the entry 0.35, find S_1, and state which cinema gains patrons.

Q8. A café-loyalty study of two cafés (order café 1, café 2) gives $T = \begin{bmatrix} \square & 0.45 \\ 0.15 & \square \end{bmatrix}$. Use the column-sum rule to complete T, state the retention proportion of each café, and find S_1 when $S_0 = \begin{bmatrix} 240 \\ 160 \end{bmatrix}$.

Q9. A bird is found in Forest (F), Wetland (W) or Grassland (G), with

$$T = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.2 \\ 0.2 & 0.1 & 0.6 \end{bmatrix} (\text{order } F, W, G), S_0 = \begin{bmatrix} 500 \\ 300 \\ 200 \end{bmatrix}.$$

Find S_1, and state which habitat's population grows the most and which falls the most.

Q10. A population is spread over three states with transition matrix (order state $1, 2, 3$)

$$T = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.6 \end{bmatrix}.$$

(a) What proportion of state 2 stays in state 2? (b) What proportion of state 1 moves to state 2? (c) Describe in words what the third column tells you. (d) Explain why each column of T must sum to 1.

Q11. A transition matrix for two states A and B (order A, B) is $T = \begin{bmatrix} 0.55 & 0.40 \\ 0.45 & 0.60 \end{bmatrix}$. Describe its transition diagram in full — the two self-loops and the two connecting arrows, with their labels — and state the retention proportion of each state.

Q12. Explain why the array $\begin{bmatrix} 0.6 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$ **cannot** be a transition matrix.

Q13. Gym members hold a Basic (B), Plus (P) or Premium (M) tier, with

$$T = \begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.2 & 0.6 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} (\text{order } B, P, M), S_0 = \begin{bmatrix} 300 \\ 300 \\ 400 \end{bmatrix}.$$

(a) Find S_1. (b) Interpret the entry 0 in the "from M" column. (c) Which tier grows over the month?

Q14. Three shops A, B, C share customers with transition matrix (order A, B, C)

$$T = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.3 & \square & 0.3 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}, S_0 = \begin{bmatrix} 200 \\ 500 \\ 300 \end{bmatrix}.$$

(a) Find the missing entry. (b) Find S_1. (c) State which shop's total is unchanged and which shop loses customers.

Q15. Two phone carriers, Telco X and Telco Y (order X, Y), have $T = \begin{bmatrix} 0.92 & 0.05 \\ 0.08 & 0.95 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 5000 \\ 5000 \end{bmatrix}$. Find S_1, state which carrier is gaining customers, and explain what the diagonal entries say about customer loyalty.

Q16. Delivery vans are based at the East (E), Central (C) or West (W) depot, with

$$T = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} (\text{order } E, C, W), S_0 = \begin{bmatrix} 400 \\ 400 \\ 200 \end{bmatrix}.$$

Find S_1, break next week's East total into those that stayed and those that arrived from each other depot, and state which depot's total is unchanged.

Q17. A population moves between three states with transition matrix (order state $1, 2, 3$)

$$T = \begin{bmatrix} 0.9 & 0.1 & 0.2 \\ 0.05 & 0.8 & 0.3 \\ 0.05 & 0.1 & 0.5 \end{bmatrix}.$$

(a) Which state has the highest retention? (b) Which state loses the greatest proportion of its members each step? (c) What proportions of that state's members move to each of the other two states?

Q18. A machine part is Good (P), Worn (Q) or Failed (R), with transition matrix (order P, Q, R)

$$T = \begin{bmatrix} 0.8 & 0 & 0.1 \\ 0.2 & 0.9 & 0 \\ 0 & 0.1 & 0.9 \end{bmatrix}.$$

(a) Draw the transition diagram. (b) State which direct transitions between different states are impossible, and identify the zero entry responsible for each. (c) Write down the retention proportion of each state.

Q19. For two states A and B (order A, B), $T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 600 \\ 400 \end{bmatrix}$. Find S_1, and comment on the two totals after one step.

Q20. A population occupies states A, B, C with transition matrix (order A, B, C)

$$T = \begin{bmatrix} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.7 & 0.4 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}, S_0 = \begin{bmatrix} 400 \\ 300 \\ 300 \end{bmatrix}.$$

(a) Find S_1 and state which state gains the most. (b) How many members move from state A to state B in this step? (c) Explain why the total population is the same before and after the step.

Answers

Q1. (a) $0.75+0.25=1$, $0.20+0.80=1$; (b) 0.75; (c) 0.25; (d) 0.20; (e) $\begin{bmatrix}420\\580\end{bmatrix}$; (f) Apex gains 20 (Bright loses 20).

Q2. (a) 0.6 = drivers who drive again, 0.4 = drivers who switch to the train; (b) $\begin{bmatrix}450\\550\end{bmatrix}$; (c) $450=300$ kept driving $+150$ switched from the train; (d) the train gains 50. **Q3.** (a) 0.30; (b) proportion of Delta customers who move to Coastal; (c) Coastal 0.85, Delta 0.70; (d) $\begin{bmatrix}260\\240\end{bmatrix}$, Coastal gains 60. **Q4.** (a) see the diagram in the solutions; (b) proportion of Zesty buyers who buy Zesty again (0.9); (c) columns $0.9+0.1=1$ and $0.25+0.75=1$. **Q5.** (a) each column sums to 1; (b) 0.6; (c) 0.2; (d) 0.1; (e) North (0.7). **Q6.** (a) $\begin{bmatrix}320\\390\\290\end{bmatrix}$; (b) $320=210$ stayed $+80$ from Mid $+30$ from South; (c) North gains 20. **Q7.** 0.35 = Odeon patrons who switch to the Palace; $S_1=\begin{bmatrix}310\\190\end{bmatrix}$; the Palace gains 10.

Q8. $T=\begin{bmatrix}0.85 & 0.45\\0.15 & 0.55\end{bmatrix}$; retention café 1 0.85, café 2 0.55; $S_1=\begin{bmatrix}276\\124\end{bmatrix}$. **Q9.** $S_1=\begin{bmatrix}370\\380\\250\end{bmatrix}$; Wetland grows the most ($+80$), Forest falls the most (-130). **Q10.** (a) 0.7; (b) 0.3; (c) of the current state-3 members, 0.3 move to state 1, 0.1 move to state 2 and 0.6 stay in state 3; (d) every member must go somewhere, so the proportions down a column account for the whole group and total 1. **Q11.** Self-loops 0.55 on A and 0.60 on B; arrow $A \to B$ labelled 0.45 and arrow $B \to A$ labelled 0.40; retention $A\,0.55$, $B\,0.60$. **Q12.** The first column sums to $0.6+0.5=1.1\neq 1$, so the destinations of the state-1 members would account for more than $100\,\%$ of them; a valid transition matrix needs every column to sum to 1. **Q13.** (a) $\begin{bmatrix}270\\360\\370\end{bmatrix}$; (b) no Premium member moves directly to Basic; (c) the Plus tier grows ($+60$).

Q14. (a) 0.7; (b) $\begin{bmatrix}230\\500\\270\end{bmatrix}$; (c) shop B is unchanged, shop C loses 30. **Q15.** $S_1=\begin{bmatrix}4850\\5150\end{bmatrix}$; Telco Y is gaining ($+150$); the high diagonals (0.92, 0.95) mean most customers stay with their carrier each month. **Q16.** $S_1=\begin{bmatrix}380\\400\\220\end{bmatrix}$; East $380=320$ stayed $+40$ from Central $+20$ from West; Central is unchanged. **Q17.** (a) state 1 (0.9); (b) state 3 (retains only 0.5); (c) of the state-3 members, 0.2 move to state 1 and 0.3 move to state 2 (together the 0.5 that leave). **Q18.** (a) see the diagram in the solutions; (b) $Q \to P$ (entry $t_{PQ}=0$), $P \to R$ (entry $t_{RP}=0$) and $R \to Q$ (entry $t_{QR}=0$); (c) $P\,0.8$, $Q\,0.9$, $R\,0.9$. **Q19.** $S_1=\begin{bmatrix}500\\500\end{bmatrix}$; both states now hold 500 — A has fallen by 100 and B has risen by 100. **Q20.** (a) $\begin{bmatrix}330\\450\\220\end{bmatrix}$, state B gains the most ($+150$); (b) $0.3\times 400=120$; (c) each column sums to 1, so every member is counted exactly once and the total stays at 1000.

Fully-worked solutions

Q1. (a) Column A: $0.75+0.25=1$; column B: $0.20+0.80=1$, so T is a valid transition matrix. (b) The diagonal entry $t_{AA}=0.75$ is the proportion of Apex shoppers who stay at Apex. (c) The remaining Apex proportion is the off-diagonal $t_{BA}=0.25$. (d) The Bright-to-Apex movement is $t_{AB}=0.20$. (e)

$$S_1=\begin{bmatrix}0.75 & 0.20\\0.25 & 0.80\end{bmatrix}\begin{bmatrix}400\\600\end{bmatrix}=\begin{bmatrix}300+120\\100+480\end{bmatrix}=\begin{bmatrix}420\\580\end{bmatrix}.$$

(f) Apex rises 400 → 420 (gains 20) while Bright falls 600 → 580 (loses 20).

Q2. (a) In the "from D" column, $0.6=t_{DD}$ is the proportion of drivers who drive again the next day, and $0.4=t_{RD}$ is the proportion of drivers who switch to the train. (b)

$$S_1=\begin{bmatrix}0.6 & 0.3\\0.4 & 0.7\end{bmatrix}\begin{bmatrix}500\\500\end{bmatrix}=\begin{bmatrix}300+150\\200+350\end{bmatrix}=\begin{bmatrix}450\\550\end{bmatrix}.$$

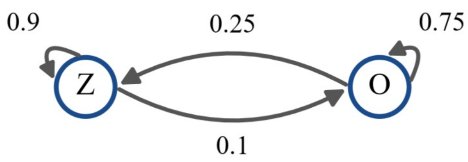
(c) The "drive" total is row D: $450=0.6(500)+0.3(500)=300$ who kept driving $+150$ who switched from the train. (d) Driving falls 500 → 450 and the train rises 500 → 550, so the train gains 50 commuters.

Q3. (a) Column D must sum to 1, so the missing entry is $1-0.70=0.30$. (b) It is $t_{CD}=0.30$, the proportion of Delta customers who move to Coastal. (c) The diagonal entries give the retention proportions: Coastal 0.85, Delta 0.70. (d)

$$S_1=\begin{bmatrix}0.85 & 0.30\\0.15 & 0.70\end{bmatrix}\begin{bmatrix}200\\300\end{bmatrix}=\begin{bmatrix}170+90\\30+210\end{bmatrix}=\begin{bmatrix}260\\240\end{bmatrix}.$$

Coastal rises 200 → 260, so Coastal gains 60 customers.

Q4. (a) The diagram has states Z and O, with a self-loop 0.9 on Z and 0.75 on O, an arrow $Z \to O$ labelled 0.1 and an arrow $O \to Z$ labelled 0.25.



(b) The entry $0.9=t_{ZZ}$ is the proportion of Zesty buyers who buy Zesty again next time (the brand's retention). (c) Column Z: $0.9+0.1=1$; column O: $0.25+0.75=1$, so T is a valid transition matrix.

Q5. (a) Column N: $0.7+0.2+0.1=1$; column M: $0.2+0.6+0.2=1$; column S: $0.1+0.3+0.6=1$. (b) The Mid retention is $t_{MM}=0.6$. (c) The North-to-Mid proportion is $t_{MN}=0.2$. (d) The South-to-North proportion is $t_{NS}=0.1$. (e) The diagonal entries are $0.7, 0.6, 0.6$, so North has the highest retention, 0.7.

Q6. (a)

$$S_1=\begin{bmatrix}0.7 & 0.2 & 0.1\\0.2 & 0.6 & 0.3\\0.1 & 0.2 & 0.6\end{bmatrix}\begin{bmatrix}300\\400\\300\end{bmatrix}=\begin{bmatrix}210+80+30\\60+240+90\\30+80+180\end{bmatrix}=\begin{bmatrix}320\\390\\290\end{bmatrix}.$$

(b) The North total is row N: $320=0.7(300)+0.2(400)+0.1(300)=210$ bikes that stayed at North $+80$ that came from Mid $+30$ that came from South. (c) North rises 300 → 320 (gains 20); Mid and South each fall by 10.

Q7. The entry $0.35=t_{PO}$ is the proportion of Odeon patrons who switch to the Palace.

$$S_1=\begin{bmatrix}0.8 & 0.35\\0.2 & 0.65\end{bmatrix}\begin{bmatrix}300\\200\end{bmatrix}=\begin{bmatrix}240+70\\60+130\end{bmatrix}=\begin{bmatrix}310\\190\end{bmatrix}.$$

The Palace rises 300 → 310, so the Palace gains 10 patrons (the Odeon loses 10).

Q8. Each column must sum to 1: the top-left entry is $1-0.15=0.85$ and the bottom-right entry is $1-0.45=0.55$, so

$$T=\begin{bmatrix}0.85 & 0.45\\ 0.15 & 0.55\end{bmatrix}.$$

The retention proportions are café 1: 0.85 and café 2: 0.55. Then

$$S_1=\begin{bmatrix}0.85 & 0.45\\ 0.15 & 0.55\end{bmatrix}\begin{bmatrix}240\\ 160\end{bmatrix}=\begin{bmatrix}204+72\\ 36+88\end{bmatrix}=\begin{bmatrix}276\\ 124\end{bmatrix}.$$

Q9.

$$S_1=\begin{bmatrix}0.6 & 0.1 & 0.2\\ 0.2 & 0.8 & 0.2\\ 0.2 & 0.1 & 0.6\end{bmatrix}\begin{bmatrix}500\\ 300\\ 200\end{bmatrix}=\begin{bmatrix}300+30+40\\ 100+240+40\\ 100+30+120\end{bmatrix}=\begin{bmatrix}370\\ 380\\ 250\end{bmatrix}.$$

The changes are Forest 500 → 370 (down 130), Wetland 300 → 380 (up 80) and Grassland 200 → 250 (up 50). So the Wetland population grows the most ($+80$) and the Forest population falls the most (-130).

Q10. (a) The diagonal entry $t_{22}=0.7$ is the proportion of state 2 that stays in state 2. (b) The entry $t_{21}=0.3$ (row 2, column 1) is the proportion of state 1 that moves to state 2. (c) The third column, $\begin{bmatrix}0.3\\ 0.1\\ 0.6\end{bmatrix}$, describes how this step's state-3 members split next step: 0.3 move to state 1, 0.1 move to state 2 and 0.6 stay in state 3. (d) A column lists every destination of a state's current members. Since each member must go to exactly one state (staying counts as a destination), the proportions account for the whole group and therefore add to 1.

Q11. Reading T column by column: column A gives a self-loop 0.55 on A and an arrow $A\to B$ labelled 0.45; column B gives an arrow $B\to A$ labelled 0.40 and a self-loop 0.60 on B. The retention proportions are the diagonal entries, $A:0.55$ and $B:0.60$. (Each state's two outgoing labels add to 1: $0.55+0.45=1$ and $0.60+0.40=1$.)

Q12. In a transition matrix every column must sum to 1. Here the first column sums to $0.6+0.5=1.1$, which would mean the two destinations of the state-1 members (60% staying and 50% moving) account for 110% of them — impossible. So the array cannot be a transition matrix. (The second column, $0.3+0.7=1$, is fine, but one bad column is enough.)

Q13. (a)

$$S_1=\begin{bmatrix}0.7 & 0.2 & 0\\ 0.2 & 0.6 & 0.3\\ 0.1 & 0.2 & 0.7\end{bmatrix}\begin{bmatrix}300\\ 300\\ 400\end{bmatrix}=\begin{bmatrix}210+60+0\\ 60+180+120\\ 30+60+280\end{bmatrix}=\begin{bmatrix}270\\ 360\\ 370\end{bmatrix}.$$

(b) The entry $0=t_{BM}$ (row B, "from M" column) means none of the Premium members move directly to Basic. (c) The changes are Basic 300 → 270 (down 30), Plus 300 → 360 (up 60) and Premium 400 → 370 (down 30), so the Plus tier grows over the month.

Q14. (a) Column B must sum to 1, so the missing entry is $1-(0.1+0.2)=0.7$. (b)

$$S_1=\begin{bmatrix}0.6 & 0.1 & 0.2\\ 0.3 & 0.7 & 0.3\\ 0.1 & 0.2 & 0.5\end{bmatrix}\begin{bmatrix}200\\ 500\\ 300\end{bmatrix}=\begin{bmatrix}120+50+60\\ 60+350+90\\ 20+100+150\end{bmatrix}=\begin{bmatrix}230\\ 500\\ 270\end{bmatrix}.$$

(c) Shop B is unchanged (500 → 500); shop A gains 30 and shop C loses 30 (300 → 270).

Q15.

$$S_1=\begin{bmatrix}0.92 & 0.05\\ 0.08 & 0.95\end{bmatrix}\begin{bmatrix}5000\\ 5000\end{bmatrix}=\begin{bmatrix}4600+250\\ 400+4750\end{bmatrix}=\begin{bmatrix}4850\\ 5150\end{bmatrix}.$$

Telco X falls 5000 → 4850 and Telco Y rises 5000 → 5150, so Telco Y is gaining customers ($+150$). The diagonal entries 0.92 and 0.95 are large, so each month most customers stay with their current carrier — customer loyalty is high, and change from step to step is slow.

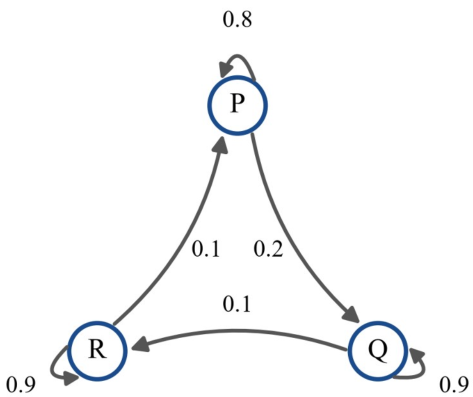
Q16.

$$S_1=\begin{bmatrix}0.8 & 0.1 & 0.1\\ 0.1 & 0.8 & 0.2\\ 0.1 & 0.1 & 0.7\end{bmatrix}\begin{bmatrix}400\\ 400\\ 200\end{bmatrix}=\begin{bmatrix}320+40+20\\ 40+320+40\\ 40+40+140\end{bmatrix}=\begin{bmatrix}380\\ 400\\ 220\end{bmatrix}.$$

The East total is row E: $380=0.8(400)+0.1(400)+0.1(200)=320$ vans that stayed at East $+40$ from Central $+20$ from West. Comparing with S_0, Central is unchanged (400 → 400), East loses 20 and West gains 20.

Q17. (a) The diagonal entries are $0.9, 0.8, 0.5$, so state 1 has the highest retention, 0.9. (b) State 3 retains only 0.5 of its members, so it loses the greatest proportion, $1-0.5=0.5$, each step. (c) The "from state 3" column is $\begin{bmatrix}0.2\\ 0.3\\ 0.5\end{bmatrix}$: of the state-3 members, a proportion 0.2 moves to state 1 and 0.3 moves to state 2. These two account for the 0.5 that leave, since $0.2+0.3=0.5$.

Q18. (a) The diagram has states P, Q, R, self-loops 0.8, 0.9, 0.9, and the three arrows $P\to Q$ (label 0.2), $Q\to R$ (label 0.1) and $R\to P$ (label 0.1).



(b) A zero off-diagonal entry marks an impossible direct transition. Here $t_{PQ}=0$ means $Q\to P$ is impossible, $t_{RP}=0$ means $P\to R$ is impossible, and $t_{QR}=0$ means $R\to Q$ is impossible. (c) The retention proportions are the diagonal entries: $P:0.8$, $Q:0.9$, $R:0.9$.

Q19.

$$S_1=\begin{bmatrix}0.7 & 0.2\\ 0.3 & 0.8\end{bmatrix}\begin{bmatrix}600\\ 400\end{bmatrix}=\begin{bmatrix}420+80\\ 180+320\end{bmatrix}=\begin{bmatrix}500\\ 500\end{bmatrix}.$$

After one step both states hold 500: state A has fallen from 600 to 500 (down 100) and state B has risen from 400 to 500 (up 100). The total stays at 1000.

Q20. (a)

$$S_1=\begin{bmatrix}0.6 & 0.2 & 0.1\\ 0.3 & 0.7 & 0.4\\ 0.1 & 0.1 & 0.5\end{bmatrix}\begin{bmatrix}400\\ 300\\ 300\end{bmatrix}=\begin{bmatrix}240+60+30\\ 120+210+120\\ 40+30+150\end{bmatrix}=\begin{bmatrix}330\\ 450\\ 220\end{bmatrix}.$$

The changes are $A: 400 \rightarrow 330\ (-70)$, $B: 300 \rightarrow 450\ (+150)$ and $C: 300 \rightarrow 220\ (-80)$, so state B gains the most. (b) The number moving from state A to state B is $t_{BA} \times (S_0)_A = 0.3 \times 400 = 120$. (c) Because every column of T sums to 1, each member of every state is sent to exactly one destination and none is lost or added; the total is therefore conserved at $400 + 300 + 300 = 1000$, which matches $330 + 450 + 220 = 1000$.

Theory

Many situations involve a fixed set of **states** that individuals move between at regular time steps: drivers switching between two petrol stations from week to week, animals moving between zones of a park each month, or customers changing phone plans each year. If we know how many are in each state now, and the **proportions** that move between states in one step, we can predict how the numbers change over time. Matrices make this prediction routine.

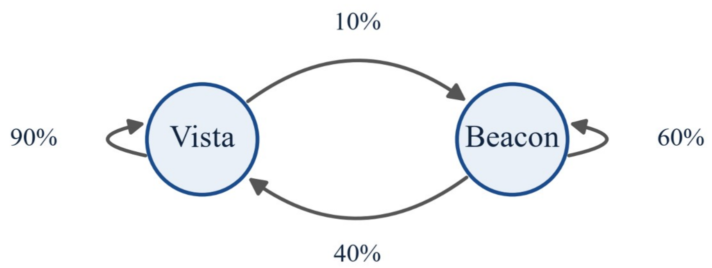
State matrices and transition matrices.

The **state matrix** S_n is a *column* matrix listing how many are in each state after n steps. The starting distribution is the **initial state matrix** S_0. The **transition matrix** T is a square matrix that records the proportions moving between states in one step, with the convention

$$\text{entry in row } i\text{, column } j = \text{the proportion of state } j \text{ that moves to state } i.$$

So each **column** describes where a state goes *from*, and each **row** describes where a state is fed *to*. Because everyone in a given state must go somewhere, **every column of T adds to** 1.

For example, suppose drivers fill up at one of two petrol stations, *Vista* (V) and *Beacon* (B), each week. Of those who use Vista this week, $90\,\%$ return to Vista and $10\,\%$ switch to Beacon; of those who use Beacon, $40\,\%$ switch to Vista and $60\,\%$ stay. The transition diagram below records these proportions — each arrow runs *from* the current station *to* next week's station, and the looped arrows are the drivers who stay.



Reading each **column** as "from", the transition matrix (in the order V, B) is

$$T = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}.$$

The first column $\begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$ says that of the Vista drivers, 0.9 stay at Vista and 0.1 move to Beacon; the second column does the same for Beacon. Both columns add to 1.

The matrix recurrence relation.

If $S_n = \begin{bmatrix} a \\ b \end{bmatrix}$ is this week's state matrix, then next week's numbers are found by multiplying by T:

$$S_{n+1} = T\,S_n.$$

This is a **first-order matrix recurrence relation**: each state matrix depends only on the one immediately before it (the next state relies only on the current state). Together with the initial state matrix S_0 it defines the whole sequence $S_1, S_2, S_3, \ldots$

Generating the sequence by repeated multiplication.

Suppose 1000 drivers are split evenly to begin with, $S_0=\begin{bmatrix}500\\500\end{bmatrix}$. Applying the recurrence one step at a time,

$$S_1=T\,S_0=\begin{bmatrix}0.9 & 0.4\\0.1 & 0.6\end{bmatrix}\begin{bmatrix}500\\500\end{bmatrix}=\begin{bmatrix}0.9(500)+0.4(500)\\0.1(500)+0.6(500)\end{bmatrix}=\begin{bmatrix}650\\350\end{bmatrix},$$

$$S_2=T\,S_1=\begin{bmatrix}0.9(650)+0.4(350)\\0.1(650)+0.6(350)\end{bmatrix}=\begin{bmatrix}725\\275\end{bmatrix}, S_3=T\,S_2=\begin{bmatrix}762.5\\237.5\end{bmatrix}.$$

Notice that each state matrix still totals 1000: the drivers are only moving between stations, not appearing or disappearing.

Jumping ahead with a matrix power.

Stepping through every term is slow if we want the state after many steps. Since $S_1=T\,S_0$ and $S_2=T\,S_1=T\,(T\,S_0)=T^2\,S_0$, and so on, in general

$$S_n=T^n\,S_0.$$

So to find the state after n steps we raise T to the power n (this is where a CAS is used — a matrix power by hand is tedious) and multiply by S_0. For the drivers,

$$S_{10}=T^{10}\,S_0\approx\begin{bmatrix}799.7\\200.3\end{bmatrix}.$$

The long-run (steady) state.

Look at the sequence $650, 725, 762.5, 781.25, \ldots$ for the Vista drivers: the numbers keep increasing but by smaller and smaller amounts, and after enough steps there is **no noticeable change** from one state matrix to the next. The state matrix settles down to a fixed distribution called the **steady state** (or **equilibrium state**). Here it settles to

$$S=\begin{bmatrix}800\\200\end{bmatrix}.$$

The steady state has three important features.

- **It is unchanged by a further step:** applying T once more leaves it fixed, so $T\,S=S$. Checking, $\begin{bmatrix}0.9 & 0.4\\0.1 & 0.6\end{bmatrix}\begin{bmatrix}800\\200\end{bmatrix}=\begin{bmatrix}720+80\\80+120\end{bmatrix}=\begin{bmatrix}800\\200\end{bmatrix}$.
- **It does not depend on where you started.** A transition matrix is called **regular** if some power of it has *no zero entries* — that is, for some number of steps, every state can be reached from every state in exactly that many steps. (A transition matrix whose entries are *all* positive is regular straight away, since the first power already has no zeros.) For a regular transition matrix the steady state is unique and is reached no matter what S_0 is. If only 100 drivers had started at Vista, $S_0=\begin{bmatrix}100\\900\end{bmatrix}$, the sequence $\begin{bmatrix}450\\550\end{bmatrix}, \begin{bmatrix}625\\375\end{bmatrix}, \begin{bmatrix}712.5\\287.5\end{bmatrix}, \ldots$ still approaches $\begin{bmatrix}800\\200\end{bmatrix}$.
- **It is found by iterating far enough.** Keep multiplying by T (or compute $T^n\,S_0$ for a large n) until the entries stop changing to the accuracy you need.

Interpreting the steady state. In the long run about 800 drivers use Vista and 200 use Beacon each week — an $80\,\%$ / $20\,\%$ split — regardless of how the drivers were divided at the start.

Points to note.

- Each column of a transition matrix adds to 1, and every state matrix keeps the same total. These are quick checks that T and each S_n are set up correctly.
- Order matters: $S_n = T^n S_0$, with T on the **left** of the state matrix. Writing the product the other way around is not even defined.
- Keep full accuracy while iterating and round only in the final interpretation (you cannot have half a driver, so a state matrix entry of 762.5 would be reported as "about 763").
- The steady state is identified **informally**: keep iterating (or compute $T^n S_0$ for a large n on CAS) until there is no noticeable change from one state matrix to the next. Once you have it — or if one is given to you — substituting it into $T\,S = S$ is a quick check that it really is the steady state.

Worked examples

Example 1 — scaffolded

A car-share company keeps its cars at two depots, City (C) and Airport (A). Each day, of the cars at City $70\,\%$ stay at City and $30\,\%$ are driven to Airport; of the cars at Airport $40\,\%$ go to City and $60\,\%$ stay. This morning 700 cars are at City and none at Airport. Find the state matrix after 1 day and after 2 days.

Working	Comment
Order the states C, A. $T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$.	Column C = "from City": 0.7 stay, 0.3 leave. Column A = "from Airport".
$S_0 = \begin{bmatrix} 700 \\ 0 \end{bmatrix}$.	Initial state matrix (a column matrix).
Each column adds to 1: $0.7+0.3=1$, $0.4+0.6=1$.	Check the transition matrix is valid.
$S_1 = T\,S_0 = \begin{bmatrix} 0.7(700)+0.4(0) \\ 0.3(700)+0.6(0) \end{bmatrix} = \begin{bmatrix} 490 \\ 210 \end{bmatrix}$.	Apply $S_{n+1} = T\,S_n$ once.
$S_2 = T\,S_1 = \begin{bmatrix} 0.7(490)+0.4(210) \\ 0.3(490)+0.6(210) \end{bmatrix} = \begin{bmatrix} 427 \\ 273 \end{bmatrix}$.	Multiply the previous state matrix by T again.

After 1 day there are 490 cars at City and 210 at Airport; after 2 days, 427 at City and 273 at Airport. Each total is still 700.

Example 2 — scaffolded

Subscribers to a coffee-pod service choose between two blends each month, *Roast* (B) and *Mild* (M). Of the Roast subscribers $60\,\%$ keep Roast and $40\,\%$ switch to Mild; of the Mild subscribers $50\,\%$ switch to Roast and $50\,\%$ stay. Initially 900 subscribers take Roast and none take Mild.

Working	Comment
$T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$, $S_0 = \begin{bmatrix} 900 \\ 0 \end{bmatrix}$, recurrence $S_{n+1} = T\,S_n$.	Order R, M; columns add to 1.
$S_1 = T\,S_0 = \begin{bmatrix} 0.6(900)+0.5(0) \\ 0.4(900)+0.5(0) \end{bmatrix} = \begin{bmatrix} 540 \\ 360 \end{bmatrix}$.	One step of repeated multiplication.
$S_2 = T\,S_1 = \begin{bmatrix} 0.6(540)+0.5(360) \\ 0.4(540)+0.5(360) \end{bmatrix} = \begin{bmatrix} 504 \\ 396 \end{bmatrix}$.	Another step.
$S_4 = T^4 S_0 \approx \begin{bmatrix} 500.04 \\ 399.96 \end{bmatrix}$.	Jump ahead with a matrix power on CAS instead of stepping.
Continuing, the state matrices settle to $S = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$.	The steady state — no noticeable change beyond here.

Working	Comment
Check $T\,S = \begin{bmatrix} 0.6(500)+0.5(400) \\ 0.4(500)+0.5(400) \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \end{bmatrix} = S$.	A steady state is unchanged by a further step.

In the long run about 500 subscribers take Roast and 400 take Mild.

Example 3 — unscaffolded

A resort models its daily weather with three states — Fine (F), Cloudy (C) and Rainy (B) — using the transition matrix (columns and rows in the order $F\,,C\,,R$)

$$T = \begin{bmatrix} 0.6 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}.$$

The state matrix gives the percentage chance of each type of weather. Today is Fine, so $S_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ (a $100\,\%$ chance of Fine). Find the distribution for the next two days and describe the long-run pattern.

Applying $S_{n+1} = T\,S_n$,

$$S_1 = T\,S_0 = \begin{bmatrix} 0.6(100) \\ 0.3(100) \\ 0.1(100) \end{bmatrix} = \begin{bmatrix} 60 \\ 30 \\ 10 \end{bmatrix}, S_2 = T\,S_1 = \begin{bmatrix} 0.6(60)+0.3(30)+0.3(10) \\ 0.3(60)+0.4(30)+0.3(10) \\ 0.1(60)+0.3(30)+0.4(10) \end{bmatrix} = \begin{bmatrix} 48 \\ 33 \\ 19 \end{bmatrix}.$$

So tomorrow there is a $60\,\%$ chance of Fine, $30\,\%$ Cloudy, $10\,\%$ Rainy; the day after, $48\,\%$, $33\,\%$, $19\,\%$. Every entry of T is positive, so T is regular and the state matrices settle to a single steady state independent of today's weather. Iterating far enough (on CAS, $T^n\,S_0$ for a large n) gives

$$S \approx \begin{bmatrix} 42.9 \\ 33.3 \\ 23.8 \end{bmatrix}.$$

$\therefore$ in the long run about $43\,\%$ of days are Fine, $33\,\%$ Cloudy and $24\,\%$ Rainy, whatever the weather is today.

Example 4 — unscaffolded

Two local newspapers, the *Herald* (H) and the *Gazette* (G), share 300 readers. Each week $90\,\%$ of Herald readers stay with the Herald and $10\,\%$ switch to the Gazette, while $20\,\%$ of Gazette readers switch to the Herald and $80\,\%$ stay. Show that the long-run readership is the same whether every reader starts with the Herald or every reader starts with the Gazette.

The transition matrix (order $H\,,G$) is $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$, whose entries are all positive, so it is regular.

Starting from everyone at the Herald, $S_0 = \begin{bmatrix} 300 \\ 0 \end{bmatrix}$:

$$S_1 = \begin{bmatrix} 270 \\ 30 \end{bmatrix}, S_2 = \begin{bmatrix} 249 \\ 51 \end{bmatrix}, S_3 = \begin{bmatrix} 234.3 \\ 65.7 \end{bmatrix}, \ldots \to \begin{bmatrix} 200 \\ 100 \end{bmatrix}.$$

Starting from everyone at the Gazette, $S_0 = \begin{bmatrix} 0 \\ 300 \end{bmatrix}$:

$$S_1=\begin{bmatrix}60\\240\end{bmatrix}, S_2=\begin{bmatrix}102\\198\end{bmatrix}, \ldots \to \begin{bmatrix}200\\100\end{bmatrix}.$$

Both sequences approach the same steady state $\begin{bmatrix}200\\100\end{bmatrix}$, which satisfies $T\,S=\begin{bmatrix}0.9(200)+0.2(100)\\0.1(200)+0.8(100)\end{bmatrix}=\begin{bmatrix}200\\100\end{bmatrix}$.

$\therefore$ because T is regular the steady state does not depend on the starting split: in the long run about 200 read the Herald and 100 read the Gazette.

Practice questions

Scaffolded

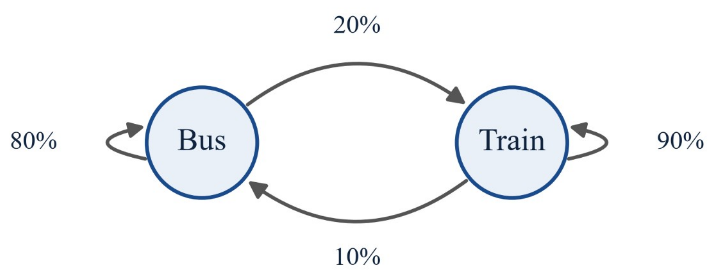
Q1. A reserve has two lakes, North (N) and South (S). Each week 75 % of the birds at North stay and 25 % fly to South; of those at South, 25 % fly to North and 75 % stay. Initially 800 birds are at North and none at South. (a) Write the transition matrix T (order N, S) and S_0. (b) Find S_1. (c) Find S_2. (d) Find S_3. (e) State the steady state the numbers are approaching.

Q2. Each month, customers of two phone plans switch. Of Plan A customers 80 % stay and 20 % move to Plan B; of Plan B customers 25 % move to Plan A and 75 % stay. Initially 450 customers are on Plan A and none on Plan B, so $T=\begin{bmatrix}0.8 & 0.25\\0.2 & 0.75\end{bmatrix}$, $S_0=\begin{bmatrix}450\\0\end{bmatrix}$. (a) Write the recurrence relation for this model. (b) Find S_1. (c) Find S_2. (d) Use $S_4=T^4S_0$ (with CAS) to find the state after 4 months, to one decimal place. (e) The steady state is $\begin{bmatrix}250\\200\end{bmatrix}$. Verify that $T\,S=S$.

Q3. Two brands of toothpaste, X and Y, are modelled by $T=\begin{bmatrix}0.55 & 0.3\\0.45 & 0.7\end{bmatrix}$ (order X, Y) with $S_0=\begin{bmatrix}500\\0\end{bmatrix}$. (a) Find S_1. (b) Find S_2. (c) Find S_3 (to four decimal places where needed). (d) The steady state is $\begin{bmatrix}200\\300\end{bmatrix}$. Verify that $T\,S=S$.

Q4. A subscription service is modelled by $T=\begin{bmatrix}0.9 & 0.3\\0.1 & 0.7\end{bmatrix}$ (states: Subscribed, Not subscribed) with $S_0=\begin{bmatrix}400\\0\end{bmatrix}$. (a) Write the recurrence relation. (b) Find S_1. (c) Find S_2. (d) Use matrix powers (CAS) to find $S_3=T^3S_0$ and $S_6=T^6S_0$, to one decimal place. (e) State the steady state.

Q5. The transition diagram below models how commuters travel each day between two modes, Bus and Train.



(a) Write the transition matrix T (order Bus, Train).

(b) Initially 600 commuters take the Bus and none take the Train, so $S_0=\begin{bmatrix}600\\0\end{bmatrix}$. Find S_1.

(c) Find S_2.

(d) The steady state is $\begin{bmatrix} 200 \\ 400 \end{bmatrix}$. Verify that $TS = S$.

Q6. Each day, 1000 students choose one of three lunch options — Canteen (C), Café (F) or Home (H). The transition matrix (order C, F, H) is

$$T = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}, S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Write the recurrence relation. (b) Find S_1. (c) Find S_2. (d) The steady state is approximately $\begin{bmatrix} 333.3 \\ 333.3 \\ 333.3 \end{bmatrix}$. Interpret this in context.

Unscaffolded

Q7. Two streaming services share 1000 users. Each month $T = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}$ (order A, B) and $S_0 = \begin{bmatrix} 500 \\ 500 \end{bmatrix}$. Find S_1 and S_2.

Q8. A system has transition matrix $T = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 700 \\ 0 \end{bmatrix}$. Use $S_5 = T^5 S_0$ (CAS) to find the state after 5 steps, to three decimal places, and state the steady state.

Q9. Two rival apps share 1000 users, with $T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ (order App A, App B). Find the steady state, and show that it is the same whether all users start on App A ($S_0 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$) or all start on App B ($S_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$) by computing S_1 and S_2 in each case.

Q10. Each year, 75 % of households subscribed to a service renew and 25 % cancel (becoming "not subscribed"); of the non-subscribers, 20 % subscribe and 80 % stay unsubscribed. There are 900 households, all subscribed to begin with. Set up T (order Subscribed, Not subscribed) and S_0, and find the state after 1 year and after 2 years.

Q11. Three supermarkets share 1200 customers, modelled by

$$T = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}, S_0 = \begin{bmatrix} 600 \\ 300 \\ 300 \end{bmatrix}.$$

Find S_1 and S_2, and the steady state (to one decimal place where needed).

Q12. The transition matrix $T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$ models cars moving between two depots. Explain what it means for a state matrix S to be the steady state, and verify that $S = \begin{bmatrix} 400 \\ 300 \end{bmatrix}$ is the steady state by showing that $TS = S$.

Q13. Two pharmacies share 1200 customers, with $T = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$ (order Pharmacy A, Pharmacy B). Find the steady state, and show that starting from $\begin{bmatrix} 1200 \\ 0 \end{bmatrix}$ and from $\begin{bmatrix} 0 \\ 1200 \end{bmatrix}$ both approach it by computing S_1, S_2 and S_3 for each.

Q14. For $T=\begin{bmatrix}0.7 & 0.4\\0.3 & 0.6\end{bmatrix}$ and $S_0=\begin{bmatrix}500\\0\end{bmatrix}$, compute T^2 and use it to find $S_2=T^2 S_0$ directly. Explain what the columns of T^2 represent.

Q15. A city has three suburbs and 900 residents move between them each year, modelled by

$$T=\begin{bmatrix}0.8 & 0.1 & 0.1\\0.1 & 0.7 & 0.2\\0.1 & 0.2 & 0.7\end{bmatrix}, S_0=\begin{bmatrix}500\\300\\100\end{bmatrix}.$$

Find S_1 and S_2, and state the steady state.

Q16. Two clubs share 800 members, with $T=\begin{bmatrix}0.5 & 0.3\\0.5 & 0.7\end{bmatrix}$ and $S_0=\begin{bmatrix}800\\0\end{bmatrix}$. Generate S_1, S_2 and S_3, use them to estimate the steady state, and confirm your estimate by checking that $T\,S=S$.

Q17. A national park has two zones, North and South. Each month $10\,\%$ of the animals in North move to South and $15\,\%$ of those in South move to North (the rest stay). There are 1000 animals, all in North to begin with. Find the state after 1 month and after 2 months, and the long-run distribution.

Q18. For $T=\begin{bmatrix}0.6 & 0.3\\0.4 & 0.7\end{bmatrix}$ and $S_0=\begin{bmatrix}700\\0\end{bmatrix}$, find S_1, S_2 and S_3. Comment on how the change from one state matrix to the next behaves as n increases, and state the steady state.

Q19. For a regular transition matrix, explain in words: (a) what is meant by the steady state (equilibrium state); (b) why the steady state does not depend on the initial state matrix S_0; (c) how you could find the steady state using only repeated matrix multiplication.

Q20. Three mobile networks share 2000 subscribers, modelled by

$$T=\begin{bmatrix}0.7 & 0.2 & 0.2\\0.2 & 0.6 & 0.2\\0.1 & 0.2 & 0.6\end{bmatrix}, S_0=\begin{bmatrix}1000\\500\\500\end{bmatrix}.$$

(a) Find S_1 and S_2. (b) Use $S_{15}=T^{15} S_0$ (CAS) to find the long-run state, to one decimal place. (c) Interpret the long-run state in context.

Answers

Q1. (a) $T=\begin{bmatrix}0.75 & 0.25\\0.25 & 0.75\end{bmatrix}$, $S_0=\begin{bmatrix}800\\0\end{bmatrix}$; (b) $\begin{bmatrix}600\\200\end{bmatrix}$; (c) $\begin{bmatrix}500\\300\end{bmatrix}$; (d) $\begin{bmatrix}450\\350\end{bmatrix}$; (e) $\begin{bmatrix}400\\400\end{bmatrix}$. **Q2.** (a) $S_{n+1}=T\,S_n$; (b) $\begin{bmatrix}360\\90\end{bmatrix}$; (c) $\begin{bmatrix}310.5\\139.5\end{bmatrix}$; (d) $\begin{bmatrix}268.3\\181.7\end{bmatrix}$; (e) $T\,S=\begin{bmatrix}250\\200\end{bmatrix}=S$. ✓ **Q3.** (a) $\begin{bmatrix}275\\225\end{bmatrix}$; (b) $\begin{bmatrix}218.75\\281.25\end{bmatrix}$; (c) $\begin{bmatrix}204.6875\\295.3125\end{bmatrix}$; (d) $T\,S=\begin{bmatrix}200\\300\end{bmatrix}=S$. ✓ **Q4.** (a) $S_{n+1}=T\,S_n$; (b) $\begin{bmatrix}360\\40\end{bmatrix}$; (c) $\begin{bmatrix}336\\64\end{bmatrix}$; (d) $S_3\approx\begin{bmatrix}321.6\\78.4\end{bmatrix}$, $S_6\approx\begin{bmatrix}304.7\\95.3\end{bmatrix}$; (e) $\begin{bmatrix}300\\100\end{bmatrix}$. **Q5.** (a) $T=\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$; (b) $\begin{bmatrix}480\\120\end{bmatrix}$; (c) $\begin{bmatrix}396\\204\end{bmatrix}$; (d) $T\,S=\begin{bmatrix}200\\400\end{bmatrix}=S$. ✓ **Q6.** (a) $S_{n+1}=T\,S_n$; (b) $\begin{bmatrix}700\\200\\100\end{bmatrix}$; (c) $\begin{bmatrix}530\\280\\190\end{bmatrix}$; (d) in the long run the students split roughly equally, about 333 choosing each of Canteen, Café and Home. **Q7.** $S_1=\begin{bmatrix}475\\525\end{bmatrix}$, $S_2=\begin{bmatrix}456.25\\543.75\end{bmatrix}$. **Q8.** $S_5\approx\begin{bmatrix}201.215\\498.785\end{bmatrix}$; steady state $\begin{bmatrix}200\\500\end{bmatrix}$. **Q9.** Steady state $\begin{bmatrix}400\\600\end{bmatrix}$. From $\begin{bmatrix}1000\\0\end{bmatrix}$: $\begin{bmatrix}700\\300\end{bmatrix}$, $\begin{bmatrix}550\\450\end{bmatrix}$. From $\begin{bmatrix}0\\1000\end{bmatrix}$: $\begin{bmatrix}200\\800\end{bmatrix}$, $\begin{bmatrix}300\\700\end{bmatrix}$. Both head to $\begin{bmatrix}400\\600\end{bmatrix}$. **Q10.** $T=\begin{bmatrix}0.75 & 0.2\\0.25 & 0.8\end{bmatrix}$, $S_0=\begin{bmatrix}900\\0\end{bmatrix}$; $S_1=\begin{bmatrix}675\\225\end{bmatrix}$, $S_2=\begin{bmatrix}551.25\\348.75\end{bmatrix}$. **Q11.** $S_1=\begin{bmatrix}450\\450\\300\end{bmatrix}$, $S_2=\begin{bmatrix}405\\495\\300\end{bmatrix}$; steady state $\approx\begin{bmatrix}385.7\\514.3\\300\end{bmatrix}$. **Q12.** A steady state S is a state matrix that a further step leaves unchanged ($T\,S=S$); the sequence settles to it. $T\,S=\begin{bmatrix}400\\300\end{bmatrix}=S$. ✓ **Q13.** Steady state $\begin{bmatrix}300\\900\end{bmatrix}$. From $\begin{bmatrix}1200\\0\end{bmatrix}$: $\begin{bmatrix}480\\720\end{bmatrix}$, $\begin{bmatrix}336\\864\end{bmatrix}$, $\begin{bmatrix}307.2\\892.8\end{bmatrix}$. From $\begin{bmatrix}0\\1200\end{bmatrix}$: $\begin{bmatrix}240\\960\end{bmatrix}$, $\begin{bmatrix}288\\912\end{bmatrix}$, $\begin{bmatrix}297.6\\902.4\end{bmatrix}$. **Q14.** $T^2=\begin{bmatrix}0.61 & 0.52\\0.39 & 0.48\end{bmatrix}$, $S_2=\begin{bmatrix}305\\195\end{bmatrix}$; the columns of T^2 give the proportions moving between the states over **two** steps. **Q15.** $S_1=\begin{bmatrix}440\\280\\180\end{bmatrix}$, $S_2=\begin{bmatrix}398\\276\\226\end{bmatrix}$; steady state $\begin{bmatrix}300\\300\\300\end{bmatrix}$ (an equal split). **Q16.** $S_1=\begin{bmatrix}400\\400\end{bmatrix}$, $S_2=\begin{bmatrix}320\\480\end{bmatrix}$, $S_3=\begin{bmatrix}304\\496\end{bmatrix}$; steady state $\begin{bmatrix}300\\500\end{bmatrix}$. **Q17.** $T=\begin{bmatrix}0.9 & 0.15\\0.1 & 0.85\end{bmatrix}$; $S_1=\begin{bmatrix}900\\100\end{bmatrix}$, $S_2=\begin{bmatrix}825\\175\end{bmatrix}$; long-run $\begin{bmatrix}600\\400\end{bmatrix}$. **Q18.** $S_1=\begin{bmatrix}420\\280\end{bmatrix}$, $S_2=\begin{bmatrix}336\\364\end{bmatrix}$, $S_3=\begin{bmatrix}310.8\\389.2\end{bmatrix}$; the changes shrink each step (by a factor of 0.3), approaching the steady state $\begin{bmatrix}300\\400\end{bmatrix}$. **Q19.** (a) the fixed distribution the state matrices settle to, unchanged by a further step ($T\,S=S$); (b) because T is regular, so the same distribution is reached from any start; (c) keep multiplying by T until the state matrix stops changing. **Q20.** (a) $S_1=\begin{bmatrix}900\\600\\500\end{bmatrix}$, $S_2=\begin{bmatrix}850\\640\\510\end{bmatrix}$; (b) $\approx\begin{bmatrix}800\\666.7\\533.3\end{bmatrix}$; (c) in the long run about 800, 667 and 533 subscribers use the three networks.

Fully-worked solutions

Q1. (a) Each column is "from" a lake: $T=\begin{bmatrix}0.75 & 0.25\\0.25 & 0.75\end{bmatrix}$, $S_0=\begin{bmatrix}800\\0\end{bmatrix}$. (b) $S_1=T\,S_0=\begin{bmatrix}0.75(800)+0.25(0)\\0.25(800)+0.75(0)\end{bmatrix}=\begin{bmatrix}600\\200\end{bmatrix}$. (c) $S_2=T\,S_1=\begin{bmatrix}0.75(600)+0.25(200)\\0.25(600)+0.75(200)\end{bmatrix}=\begin{bmatrix}500\\300\end{bmatrix}$. (d) $S_3=T\,S_2=\begin{bmatrix}0.75(500)+0.25(300)\\0.25(500)+0.75(300)\end{bmatrix}=\begin{bmatrix}450\\350\end{bmatrix}$. (e) The gap between the two lakes halves each week, so the numbers approach the steady state $\begin{bmatrix}400\\400\end{bmatrix}$ (an even split).

Q2. (a) $S_{n+1}=T\,S_n$. (b) $S_1=\begin{bmatrix}0.8(450)+0.25(0)\\0.2(450)+0.75(0)\end{bmatrix}=\begin{bmatrix}360\\90\end{bmatrix}$. (c) $S_2=\begin{bmatrix}0.8(360)+0.25(90)\\0.2(360)+0.75(90)\end{bmatrix}=\begin{bmatrix}288+22.5\\72+67.5\end{bmatrix}=\begin{bmatrix}310.5\\139.5\end{bmatrix}$. (d) On CAS, $S_4=T^4S_0\approx\begin{bmatrix}268.3\\181.7\end{bmatrix}$. (e) $T\,S=\begin{bmatrix}0.8(250)+0.25(200)\\0.2(250)+0.75(200)\end{bmatrix}=\begin{bmatrix}200+50\\50+150\end{bmatrix}=\begin{bmatrix}250\\200\end{bmatrix}=S$, as required.

Q3. (a) $S_1=\begin{bmatrix}0.55(500)+0.3(0)\\0.45(500)+0.7(0)\end{bmatrix}=\begin{bmatrix}275\\225\end{bmatrix}$. (b) $S_2=\begin{bmatrix}0.55(275)+0.3(225)\\0.45(275)+0.7(225)\end{bmatrix}=\begin{bmatrix}151.25+67.5\\123.75+157.5\end{bmatrix}=\begin{bmatrix}218.75\\281.25\end{bmatrix}$. (c) $S_3=\begin{bmatrix}0.55(218.75)+0.3(281.25)\\0.45(218.75)+0.7(281.25)\end{bmatrix}=\begin{bmatrix}204.6875\\295.3125\end{bmatrix}$. (d) $T\,S=\begin{bmatrix}0.55(200)+0.3(300)\\0.45(200)+0.7(300)\end{bmatrix}=\begin{bmatrix}110+90\\90+210\end{bmatrix}=\begin{bmatrix}200\\300\end{bmatrix}=S$. ✓

Q4. (a) $S_{n+1}=T\,S_n$. (b) $S_1=\begin{bmatrix}0.9(400)+0.3(0)\\0.1(400)+0.7(0)\end{bmatrix}=\begin{bmatrix}360\\40\end{bmatrix}$. (c) $S_2=\begin{bmatrix}0.9(360)+0.3(40)\\0.1(360)+0.7(40)\end{bmatrix}=\begin{bmatrix}324+12\\36+28\end{bmatrix}=\begin{bmatrix}336\\64\end{bmatrix}$. (d) With CAS, $S_3=T^3S_0\approx\begin{bmatrix}321.6\\78.4\end{bmatrix}$ and $S_6=T^6S_0\approx\begin{bmatrix}304.7\\95.3\end{bmatrix}$. (e) The numbers settle to the steady state $\begin{bmatrix}300\\100\end{bmatrix}$ (indeed $T\begin{bmatrix}300\\100\end{bmatrix}=\begin{bmatrix}270+30\\30+70\end{bmatrix}=\begin{bmatrix}300\\100\end{bmatrix}$).

Q5. (a) Reading each arrow *from* its starting mode: of Bus users $80\,\%$ stay and $20\,\%$ switch to Train; of Train users $10\,\%$ switch to Bus and $90\,\%$ stay, so $T=\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$. (b) $S_1=\begin{bmatrix}0.8(600)+0.1(0)\\0.2(600)+0.9(0)\end{bmatrix}=\begin{bmatrix}480\\120\end{bmatrix}$. (c) $S_2=\begin{bmatrix}0.8(480)+0.1(120)\\0.2(480)+0.9(120)\end{bmatrix}=\begin{bmatrix}384+12\\96+108\end{bmatrix}=\begin{bmatrix}396\\204\end{bmatrix}$. (d) $T\,S=\begin{bmatrix}0.8(200)+0.1(400)\\0.2(200)+0.9(400)\end{bmatrix}=\begin{bmatrix}160+40\\40+360\end{bmatrix}=\begin{bmatrix}200\\400\end{bmatrix}=S$. ✓

Q6. (a) $S_{n+1}=T\,S_n$. (b) $S_1=T\,S_0=\begin{bmatrix}0.7(1000)\\0.2(1000)\\0.1(1000)\end{bmatrix}=\begin{bmatrix}700\\200\\100\end{bmatrix}$. (c) $S_2=T\,S_1=\begin{bmatrix}0.7(700)+0.1(200)+0.2(100)\\0.2(700)+0.6(200)+0.2(100)\\0.1(700)+0.3(200)+0.6(100)\end{bmatrix}=\begin{bmatrix}490+20+20\\140+120+20\\70+60+60\end{bmatrix}=\begin{bmatrix}530\\280\\190\end{bmatrix}$. (d) In the long run the three options attract roughly equal numbers — about 333 students each choose the Canteen, the Café and Home.

Q7. $S_1=T\,S_0=\begin{bmatrix}0.85(500)+0.1(500)\\0.15(500)+0.9(500)\end{bmatrix}=\begin{bmatrix}425+50\\75+450\end{bmatrix}=\begin{bmatrix}475\\525\end{bmatrix}$, then $S_2=T\,S_1=\begin{bmatrix}0.85(475)+0.1(525)\\0.15(475)+0.9(525)\end{bmatrix}=\begin{bmatrix}403.75+52.5\\71.25+472.5\end{bmatrix}=\begin{bmatrix}456.25\\543.75\end{bmatrix}$.

Q8. Using CAS, $S_5 = T^5 S_0 \approx \begin{bmatrix} 201.215 \\ 498.785 \end{bmatrix}$. The state matrices are settling to the steady state $\begin{bmatrix} 200 \\ 500 \end{bmatrix}$, which satisfies $T\begin{bmatrix} 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 100+100 \\ 100+400 \end{bmatrix} = \begin{bmatrix} 200 \\ 500 \end{bmatrix}$.

Q9. Starting from all on App A, $\begin{bmatrix} 1000 \\ 0 \end{bmatrix} \to \begin{bmatrix} 700 \\ 300 \end{bmatrix} \to \begin{bmatrix} 550 \\ 450 \end{bmatrix} \to \ldots$; starting from all on App B, $\begin{bmatrix} 0 \\ 1000 \end{bmatrix} \to \begin{bmatrix} 200 \\ 800 \end{bmatrix} \to \begin{bmatrix} 300 \\ 700 \end{bmatrix} \to \ldots$. Continuing either sequence (on CAS, $S_n = T^n S_0$ for a large n) there is no noticeable change beyond $\begin{bmatrix} 400 \\ 600 \end{bmatrix}$, so that is the steady state — and a further step leaves it unchanged: $T\begin{bmatrix} 400 \\ 600 \end{bmatrix} = \begin{bmatrix} 280+120 \\ 120+480 \end{bmatrix} = \begin{bmatrix} 400 \\ 600 \end{bmatrix}$. ✓ T is regular (all entries positive), so the same steady state is reached from either start.

Q10. Each column is "from" a state: renewers/cancellers form column 1, and the non-subscribers' behaviour forms column 2, so $T = \begin{bmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{bmatrix}$, $S_0 = \begin{bmatrix} 900 \\ 0 \end{bmatrix}$. Then $S_1 = \begin{bmatrix} 0.75(900)+0.2(0) \\ 0.25(900)+0.8(0) \end{bmatrix} = \begin{bmatrix} 675 \\ 225 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 0.75(675)+0.2(225) \\ 0.25(675)+0.8(225) \end{bmatrix} = \begin{bmatrix} 506.25+45 \\ 168.75+180 \end{bmatrix} = \begin{bmatrix} 551.25 \\ 348.75 \end{bmatrix}$. So after 1 year about 675 households are subscribed and after 2 years about 551.

Q11. $S_1 = T\,S_0 = \begin{bmatrix} 0.5(600)+0.2(300)+0.3(300) \\ 0.3(600)+0.6(300)+0.3(300) \\ 0.2(600)+0.2(300)+0.4(300) \end{bmatrix} = \begin{bmatrix} 300+60+90 \\ 180+180+90 \\ 120+60+120 \end{bmatrix} = \begin{bmatrix} 450 \\ 450 \\ 300 \end{bmatrix}$, then $S_2 = T\,S_1 = \begin{bmatrix} 0.5(450)+0.2(450)+0.3(300) \\ 0.3(450)+0.6(450)+0.3(300) \\ 0.2(450)+0.2(450)+0.4(300) \end{bmatrix} = \begin{bmatrix} 225+90+90 \\ 135+270+90 \\ 90+90+120 \end{bmatrix} = \begin{bmatrix} 405 \\ 495 \\ 300 \end{bmatrix}$. Iterating with CAS ($T^n S_0$ for a large n, until there is no noticeable change) gives the steady state $\approx \begin{bmatrix} 385.7 \\ 514.3 \\ 300 \end{bmatrix}$. (The third supermarket sits at 300 throughout, because its inflow and outflow are already balanced.)

Q12. The steady state is the state matrix the sequence settles to, at which a further transition produces no change: applying T leaves it fixed, i.e. $T\,S = S$. Verifying, $T\,S = \begin{bmatrix} 0.7(400)+0.4(300) \\ 0.3(400)+0.6(300) \end{bmatrix} = \begin{bmatrix} 280+120 \\ 120+180 \end{bmatrix} = \begin{bmatrix} 400 \\ 300 \end{bmatrix} = S$, so $\begin{bmatrix} 400 \\ 300 \end{bmatrix}$ is the steady state.

Q13. From $\begin{bmatrix} 1200 \\ 0 \end{bmatrix}$: $S_1 = \begin{bmatrix} 0.4(1200)+0.2(0) \\ 0.6(1200)+0.8(0) \end{bmatrix} = \begin{bmatrix} 480 \\ 720 \end{bmatrix}$, $S_2 = \begin{bmatrix} 192+144 \\ 288+576 \end{bmatrix} = \begin{bmatrix} 336 \\ 864 \end{bmatrix}$, $S_3 = \begin{bmatrix} 134.4+172.8 \\ 201.6+691.2 \end{bmatrix} = \begin{bmatrix} 307.2 \\ 892.8 \end{bmatrix}$.
From $\begin{bmatrix} 0 \\ 1200 \end{bmatrix}$: $S_1 = \begin{bmatrix} 240 \\ 960 \end{bmatrix}$, $S_2 = \begin{bmatrix} 96+192 \\ 144+768 \end{bmatrix} = \begin{bmatrix} 288 \\ 912 \end{bmatrix}$, $S_3 = \begin{bmatrix} 115.2+182.4 \\ 172.8+729.6 \end{bmatrix} = \begin{bmatrix} 297.6 \\ 902.4 \end{bmatrix}$. Each sequence is closing in on the same distribution from opposite sides, and continuing the iteration (on CAS, $T^n S_0$ for a large n) shows no noticeable change beyond $\begin{bmatrix} 300 \\ 900 \end{bmatrix}$, so that is the steady state — and it is unchanged by a further step: $T\begin{bmatrix} 300 \\ 900 \end{bmatrix} = \begin{bmatrix} 120+180 \\ 180+720 \end{bmatrix} = \begin{bmatrix} 300 \\ 900 \end{bmatrix}$. ✓ T is regular (all entries positive), so the same steady state is reached from either start.

Q14. $T^2=\begin{bmatrix}0.7&0.4\\0.3&0.6\end{bmatrix}\begin{bmatrix}0.7&0.4\\0.3&0.6\end{bmatrix}=\begin{bmatrix}0.49+0.12&0.28+0.24\\0.21+0.18&0.12+0.36\end{bmatrix}=\begin{bmatrix}0.61&0.52\\0.39&0.48\end{bmatrix}$. Then $S_2=T^2S_0=\begin{bmatrix}0.61(500)+0.52(0)\\0.39(500)+0.48(0)\end{bmatrix}=\begin{bmatrix}305\\195\end{bmatrix}$, the same as stepping twice. The columns of T^2 give the proportions moving between the states over **two** steps (its columns also add to 1).

Q15. $S_1=T\,S_0=\begin{bmatrix}0.8(500)+0.1(300)+0.1(100)\\0.1(500)+0.7(300)+0.2(100)\\0.1(500)+0.2(300)+0.7(100)\end{bmatrix}=\begin{bmatrix}400+30+10\\50+210+20\\50+60+70\end{bmatrix}=\begin{bmatrix}440\\280\\180\end{bmatrix}$, then $S_2=T\,S_1=\begin{bmatrix}0.8(440)+0.1(280)+0.1(180)\\0.1(440)+0.7(280)+0.2(180)\\0.1(440)+0.2(280)+0.7(180)\end{bmatrix}=\begin{bmatrix}352+28+18\\44+196+36\\44+56+126\end{bmatrix}=\begin{bmatrix}398\\276\\226\end{bmatrix}$. Iterating far enough gives the steady state $\begin{bmatrix}300\\300\\300\end{bmatrix}$ — an equal split of the 900 residents (convergence is slow here, so many steps are needed).

Q16. $S_1=\begin{bmatrix}0.5(800)+0.3(0)\\0.5(800)+0.7(0)\end{bmatrix}=\begin{bmatrix}400\\400\end{bmatrix}$, $S_2=\begin{bmatrix}0.5(400)+0.3(400)\\0.5(400)+0.7(400)\end{bmatrix}=\begin{bmatrix}320\\480\end{bmatrix}$, $S_3=\begin{bmatrix}0.5(320)+0.3(480)\\0.5(320)+0.7(480)\end{bmatrix}=\begin{bmatrix}304\\496\end{bmatrix}$. The changes are shrinking quickly (-400, then -80, then -16 in the first entry), so the values are levelling off near $\begin{bmatrix}300\\500\end{bmatrix}$ — the estimated steady state. Checking it, $T\,S=\begin{bmatrix}0.5(300)+0.3(500)\\0.5(300)+0.7(500)\end{bmatrix}=\begin{bmatrix}150+150\\150+350\end{bmatrix}=\begin{bmatrix}300\\500\end{bmatrix}=S$, ✓ so the estimate is confirmed: a further step leaves it unchanged.

Q17. Of the North animals $90\,\%$ stay and $10\,\%$ move to South; of the South animals $15\,\%$ move to North and $85\,\%$ stay, so $T=\begin{bmatrix}0.9&0.15\\0.1&0.85\end{bmatrix}$, $S_0=\begin{bmatrix}1000\\0\end{bmatrix}$. Then $S_1=\begin{bmatrix}0.9(1000)+0.15(0)\\0.1(1000)+0.85(0)\end{bmatrix}=\begin{bmatrix}900\\100\end{bmatrix}$ and $S_2=\begin{bmatrix}0.9(900)+0.15(100)\\0.1(900)+0.85(100)\end{bmatrix}=\begin{bmatrix}810+15\\90+85\end{bmatrix}=\begin{bmatrix}825\\175\end{bmatrix}$. Continuing the iteration (on CAS, $T^n S_0$ for a large n) the entries stop changing noticeably at the long-run distribution $\begin{bmatrix}600\\400\end{bmatrix}$, which a further month leaves unchanged: $T\begin{bmatrix}600\\400\end{bmatrix}=\begin{bmatrix}540+60\\60+340\end{bmatrix}=\begin{bmatrix}600\\400\end{bmatrix}$. ✓

Q18. $S_1=\begin{bmatrix}0.6(700)\\0.4(700)\end{bmatrix}=\begin{bmatrix}420\\280\end{bmatrix}$, $S_2=\begin{bmatrix}0.6(420)+0.3(280)\\0.4(420)+0.7(280)\end{bmatrix}=\begin{bmatrix}252+84\\168+196\end{bmatrix}=\begin{bmatrix}336\\364\end{bmatrix}$, $S_3=\begin{bmatrix}0.6(336)+0.3(364)\\0.4(336)+0.7(364)\end{bmatrix}=\begin{bmatrix}201.6+109.2\\134.4+254.8\end{bmatrix}=\begin{bmatrix}310.8\\389.2\end{bmatrix}$. The first component changes by -280, then -84, then -25.2: each change is 0.3 times the previous one, so the state matrices move by smaller and smaller amounts and close in on the steady state $\begin{bmatrix}300\\400\end{bmatrix}$.

Q19. (a) The steady state (equilibrium state) is the fixed distribution that the sequence of state matrices settles to: after enough steps there is no noticeable change from one state matrix to the next, and a further transition leaves it unchanged, so it satisfies $T\,S=S$. (b) When T is regular (some power of it has no zero entries, so after that many steps every state can be reached from every state), the effect of the starting distribution washes out as the process runs, and the same steady state is reached from any initial state matrix S_0. (c) Choose any S_0 with the correct total and repeatedly multiply by T — computing $S_1 = T\,S_0$, $S_2 = T\,S_1$, and so on (equivalently $S_n = T^n S_0$ for large n) — until the entries of the state matrix stop changing to the required accuracy; that stable state matrix is the steady state.

Q20. (a) $S_1 = T\,S_0 = \begin{bmatrix} 0.7(1000)+0.2(500)+0.2(500) \\ 0.2(1000)+0.6(500)+0.2(500) \\ 0.1(1000)+0.2(500)+0.6(500) \end{bmatrix} = \begin{bmatrix} 700+100+100 \\ 200+300+100 \\ 100+100+300 \end{bmatrix} = \begin{bmatrix} 900 \\ 600 \\ 500 \end{bmatrix}$, then

$S_2 = T\,S_1 = \begin{bmatrix} 0.7(900)+0.2(600)+0.2(500) \\ 0.2(900)+0.6(600)+0.2(500) \\ 0.1(900)+0.2(600)+0.6(500) \end{bmatrix} = \begin{bmatrix} 630+120+100 \\ 180+360+100 \\ 90+120+300 \end{bmatrix} = \begin{bmatrix} 850 \\ 640 \\ 510 \end{bmatrix}$. (b) With CAS, $S_{15} = T^{15} S_0 \approx \begin{bmatrix} 800.0 \\ 666.7 \\ 533.3 \end{bmatrix}$.

(c) In the long run the 2000 subscribers settle at about 800 on the first network, 667 on the second and 533 on the third, regardless of the starting split.

Chapter 9 Transition matrices — 9D Modelling populations with culling and restocking

Theory

An earlier section modelled a system that moves between a fixed number of **states** using the matrix recurrence relation

$$S_0 = \text{initial state matrix}\,, S_{n+1} = T\,S_n\,,$$

where T is the **transition matrix** (each of its columns adds to 1) and S_n is a **column state matrix** listing how many are in each state after n steps. Because the columns of a transition matrix each add to 1, multiplying by T only **redistributes** the population between the states — it never changes the overall total.

Many real systems are not closed in this way. Each period a fixed number of individuals is **added** or **removed** from the outside: a lake is **restocked** with fish, a herd is **culled**, animals are bred and released, or there is regular **immigration** and **emigration**. These fixed external changes are collected into a **constant matrix** B (a column matrix of the same order as S_n), and the recurrence becomes

$$S_0 = \text{initial state matrix}\,, S_{n+1} = T\,S_n + B\,.$$

At each step you first apply the transition proportions T to the current state, and **then add** B.

Reading the matrix B. Each entry of B is the net external change to that state every period, applied after the transitions:

- a **positive** entry is an **addition** — restocking, births released into the system, or immigration into that state;
- a **negative** entry is a **removal** — culling, harvesting, or emigration out of that state.

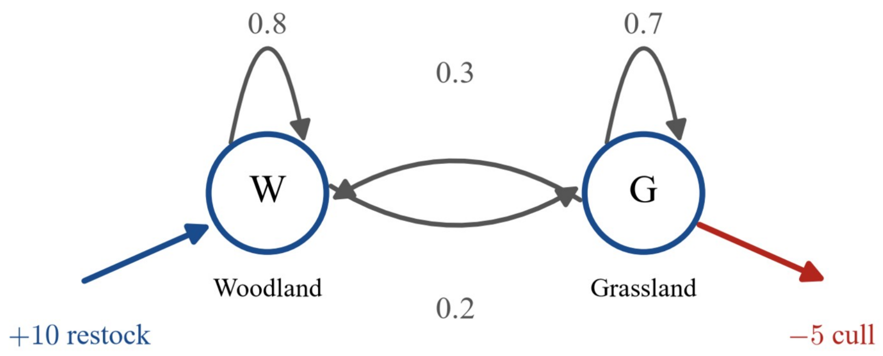
For example, $B = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$ adds 10 to the first state (a restock) and removes 5 from the second state (a cull) every period.

Generating the sequence. As before, the sequence of state matrices is produced by **iteration** — repeatedly substituting the current state into the rule:

$$S_1 = T\,S_0 + B\,, S_2 = T\,S_1 + B\,, S_3 = T\,S_2 + B\,, \ldots$$

Each step is a matrix product $T\,S_n$ followed by the addition of B. (Unlike the pure case, you cannot get S_n from a single power $T^n S_0$, because the constant B is added afresh at every step.)

The transition/state diagram below shows the idea for a conservation park in which a species moves each month between the **Woodland** (W) and the **Grassland** (G). The grey arrows are the transition proportions (these form T); the coloured arrows are the external flows carried by B — a restock of $+10$ added to the Woodland and a cull of -5 removed from the Grassland.



Here

$$T=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}, B=\begin{bmatrix}10\\-5\end{bmatrix}, S_0=\begin{bmatrix}100\\100\end{bmatrix}.$$

Applying the rule once,

$$S_1=T\,S_0+B=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}\begin{bmatrix}100\\100\end{bmatrix}+\begin{bmatrix}10\\-5\end{bmatrix}=\begin{bmatrix}110\\90\end{bmatrix}+\begin{bmatrix}10\\-5\end{bmatrix}=\begin{bmatrix}120\\85\end{bmatrix}.$$

Why the long-run behaviour changes. With the pure rule $S_{n+1}=T\,S_n$ (and a regular transition matrix) the state settles to a **steady state** and the total stays fixed. Adding B can change this completely. Since T leaves the total unchanged, applying $S_{n+1}=T\,S_n+B$ changes the total each period by exactly the **sum of the entries of** B:

- if the entries of B add to a **positive** number, the total **grows** without bound;
- if they add to a **negative** number, the total **declines** (a population may fall towards zero — extinction);
- if they add to **zero**, the total is unchanged, but the state usually settles to a **new equilibrium** that is different from the pure-transition steady state.

In the park example the entries of B add to $10+(-5)=5$, so the total rises by 5 each month ($200 \rightarrow 205 \rightarrow 210 \rightarrow \ldots$) — the population grows steadily, whereas without restocking and culling the total would have stayed at 200.

Equilibrium (steady) state. If the state settles so that there is no noticeable change from one step to the next, then $S_{n+1}=S_n$; writing this common value as S, the equilibrium satisfies

$$S=T\,S+B.$$

We identify such a steady state **informally**, either by iterating until the entries stop changing or by checking that a proposed matrix S satisfies $S=T\,S+B$.

Using CAS. Generating a long sequence of state matrices by hand is tedious. In practice you store T, B and S_0 in your CAS calculator and apply $S_{n+1}=T\,S_n+B$ repeatedly, reading off successive state matrices. Learn the by-hand steps below so that you understand each calculation and can check the output is reasonable.

Worked examples

Example 1 — scaffolded

A wildlife sanctuary keeps birds in an **Aviary** (A) and an adjoining **Open range** (R). Each week $90\,\%$ of the aviary birds stay and $10\,\%$ move to the range, while $20\,\%$ of the range birds move to the aviary and $80\,\%$ stay. Every week 8 birds are added to the aviary and 4 are removed from the range. Initially there are 50 birds in the aviary and 30 on the range. Find the state matrices S_1, S_2 and S_3.

Working	Comment
$T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$, $B=\begin{bmatrix}8\\-4\end{bmatrix}$, $S_0=\begin{bmatrix}50\\30\end{bmatrix}$.	Columns of T add to 1; $+8$ to the aviary, -4 from the range.
$S_1=T\,S_0+B=\begin{bmatrix}45+6\\5+24\end{bmatrix}+\begin{bmatrix}8\\-4\end{bmatrix}=\begin{bmatrix}59\\25\end{bmatrix}$.	Multiply, then add B.
$S_2=T\,S_1+B=\begin{bmatrix}53.1+5\\5.9+20\end{bmatrix}+\begin{bmatrix}8\\-4\end{bmatrix}=\begin{bmatrix}66.1\\21.9\end{bmatrix}$.	Feed S_1 back into the rule.
$S_3=T\,S_2+B=\begin{bmatrix}59.49+4.38\\6.61+17.52\end{bmatrix}+\begin{bmatrix}8\\-4\end{bmatrix}=\begin{bmatrix}71.87\\20.13\end{bmatrix}$.	Iterate once more.

The totals are $80,84,88,92$: the population rises by 4 each week, matching the entries of B, which add to $8+(-4)=4$.

Example 2 — scaffolded

Each spring a park ranger surveys deer in the **Valley** (V) and on the **Ridge** (R). $70\,\%$ of the valley deer stay in the valley and $30\,\%$ move to the ridge; $40\,\%$ of the ridge deer move to the valley and $60\,\%$ stay. Each spring 12 deer are released into the valley and 12 are culled from the ridge. Initially there are 200 deer in the valley and 100 on the ridge. (a) Write down T and B. (b) Write the recurrence relation. (c) Find S_1 and S_2. (d) Interpret B.

Working	Comment
$T=\begin{bmatrix}0.7 & 0.4\\0.3 & 0.6\end{bmatrix}$, $B=\begin{bmatrix}12\\-12\end{bmatrix}$.	Each column of T adds to 1; $+12$ released to valley, -12 culled from ridge.
$S_0=\begin{bmatrix}200\\100\end{bmatrix}$, $S_{n+1}=T\,S_n+B$.	The recurrence with its initial state.
$S_1=\begin{bmatrix}140+40\\60+60\end{bmatrix}+\begin{bmatrix}12\\-12\end{bmatrix}=\begin{bmatrix}192\\108\end{bmatrix}$.	$T\,S_0$, then add B.
$S_2=\begin{bmatrix}134.4+43.2\\57.6+64.8\end{bmatrix}+\begin{bmatrix}12\\-12\end{bmatrix}=\begin{bmatrix}189.6\\110.4\end{bmatrix}$.	Feed S_1 back in.
The entries of B add to $12+(-12)=0$.	Restocking exactly balances culling.

(d) Each spring 12 deer are added to the valley (a restock) and 12 are removed from the ridge (a cull). Because these balance, the total number of deer stays at 300 (indeed $192+108=189.6+110.4=300$); only the split between the two locations changes.

Example 3 — unscaffolded

A lake's fish are modelled in two zones, the deep zone (state 1) and the shallow zone (state 2), with transition matrix $T=\begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}$. Each month the lake is restocked with 100 fish added to the deep zone, while 40 fish are netted (culled) from the shallow zone, so $B=\begin{bmatrix}100\\-40\end{bmatrix}$. Starting from $S_0=\begin{bmatrix}500\\500\end{bmatrix}$, find S_1 and S_2, and describe the long-run behaviour of the total number of fish.

Applying the rule twice,

$$S_1=T\,S_0+B=\begin{bmatrix}300+250\\200+250\end{bmatrix}+\begin{bmatrix}100\\-40\end{bmatrix}=\begin{bmatrix}650\\410\end{bmatrix},$$

$$S_2=T\,S_1+B=\begin{bmatrix}390+205\\260+205\end{bmatrix}+\begin{bmatrix}100\\-40\end{bmatrix}=\begin{bmatrix}695\\425\end{bmatrix}.$$

The starting total is 1000. Because each column of T adds to 1, multiplying by T does not change the total, so every month the total changes by the sum of the entries of B, namely $100+(-40)=60$. The totals are therefore $1000, 1060, 1120, \ldots$, increasing by 60 each month without ever levelling off.

∴ the restock ($+100$) outweighs the cull (-40), so the fish population grows steadily by 60 each month and increases without bound — quite unlike the pure model $S_{n+1}=T\,S_n$, whose total would have stayed fixed at 1000.

Example 4 — unscaffolded

A sanctuary manages a bird species across a **Reserve** (X) and a **Wetland** (Y) with monthly transition matrix $T=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$. Each month 18 birds are added to the reserve and 18 are removed from the wetland, so $B=\begin{bmatrix}18\\-18\end{bmatrix}$. It is claimed that in the long run the population settles at 180 in the reserve and 60 in the wetland. Show that $S=\begin{bmatrix}180\\60\end{bmatrix}$ is an equilibrium state, and confirm the population moves towards it from $S_0=\begin{bmatrix}240\\0\end{bmatrix}$.

At an equilibrium the state does not change, so it must satisfy $S=T\,S+B$. Testing the claimed matrix,

$$T\,S+B=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}\begin{bmatrix}180\\60\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}144+18\\36+42\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}162\\78\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}180\\60\end{bmatrix}=S,$$

so $S=\begin{bmatrix}180\\60\end{bmatrix}$ is indeed an equilibrium. The entries of B add to $18+(-18)=0$, so the total stays fixed at $180+60=240$. Iterating from $S_0=\begin{bmatrix}240\\0\end{bmatrix}$,

$$S_1=\begin{bmatrix}192\\48\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}210\\30\end{bmatrix}, S_2=\begin{bmatrix}177\\63\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}195\\45\end{bmatrix}, S_3=\begin{bmatrix}169.5\\70.5\end{bmatrix}+\begin{bmatrix}18\\-18\end{bmatrix}=\begin{bmatrix}187.5\\52.5\end{bmatrix}.$$

Each total is 240, and the reserve figures $210, 195, 187.5, \ldots$ close in on 180 while the wetland figures $30, 45, 52.5, \ldots$ close in on 60.

$\therefore S=\begin{bmatrix}180\\60\end{bmatrix}$ satisfies $S=T\,S+B$ and is the long-run equilibrium state, which the population approaches while its total remains constant at 240.

Practice questions

Scaffolded

Q1. For $T=\begin{bmatrix}0.6 & 0.3\\0.4 & 0.7\end{bmatrix}$, $S_0=\begin{bmatrix}100\\50\end{bmatrix}$ and $B=\begin{bmatrix}10\\-4\end{bmatrix}$, find S_1. (a) Evaluate the product $T\,S_0$. (b) Add B to obtain $S_1=T\,S_0+B$.

Q2. Using the same T, B and starting state as in Q1, continue the sequence. (a) Find $S_2=T\,S_1+B$. (b) Find $S_3=T\,S_2+B$. (c) State the total at each of S_0, S_1, S_2, S_3 and the amount by which the total changes each step.

Q3. A nature reserve holds animals in an **East** section (E) and a **West** section (W). Each year 85% of the east animals stay and 15% move west, while 25% of the west animals move east and 75% stay. Each year 20 animals are added to the east and 10 are culled from the west. Initially $S_0=\begin{bmatrix}300\\200\end{bmatrix}$. (a) Write down the transition matrix T. (b) Write down the constant matrix B. (c) Write the recurrence relation $S_{n+1}=T\,S_n+B$ with its initial state. (d) Find S_1.

Q4. In a model $S_{n+1}=T\,S_n+B$ for a bird conservation program, state 1 is the captive population and state 2 is the wild population, and $B=\begin{bmatrix}25\\-10\end{bmatrix}$ (birds per year). (a) Interpret the entry 25. (b) Interpret the entry -10. (c) By how much does the total number of birds change each year? (d) State whether the total population grows, declines or stays constant in the long run.

Q5. For $T=\begin{bmatrix}0.7 & 0.2\\0.3 & 0.8\end{bmatrix}$ and $B=\begin{bmatrix}10\\-10\end{bmatrix}$, show that $S=\begin{bmatrix}100\\100\end{bmatrix}$ is an equilibrium state. (a) Evaluate $T\,S$. (b) Add B to obtain $T\,S+B$. (c) Compare $T\,S+B$ with S and state your conclusion. (d) Explain why the total stays constant here.

Q6. Three regions A, B and C have transition matrix $T=\begin{bmatrix}0.7 & 0.2 & 0.1\\0.2 & 0.6 & 0.3\\0.1 & 0.2 & 0.6\end{bmatrix}$, with 10 animals added to region A and 10 culled from region C each year, so $B=\begin{bmatrix}10\\0\\-10\end{bmatrix}$. Starting from $S_0=\begin{bmatrix}100\\100\\100\end{bmatrix}$, find $S_1=T\,S_0+B$.

Unscaffolded

Q7. A population moves between two states with $T=\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$ and $B=\begin{bmatrix}5\\-3\end{bmatrix}$. Starting from $S_0=\begin{bmatrix}60\\40\end{bmatrix}$, find S_1 and S_2.

Q8. For $T=\begin{bmatrix}0.5 & 0.4\\0.5 & 0.6\end{bmatrix}$, $B=\begin{bmatrix}10\\-4\end{bmatrix}$ and $S_0=\begin{bmatrix}40\\60\end{bmatrix}$, find S_1, S_2 and S_3.

Q9. A fish farm has two tanks. Each day $90\,\%$ of the fish in Tank 1 stay and $10\,\%$ are moved to Tank 2, while $20\,\%$ of the fish in Tank 2 are moved to Tank 1 and $80\,\%$ stay. Each day 30 fish are added to Tank 1 and 15 are harvested from Tank 2. Initially $S_0=\begin{bmatrix}200\\300\end{bmatrix}$. Write down T and B, and find the state matrix S_1 after one day.

Q10. For $T=\begin{bmatrix}0.7 & 0.3\\0.3 & 0.7\end{bmatrix}$, $B=\begin{bmatrix}20\\10\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\100\end{bmatrix}$: (a) find S_1 and S_2; (b) state by how much the total changes each step; (c) describe the long-run behaviour of the total population.

Q11. A pest species is modelled by $T=\begin{bmatrix}0.8 & 0.2\\0.2 & 0.8\end{bmatrix}$ with a culling program $B=\begin{bmatrix}-30\\-30\end{bmatrix}$ applied each year, starting from $S_0=\begin{bmatrix}300\\300\end{bmatrix}$. (a) Find S_1 and S_2. (b) By how much does the total change each year? (c) After how many years does the total reach 0? (d) Interpret this outcome in context.

Q12. For $T=\begin{bmatrix}0.6 & 0.2\\0.4 & 0.8\end{bmatrix}$ and $B=\begin{bmatrix}10\\-10\end{bmatrix}$, verify that $S=\begin{bmatrix}100\\150\end{bmatrix}$ is an equilibrium state, interpret the matrix B, and state the constant total.

Q13. A species is modelled with $T=\begin{bmatrix}0.75 & 0.25\\0.25 & 0.75\end{bmatrix}$ and $S_0=\begin{bmatrix}200\\0\end{bmatrix}$. (a) Under the pure rule $S_{n+1}=T\,S_n$, find S_1. (b) Under $S_{n+1}=T\,S_n+B$ with $B=\begin{bmatrix}10\\-10\end{bmatrix}$, find S_1. (c) Compare the totals in (a) and (b), and explain the result. (d) If instead $B=\begin{bmatrix}10\\10\end{bmatrix}$, describe what happens to the total in the long run.

Q14. Three habitats A, B and C have $T=\begin{bmatrix}0.6 & 0.1 & 0.2\\0.3 & 0.7 & 0.2\\0.1 & 0.2 & 0.6\end{bmatrix}$ and $B=\begin{bmatrix}10\\-5\\0\end{bmatrix}$, starting from $S_0=\begin{bmatrix}50\\50\\50\end{bmatrix}$. Find S_1 and S_2.

Q15. A two-state population has transition matrix $T=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$ and initial state $S_0=\begin{bmatrix}100\\50\end{bmatrix}$. After one period the state is measured to be $S_1=\begin{bmatrix}101\\51\end{bmatrix}$. Assuming the model $S_{n+1}=T\,S_n+B$, find the constant matrix B and interpret its entries.

Q16. A system evolves by $S_{n+1}=T\,S_n+B$, where T is a transition matrix (each column adding to 1). For each constant matrix below, state whether the total population grows, stays constant, or declines in the long run, and explain your reasoning. (a) $B_1=\begin{bmatrix}5\\3\end{bmatrix}$. (b) $B_2=\begin{bmatrix}7\\-7\end{bmatrix}$. (c) $B_3=\begin{bmatrix}-4\\-9\end{bmatrix}$.

Q17. A city's population is modelled by two states, the **city centre** (state 1) and the **suburbs** (state 2), using $S_{n+1}=T\,S_n+B$ with $B=\begin{bmatrix}-200\\350\end{bmatrix}$ (people per year). (a) Interpret each entry of B in context. (b) Find the net change in the total population each year. (c) State whether the total population grows or declines over time.

Q18. A managed herd has $T=\begin{bmatrix}0.85 & 0.1\\0.15 & 0.9\end{bmatrix}$ and, each year, 50 animals are added to state 1 with none added to state 2, so $B=\begin{bmatrix}50\\0\end{bmatrix}$. Starting from $S_0=\begin{bmatrix}400\\600\end{bmatrix}$, find the state matrix after two years, and comment on the total.

Q19. A koala population lives in two areas. Each year 90% of the Area 1 koalas stay and 10% move to Area 2, while 20% of the Area 2 koalas move to Area 1 and 80% stay. Each year 12 koalas are released into Area 1 and 12 are relocated away from (removed from) Area 2. (a) Write down T and B. (b) Show that $S=\begin{bmatrix}240\\60\end{bmatrix}$ satisfies $S=T\,S+B$. (c) Starting from $S_0=\begin{bmatrix}300\\0\end{bmatrix}$, find S_1 and S_2, and comment on how the population is changing.

Q20. A national park manages koalas in the **North** (N) and **South** (S) sections. Each year 80% of the north koalas stay and 20% move south, while 30% of the south koalas move north and 70% stay. Each year 24 koalas are relocated into the north and 24 are culled from the south. Initially $S_0=\begin{bmatrix}300\\180\end{bmatrix}$. (a) Write down T, B and the recurrence relation. (b) Find S_1 and S_2. (c) State the net change in the total each year and hence the constant total. (d) Verify that $S=\begin{bmatrix}336\\144\end{bmatrix}$ is the long-run equilibrium state.

Answers

Q1. (a) $T S_0=\begin{bmatrix}75\\75\end{bmatrix}$; (b) $S_1=\begin{bmatrix}85\\71\end{bmatrix}$. **Q2.** (a) $S_2=\begin{bmatrix}82.3\\79.7\end{bmatrix}$; (b) $S_3=\begin{bmatrix}83.29\\84.71\end{bmatrix}$; (c) totals $150,156,162,168$; up by 6 each step. **Q3.** (a) $T=\begin{bmatrix}0.85 & 0.25\\0.15 & 0.75\end{bmatrix}$; (b) $B=\begin{bmatrix}20\\-10\end{bmatrix}$; (c) $S_{n+1}=T S_n+B$, $S_0=\begin{bmatrix}300\\200\end{bmatrix}$; (d) $S_1=\begin{bmatrix}325\\185\end{bmatrix}$. **Q4.** (a) 25 captive-bred birds are added each year; (b) 10 wild birds are removed (e.g. culled) each year; (c) $+15$ per year; (d) grows. **Q5.** (a) $T S=\begin{bmatrix}90\\110\end{bmatrix}$; (b) $T S+B=\begin{bmatrix}100\\100\end{bmatrix}$; (c) equals S, so S is an equilibrium state; (d) the entries of B add to 0. **Q6.** $S_1=\begin{bmatrix}110\\110\\80\end{bmatrix}$. **Q7.** $S_1=\begin{bmatrix}57\\45\end{bmatrix}$, $S_2=\begin{bmatrix}55.1\\48.9\end{bmatrix}$. **Q8.** $S_1=\begin{bmatrix}54\\52\end{bmatrix}$, $S_2=\begin{bmatrix}57.8\\54.2\end{bmatrix}$, $S_3=\begin{bmatrix}60.58\\57.42\end{bmatrix}$. **Q9.** $T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$, $B=\begin{bmatrix}30\\-15\end{bmatrix}$, $S_1=\begin{bmatrix}270\\245\end{bmatrix}$. **Q10.** (a) $S_1=\begin{bmatrix}120\\110\end{bmatrix}$, $S_2=\begin{bmatrix}137\\123\end{bmatrix}$; (b) $+30$ each step; (c) the total grows without bound. **Q11.** (a) $S_1=\begin{bmatrix}270\\270\end{bmatrix}$, $S_2=\begin{bmatrix}240\\240\end{bmatrix}$; (b) -60 each year; (c) 10 years; (d) the cull outstrips replacement, so the population declines to extinction. **Q12.** $T S=\begin{bmatrix}90\\160\end{bmatrix}$, $T S+B=\begin{bmatrix}100\\150\end{bmatrix}=S$, so it is an equilibrium; B adds 10 to state 1 and removes 10 from state 2 each period; total constant at 250. **Q13.** (a) $\begin{bmatrix}150\\50\end{bmatrix}$; (b) $\begin{bmatrix}160\\40\end{bmatrix}$; (c) both totals are 200, because the entries of B add to 0 — only the split changes; (d) the total grows by 20 each step, without bound. **Q14.** $S_1=\begin{bmatrix}55\\55\\45\end{bmatrix}$, $S_2=\begin{bmatrix}57.5\\59\\43.5\end{bmatrix}$. **Q15.** $B=\begin{bmatrix}6\\-4\end{bmatrix}$: 6 added to state 1, 4 removed from state 2 each period. **Q16.** (a) grows (sum $8>0$); (b) stays constant (sum 0); (c) declines (sum $-13<0$). **Q17.** (a) 200 people leave the city centre each year (emigration) and 350 are added to the suburbs (arrivals/births); (b) $+150$ per year; (c) grows. **Q18.** $S_2=\begin{bmatrix}492.5\\607.5\end{bmatrix}$; the total rises by 50 each year (from 1000 to 1050 to 1100). **Q19.** (a) $T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$, $B=\begin{bmatrix}12\\-12\end{bmatrix}$; (b) $T S+B=\begin{bmatrix}228\\72\end{bmatrix}+\begin{bmatrix}12\\-12\end{bmatrix}=\begin{bmatrix}240\\60\end{bmatrix}=S$; (c) $S_1=\begin{bmatrix}282\\18\end{bmatrix}$, $S_2=\begin{bmatrix}269.4\\30.6\end{bmatrix}$, total constant at 300 and approaching $\begin{bmatrix}240\\60\end{bmatrix}$. **Q20.** (a) $T=\begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$, $B=\begin{bmatrix}24\\-24\end{bmatrix}$, $S_{n+1}=T S_n+B$; (b) $S_1=\begin{bmatrix}318\\162\end{bmatrix}$, $S_2=\begin{bmatrix}327\\153\end{bmatrix}$; (c) 0, so the total stays at 480; (d) $T S+B=\begin{bmatrix}312\\168\end{bmatrix}+\begin{bmatrix}24\\-24\end{bmatrix}=\begin{bmatrix}336\\144\end{bmatrix}=S$.

Fully-worked solutions

Q1. $T S_0=\begin{bmatrix}0.6 & 0.3\\0.4 & 0.7\end{bmatrix}\begin{bmatrix}100\\50\end{bmatrix}=\begin{bmatrix}60+15\\40+35\end{bmatrix}=\begin{bmatrix}75\\75\end{bmatrix}$. Then $S_1=T S_0+B=\begin{bmatrix}75\\75\end{bmatrix}+\begin{bmatrix}10\\-4\end{bmatrix}=\begin{bmatrix}85\\71\end{bmatrix}$.

Q2. (a) $S_2=T S_1+B=\begin{bmatrix}0.6(85)+0.3(71)\\0.4(85)+0.7(71)\end{bmatrix}+\begin{bmatrix}10\\-4\end{bmatrix}=\begin{bmatrix}51+21.3\\34+49.7\end{bmatrix}+\begin{bmatrix}10\\-4\end{bmatrix}=\begin{bmatrix}82.3\\79.7\end{bmatrix}$. (b) $S_3=T S_2+B=\begin{bmatrix}0.6(82.3)+0.3(79.7)\\0.4(82.3)+0.7(79.7)\end{bmatrix}+\begin{bmatrix}10\\-4\end{bmatrix}=\begin{bmatrix}49.38+23.91\\32.92+55.79\end{bmatrix}+\begin{bmatrix}10\\-4\end{bmatrix}=\begin{bmatrix}83.29\\84.71\end{bmatrix}$. (c) The totals are $150,156,162,168$; each step the total rises by 6, matching the entries of B, which add to $10+(-4)=6$.

Q3. (a) The stay/move proportions give $T=\begin{bmatrix}0.85 & 0.25\\ 0.15 & 0.75\end{bmatrix}$ (columns add to 1). (b) Adding 20 to the east and removing 10 from the west gives $B=\begin{bmatrix}20\\ -10\end{bmatrix}$. (c) $S_{n+1}=T\,S_n+B$ with $S_0=\begin{bmatrix}300\\ 200\end{bmatrix}$. (d) $S_1=\begin{bmatrix}0.85(300)+0.25(200)\\ 0.15(300)+0.75(200)\end{bmatrix}+\begin{bmatrix}20\\ -10\end{bmatrix}=\begin{bmatrix}255+50\\ 45+150\end{bmatrix}+\begin{bmatrix}20\\ -10\end{bmatrix}=\begin{bmatrix}305\\ 195\end{bmatrix}+\begin{bmatrix}20\\ -10\end{bmatrix}=\begin{bmatrix}325\\ 185\end{bmatrix}$.

Q4. (a) The positive entry 25 means 25 captive-bred birds are added to the captive population each year (a restock). (b) The negative entry -10 means 10 birds are removed from the wild population each year (a cull or harvest). (c) The total changes by the sum of the entries, $25+(-10)=15$ birds per year. (d) Since this sum is positive, the total population grows over time.

Q5. (a) $T\,S=\begin{bmatrix}0.7 & 0.2\\ 0.3 & 0.8\end{bmatrix}\begin{bmatrix}100\\ 100\end{bmatrix}=\begin{bmatrix}70+20\\ 30+80\end{bmatrix}=\begin{bmatrix}90\\ 110\end{bmatrix}$. (b) $T\,S+B=\begin{bmatrix}90\\ 110\end{bmatrix}+\begin{bmatrix}10\\ -10\end{bmatrix}=\begin{bmatrix}100\\ 100\end{bmatrix}$. (c) This equals S, so $S=T\,S+B$ and $\begin{bmatrix}100\\ 100\end{bmatrix}$ is an equilibrium state. (d) The entries of B add to $10+(-10)=0$, so the total is unchanged from step to step.

Q6. $T\,S_0=\begin{bmatrix}0.7(100)+0.2(100)+0.1(100)\\ 0.2(100)+0.6(100)+0.3(100)\\ 0.1(100)+0.2(100)+0.6(100)\end{bmatrix}=\begin{bmatrix}100\\ 110\\ 90\end{bmatrix}$. Then $S_1=T\,S_0+B=\begin{bmatrix}100\\ 110\\ 90\end{bmatrix}+\begin{bmatrix}10\\ 0\\ -10\end{bmatrix}=\begin{bmatrix}110\\ 110\\ 80\end{bmatrix}$.

Q7. $S_1=T\,S_0+B=\begin{bmatrix}0.8(60)+0.1(40)\\ 0.2(60)+0.9(40)\end{bmatrix}+\begin{bmatrix}5\\ -3\end{bmatrix}=\begin{bmatrix}48+4\\ 12+36\end{bmatrix}+\begin{bmatrix}5\\ -3\end{bmatrix}=\begin{bmatrix}57\\ 45\end{bmatrix}$.
$S_2=T\,S_1+B=\begin{bmatrix}0.8(57)+0.1(45)\\ 0.2(57)+0.9(45)\end{bmatrix}+\begin{bmatrix}5\\ -3\end{bmatrix}=\begin{bmatrix}45.6+4.5\\ 11.4+40.5\end{bmatrix}+\begin{bmatrix}5\\ -3\end{bmatrix}=\begin{bmatrix}55.1\\ 48.9\end{bmatrix}$.

Q8. $S_1=\begin{bmatrix}0.5(40)+0.4(60)\\ 0.5(40)+0.6(60)\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}20+24\\ 20+36\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}54\\ 52\end{bmatrix}$.
$S_2=\begin{bmatrix}0.5(54)+0.4(52)\\ 0.5(54)+0.6(52)\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}27+20.8\\ 27+31.2\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}57.8\\ 54.2\end{bmatrix}$.
$S_3=\begin{bmatrix}0.5(57.8)+0.4(54.2)\\ 0.5(57.8)+0.6(54.2)\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}28.9+21.68\\ 28.9+32.52\end{bmatrix}+\begin{bmatrix}10\\ -4\end{bmatrix}=\begin{bmatrix}60.58\\ 57.42\end{bmatrix}$.

Q9. From the description $T=\begin{bmatrix}0.9 & 0.2\\ 0.1 & 0.8\end{bmatrix}$ and $B=\begin{bmatrix}30\\ -15\end{bmatrix}$. Then
$S_1=T\,S_0+B=\begin{bmatrix}0.9(200)+0.2(300)\\ 0.1(200)+0.8(300)\end{bmatrix}+\begin{bmatrix}30\\ -15\end{bmatrix}=\begin{bmatrix}180+60\\ 20+240\end{bmatrix}+\begin{bmatrix}30\\ -15\end{bmatrix}=\begin{bmatrix}240\\ 260\end{bmatrix}+\begin{bmatrix}30\\ -15\end{bmatrix}=\begin{bmatrix}270\\ 245\end{bmatrix}$.

Q10. (a) $S_1=\begin{bmatrix}0.7(100)+0.3(100)\\ 0.3(100)+0.7(100)\end{bmatrix}+\begin{bmatrix}20\\ 10\end{bmatrix}=\begin{bmatrix}100\\ 100\end{bmatrix}+\begin{bmatrix}20\\ 10\end{bmatrix}=\begin{bmatrix}120\\ 110\end{bmatrix}$;
$S_2=\begin{bmatrix}0.7(120)+0.3(110)\\ 0.3(120)+0.7(110)\end{bmatrix}+\begin{bmatrix}20\\ 10\end{bmatrix}=\begin{bmatrix}84+33\\ 36+77\end{bmatrix}+\begin{bmatrix}20\\ 10\end{bmatrix}=\begin{bmatrix}137\\ 123\end{bmatrix}$. (b) The entries of B add to $20+10=30$, so the total rises by 30 each step (from 200 to 230 to 260). (c) Because this increase is positive and constant, the total population grows without bound.

Q11. (a) $S_1=\begin{bmatrix}0.8(300)+0.2(300)\\ 0.2(300)+0.8(300)\end{bmatrix}+\begin{bmatrix}-30\\ -30\end{bmatrix}=\begin{bmatrix}300\\ 300\end{bmatrix}+\begin{bmatrix}-30\\ -30\end{bmatrix}=\begin{bmatrix}270\\ 270\end{bmatrix}$;
$S_2=\begin{bmatrix}0.8(270)+0.2(270)\\ 0.2(270)+0.8(270)\end{bmatrix}+\begin{bmatrix}-30\\ -30\end{bmatrix}=\begin{bmatrix}270\\ 270\end{bmatrix}+\begin{bmatrix}-30\\ -30\end{bmatrix}=\begin{bmatrix}240\\ 240\end{bmatrix}$. (b) The entries of B add to $-30+(-30)=-60$, so

the total falls by 60 each year. (c) The starting total is 600, so after $600 \div 60 = 10$ years the total reaches 0. (d) The annual cull removes more animals than the transitions can sustain, so the population declines steadily to zero — the species is driven to extinction in the model.

Q12. $TS = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} 100 \\ 150 \end{bmatrix} = \begin{bmatrix} 60+30 \\ 40+120 \end{bmatrix} = \begin{bmatrix} 90 \\ 160 \end{bmatrix}$, so $TS + B = \begin{bmatrix} 90 \\ 160 \end{bmatrix} + \begin{bmatrix} 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 100 \\ 150 \end{bmatrix} = S$. Hence $S = TS + B$ and $\begin{bmatrix} 100 \\ 150 \end{bmatrix}$ is an equilibrium state. The matrix B adds 10 to state 1 and removes 10 from state 2 each period; because these balance ($10 + (-10) = 0$), the total stays constant at $100 + 150 = 250$.

Q13. (a) $S_1 = TS_0 = \begin{bmatrix} 0.75(200) + 0.25(0) \\ 0.25(200) + 0.75(0) \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$. (b) $S_1 = TS_0 + B = \begin{bmatrix} 150 \\ 50 \end{bmatrix} + \begin{bmatrix} 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 160 \\ 40 \end{bmatrix}$. (c) Both totals are 200: adding B has not changed the total because its entries add to 0; it has only shifted the split ($150:50$ becomes $160:40$). (d) With $B = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ the entries add to 20, so the total would rise by 20 each step and grow without bound.

Q14. $S_1 = TS_0 + B = \begin{bmatrix} 0.6(50) + 0.1(50) + 0.2(50) \\ 0.3(50) + 0.7(50) + 0.2(50) \\ 0.1(50) + 0.2(50) + 0.6(50) \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 45 \\ 60 \\ 45 \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 55 \\ 55 \\ 45 \end{bmatrix}$.

$S_2 = TS_1 + B = \begin{bmatrix} 0.6(55) + 0.1(55) + 0.2(45) \\ 0.3(55) + 0.7(55) + 0.2(45) \\ 0.1(55) + 0.2(55) + 0.6(45) \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 47.5 \\ 64 \\ 43.5 \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 57.5 \\ 59 \\ 43.5 \end{bmatrix}$.

Q15. From $S_1 = TS_0 + B$ we get $B = S_1 - TS_0$. Now $TS_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 80+15 \\ 20+35 \end{bmatrix} = \begin{bmatrix} 95 \\ 55 \end{bmatrix}$, so $B = \begin{bmatrix} 101 \\ 51 \end{bmatrix} - \begin{bmatrix} 95 \\ 55 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$. Thus each period 6 are added to state 1 and 4 are removed from state 2.

Q16. Because each column of T adds to 1, multiplying by T leaves the total unchanged, so each period the total changes by the sum of the entries of the constant matrix. (a) B_1 has sum $5 + 3 = 8 > 0$: the total **grows**. (b) B_2 has sum $7 + (-7) = 0$: the total **stays constant**. (c) B_3 has sum $-4 + (-9) = -13 < 0$: the total **declines**.

Q17. (a) The entry -200 means 200 people leave the city centre each year (emigration/removal), and the entry 350 means 350 people are added to the suburbs each year (new arrivals or births). (b) The net change in the total is $-200 + 350 = 150$ people per year. (c) Since this is positive, the total population grows over time.

Q18. $S_1 = TS_0 + B = \begin{bmatrix} 0.85(400) + 0.1(600) \\ 0.15(400) + 0.9(600) \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 340+60 \\ 60+540 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 400 \\ 600 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 450 \\ 600 \end{bmatrix}$.

$S_2 = TS_1 + B = \begin{bmatrix} 0.85(450) + 0.1(600) \\ 0.15(450) + 0.9(600) \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 382.5+60 \\ 67.5+540 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 442.5 \\ 607.5 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 492.5 \\ 607.5 \end{bmatrix}$. The entries of B add to 50, so the total rises by 50 each year: $1000 \rightarrow 1050 \rightarrow 1100$. The herd grows steadily.

Q19. (a) $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ -12 \end{bmatrix}$. (b) $TS = \begin{bmatrix} 0.9(240) + 0.2(60) \\ 0.1(240) + 0.8(60) \end{bmatrix} = \begin{bmatrix} 216+12 \\ 24+48 \end{bmatrix} = \begin{bmatrix} 228 \\ 72 \end{bmatrix}$, so $TS + B = \begin{bmatrix} 228 \\ 72 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 240 \\ 60 \end{bmatrix} = S$; hence $S = TS + B$ is satisfied. (c)

$S_1 = TS_0 + B = \begin{bmatrix} 0.9(300) + 0.2(0) \\ 0.1(300) + 0.8(0) \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 270 \\ 30 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 282 \\ 18 \end{bmatrix}$;

$S_2 = TS_1 + B = \begin{bmatrix} 0.9(282) + 0.2(18) \\ 0.1(282) + 0.8(18) \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 253.8+3.6 \\ 28.2+14.4 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 269.4 \\ 30.6 \end{bmatrix}$. The entries of B add to 0, so the total stays at 300; Area 1 (300 → 282 → 269.4 → …) is falling towards 240 while Area 2 (0 → 18 → 30.6 → …) is rising towards 60 — the population is approaching the equilibrium $\begin{bmatrix} 240 \\ 60 \end{bmatrix}$.

Q20. (a) $T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, $B = \begin{bmatrix} 24 \\ -24 \end{bmatrix}$, and $S_{n+1} = T\,S_n + B$ with $S_0 = \begin{bmatrix} 300 \\ 180 \end{bmatrix}$. (b) $S_1 = \begin{bmatrix} 0.8(300)+0.3(180) \\ 0.2(300)+0.7(180) \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 240+54 \\ 60+126 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 294 \\ 186 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 318 \\ 162 \end{bmatrix}$; $S_2 = \begin{bmatrix} 0.8(318)+0.3(162) \\ 0.2(318)+0.7(162) \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 254.4+48.6 \\ 63.6+113.4 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 303 \\ 177 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 327 \\ 153 \end{bmatrix}$. (c) The entries of B add to $24+(-24)=0$, so the total does not change and stays at $300+180=480$. (d) $T\,S+B = \begin{bmatrix} 0.8(336)+0.3(144) \\ 0.2(336)+0.7(144) \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 268.8+43.2 \\ 67.2+100.8 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 312 \\ 168 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \end{bmatrix} = \begin{bmatrix} 336 \\ 144 \end{bmatrix} = S$, so $S = \begin{bmatrix} 336 \\ 144 \end{bmatrix}$ satisfies $S = T\,S+B$ and is the long-run equilibrium state (with total 480, consistent with part (c)).

Theory

The matrix recurrence relation $S_{n+1} = T\,S_n$ met earlier in this chapter tracks how a system moves between a fixed set of **states** from one step to the next. When the states are the **age groups** of a single population, the recurrence takes a special form. The transition matrix is then called a **Leslie matrix**, written L, and the recurrence

$$S_{n+1} = L\,S_n$$

is used to project the population forward one breeding period (a year, a season or a generation) at a time.

Age classes and the state matrix.

Divide the population into age classes of equal width — for example the 0–1 year olds, the 1–2 year olds and the 2–3 year olds. The **state matrix** S_n is the column matrix whose entries are the numbers in each age class after n steps, youngest class first:

$$S_n = \begin{bmatrix} \text{number in class 1} \\ \text{number in class 2} \\ \text{number in class 3} \end{bmatrix}.$$

The **total population** at step n is the sum of the entries of S_n, and the **age distribution** is the list of those entries (or the proportions they form).

The Leslie matrix.

Two things happen to the population in one step: members are **born**, and survivors grow **one class older**. A Leslie matrix records both.

- The **top row** holds the **fecundity** or **birth rates** b_1, b_2, b_3: the average number of new class-1 members produced, per member of each age class, that are alive to be counted at the next census. Younger classes often have birth rate 0 because they are not yet mature.
- The **sub-diagonal** (the entries just below the leading diagonal) holds the **survival proportions** s_1, s_2: the fraction of class 1 that survives to become class 2, and the fraction of class 2 that survives to become class 3. Each survival proportion lies between 0 and 1.
- Every other entry is 0.

For three age classes,

$$L = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}.$$

Multiplying L by S_n reproduces the two processes automatically. The first row multiplies each class by its birth rate and adds the results, giving all the new class-1 members; the entry s_1 in the second row carries the class-1 survivors into class 2; and s_2 in the third row carries the class-2 survivors into class 3. There is no bottom-right survival entry, so the oldest class does not survive beyond the model.

Projecting the population.

Consider a population whose Leslie matrix and starting state are

$$L = \begin{bmatrix} 0 & 1 & 4 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}, S_0 = \begin{bmatrix} 200 \\ 100 \\ 50 \end{bmatrix}.$$

Here class-1 members do not breed ($b_1 = 0$), each class-2 member produces on average 1 offspring and each class-3 member 4; half of class 1 survive into class 2, and $40\,\%$ of class 2 survive into class 3. Applying the recurrence,

$$S_1 = L\,S_0 = \begin{bmatrix} 0 & 1 & 4 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 1(100)+4(50) \\ 0.5(200) \\ 0.4(100) \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \\ 40 \end{bmatrix},$$

so after one step there are $300+100+40=440$ animals. Repeating the multiplication on S_1 gives $S_2 = \begin{bmatrix} 260 \\ 150 \\ 40 \end{bmatrix}$ and then $S_3 = \begin{bmatrix} 310 \\ 130 \\ 60 \end{bmatrix}$, with totals 450 and 500.

To reach a later step in one calculation, apply L repeatedly. Since $S_1 = L\,S_0$, $S_2 = L\,S_1 = L(L\,S_0) = L^2\,S_0$, and in general

$$S_n = L^n\,S_0.$$

So $S_5 = L^5\,S_0$ can be found by raising L to the fifth power and multiplying by S_0, rather than stepping through every intermediate year.

Long-run behaviour.

The regular transition matrices seen earlier in this chapter drive the state towards a fixed **equilibrium** that no longer changes. A Leslie population behaves differently: it usually keeps **growing** or **declining**, but the *shape* of its age distribution settles down. In the usual case, where more than one age class breeds, the proportion of the population in each age class becomes (almost) constant after enough steps, and from then on **every entry — and therefore the total — is multiplied by the same fixed factor each step**. That factor is the **long-run growth rate**:

- if the factor is greater than 1, the population grows geometrically;
- if it equals 1, the population is **stationary** — an equilibrium in which the numbers stay constant;
- if it is less than 1, the population declines towards extinction.

For the population above, continuing the recurrence shows the total settling so that each year it is about 1.11 times the year before — a long-run growth rate of roughly $11\,\%$ per year — with the age distribution stabilising at approximately $62\,\%$ in class 1, $28\,\%$ in class 2 and $10\,\%$ in class 3. In practice the growth rate is found with technology: iterate $S_0, S_1, S_2, \ldots$ for many steps, then read off the ratio of one total to the previous total once it stops changing.

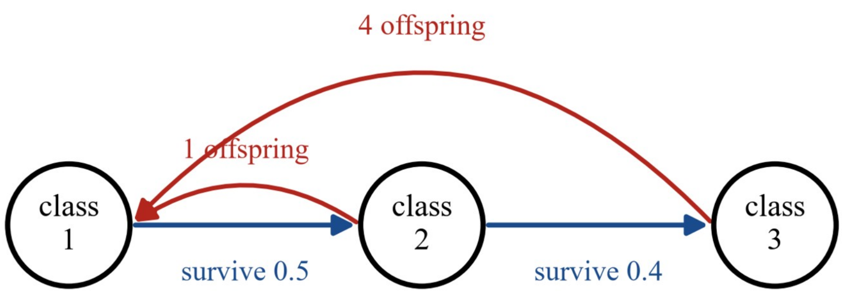
When only the oldest class breeds. This settling down is not automatic. If the only non-zero birth rate belongs to the oldest class, so that the top row of a three-class model is $\begin{bmatrix} 0 & 0 & b_3 \end{bmatrix}$, then every new class-1 member takes two steps to reach the breeding class and its own offspring arrive one step after that. Applying L three times therefore multiplies every entry by the same number, whatever the starting numbers were:

$$S_3 = L^3\,S_0 = b_3\,s_1\,s_2\,S_0.$$

The age distribution then returns to its starting shape every three steps instead of settling to a new one, and the ratio of one total to the previous total keeps cycling rather than steadying at a single value. Over each block of three steps the population is multiplied by $b_3\,s_1\,s_2$: it grows if that product is greater than 1, declines if it is less than 1, and comes back to exactly its starting numbers every three steps if it equals 1. The beetle colony of Example 2 below is a model of this kind, with $b_3\,s_1\,s_2 = 6 \times 0.25 \times 0.4 = 0.6$. So before quoting a long-run growth rate, check that the birth rates are spread over more than one age class.

Using CAS. General Mathematics is technology-active. Enter L and S_0 on your CAS and compute $S_1 = L\,S_0$, then reapply L (or evaluate $L^n S_0$) to project as many steps as you need; sum the entries for the total, and divide successive totals to estimate the long-run growth rate. The by-hand method below shows what the calculator is doing so that you can set up the matrix correctly and check that its output is reasonable.

The life-cycle diagram below matches the Leslie matrix above: blue arrows carry survival proportions from one class to the next, and red arrows carry birth rates back to class 1.



blue = survival proportions (sub-diagonal); red = birth rates to class 1 (top row)

Worked examples

Example 1 — scaffolded

A small mammal population is divided into three one-year age classes. Its Leslie matrix and this year's state matrix are

$$L = \begin{bmatrix} 0 & 2 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, S_0 = \begin{bmatrix} 100 \\ 60 \\ 40 \end{bmatrix}.$$

Find the state matrix after one year and after two years, and the total population each year.

Working	Comment
$S_1 = L\,S_0 = \begin{bmatrix} 0 & 2 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 100 \\ 60 \\ 40 \end{bmatrix}$	Apply $S_{n+1} = L\,S_n$ with $n = 0$.
Row 1: $0(100) + 2(60) + 1(40) = 160$.	Births: each class-2 animal produces 2 young, each class-3 animal 1.
Row 2: $0.6(100) = 60$.	$60\,\%$ of class 1 survive into class 2.
Row 3: $0.5(60) = 30$.	$50\,\%$ of class 2 survive into class 3.
$S_1 = \begin{bmatrix} 160 \\ 60 \\ 30 \end{bmatrix}$, total 250.	Sum the entries for the total population.
$S_2 = L\,S_1 = \begin{bmatrix} 0 & 2 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 160 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 150 \\ 96 \\ 30 \end{bmatrix}$, total 276.	Multiply again, now using S_1: row 1 $2(60) + 1(30) = 150$; row 2 $0.6(160) = 96$; row 3 $0.5(60) = 30$.

After one year $S_1=\begin{bmatrix}160\\60\\30\end{bmatrix}$ (total 250) and after two years $S_2=\begin{bmatrix}150\\96\\30\end{bmatrix}$ (total 276).

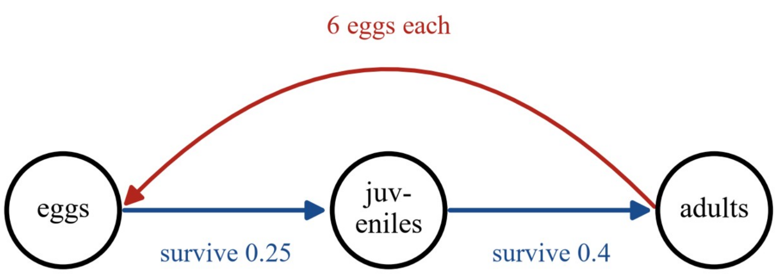
Example 2 — scaffolded

A beetle population is counted in three stages: **eggs**, **juveniles** and **adults**. Each year every adult produces on average 6 eggs, no eggs or juveniles reproduce, one quarter of the eggs survive to become juveniles, and 40 % of juveniles survive to become adults; no adult survives to the next year. A colony starts with 400 eggs, 100 juveniles and 50 adults.

(a) Write down the Leslie matrix L and the initial state matrix S_0. **(b)** Find the numbers of eggs, juveniles and adults after one year and after two years.

Working	Comment
Top row $=[0 \quad 0 \quad 6]$.	Only adults reproduce; each makes 6 eggs, so $b_1=b_2=0$, $b_3=6$.
Sub-diagonal: $s_1=0.25$, $s_2=0.4$.	One quarter of eggs reach the juvenile stage; 40 % of juveniles reach adulthood.
$L=\begin{bmatrix}0&0&6\\0.25&0&0\\0&0.4&0\end{bmatrix}$, $S_0=\begin{bmatrix}400\\100\\50\end{bmatrix}$.	The bottom-right entry is 0: adults do not survive another year.
$S_1=LS_0=\begin{bmatrix}6(50)\\0.25(400)\\0.4(100)\end{bmatrix}=\begin{bmatrix}300\\100\\40\end{bmatrix}$, total 440.	300 new eggs, 100 new juveniles, 40 new adults.
$S_2=LS_1=\begin{bmatrix}6(40)\\0.25(300)\\0.4(100)\end{bmatrix}=\begin{bmatrix}240\\75\\40\end{bmatrix}$, total 355.	Equivalently $S_2=L^2S_0$.

After one year there are 300 eggs, 100 juveniles and 40 adults; after two years 240 eggs, 75 juveniles and 40 adults.



only adults reproduce; each survival arrow ages a survivor into the next class

Example 3 — unscaffolded

A population with three age classes has Leslie matrix and initial state

$$L=\begin{bmatrix}0&1&1\\0.4&0&0\\0&0.3&0\end{bmatrix},\ S_0=\begin{bmatrix}500\\200\\100\end{bmatrix}.$$

Project the population for the next three years, and describe its long-run behaviour.

Applying $S_{n+1}=L\,S_n$ term by term,

$$S_1=\begin{bmatrix}1(200)+1(100)\\0.4(500)\\0.3(200)\end{bmatrix}=\begin{bmatrix}300\\200\\60\end{bmatrix},S_2=\begin{bmatrix}260\\120\\60\end{bmatrix},S_3=\begin{bmatrix}180\\104\\36\end{bmatrix}.$$

The totals are S_0: 800, then 560, 440 and 320. Each total is smaller than the one before, so the population is shrinking. Continuing the recurrence with technology, the ratio of one total to the previous total settles down to about 0.75: in the long run the population loses roughly a quarter of its numbers each year while the age distribution keeps the same shape.

∴ over the three years the population falls from 800 to 320, and in the long run its total is multiplied by about 0.75 each year — a decline of about $25\,\%$ per year, heading towards local extinction.

Example 4 — unscaffolded

A wildlife survey of a population with three age classes, in which **only the oldest class reproduces**, records this year's and next year's age structure as

$$S_0=\begin{bmatrix}600\\200\\100\end{bmatrix},S_1=\begin{bmatrix}800\\300\\60\end{bmatrix}.$$

Assuming a Leslie model, find the fecundity of the oldest class and the two survival proportions, write the Leslie matrix, and predict the population two years from now.

Because only the oldest class breeds, the Leslie matrix has the form $L=\begin{bmatrix}0&0&b_3\\s_1&0&0\\0&s_2&0\end{bmatrix}$, and $S_1=L\,S_0$ gives three equations. The first row is all new class-1 members: $b_3\times100=800$, so $b_3=8$. The second row carries class-1 survivors forward: $s_1\times600=300$, so $s_1=0.5$. The third row carries class-2 survivors forward: $s_2\times200=60$, so $s_2=0.3$. Hence

$$L=\begin{bmatrix}0&0&8\\0.5&0&0\\0&0.3&0\end{bmatrix}.$$

Projecting one more year,

$$S_2=L\,S_1=\begin{bmatrix}8(60)\\0.5(800)\\0.3(300)\end{bmatrix}=\begin{bmatrix}480\\400\\90\end{bmatrix},$$

a total of 970.

∴ the oldest class produces 8 offspring each, the survival proportions are $s_1=0.5$ and $s_2=0.3$, and two years from now the population is $\begin{bmatrix}480\\400\\90\end{bmatrix}$ — a total of 970.

Practice questions

Scaffolded

Q1. A population with three age classes has $L=\begin{bmatrix}0 & 2 & 3\\ 0.5 & 0 & 0\\ 0 & 0.4 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}80\\ 40\\ 20\end{bmatrix}$. (a) Find $S_1=L\,S_0$. (b) State the total population after one year.

Q2. For the Leslie matrix $L=\begin{bmatrix}0 & 1 & 2\\ 0.4 & 0 & 0\\ 0 & 0.5 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}200\\ 100\\ 50\end{bmatrix}$: (a) find S_1 and its total; (b) find S_2 and its total.

Q3. In a Leslie model of a moth population (eggs, larvae, adults), each adult lays 20 eggs, $10\,\%$ of eggs survive to the larval stage, $30\,\%$ of larvae survive to become adults, and no adult survives to the next generation. A population starts with 1000 eggs, 200 larvae and 80 adults. (a) Write down the Leslie matrix L and the initial state matrix S_0. (b) Find the population after one generation.

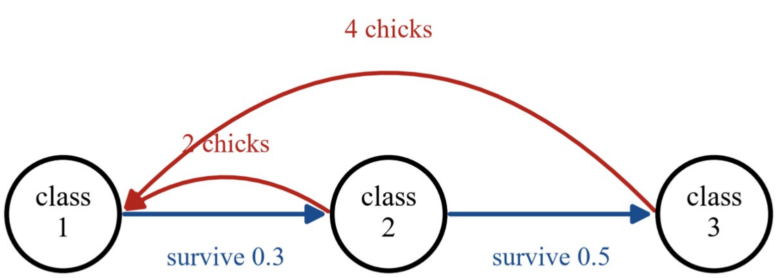
Q4. A population has $L=\begin{bmatrix}0 & 3 & 1\\ 0.5 & 0 & 0\\ 0 & 0.2 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\ 40\\ 20\end{bmatrix}$. (a) Find S_1 and its total. (b) Find S_2 and its total. (c) State whether the population grew over the two years.

Q5. A species has just two age classes, juveniles and adults, with Leslie matrix and starting state $L=\begin{bmatrix}1 & 4\\ 0.5 & 0\end{bmatrix}$, $S_0=\begin{bmatrix}60\\ 20\end{bmatrix}$. (a) Interpret each of the entries 1, 4 and 0.5. (b) Find S_1 and S_2. (c) State the total population after each of the two years.

Q6. A population has $L=\begin{bmatrix}0 & 1 & 3\\ 0.4 & 0 & 0\\ 0 & 0.5 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\ 40\\ 20\end{bmatrix}$. (a) Find S_1. (b) What do you notice when you compare S_1 with S_0? (c) Find S_2, and describe the long-term behaviour of the population and its growth rate.

Unscaffolded

Q7. The life-cycle diagram below models a bird population with three age classes: each red arrow gives the number of chicks a bird of that class raises in a year, and each blue arrow gives the proportion of a class that survives into the next class.



blue = proportion surviving into the next class;   red = chicks raised each year, per bird

There are currently 300, 90 and 40 birds in classes 1, 2 and 3. Write down the Leslie matrix L, then find next year's state matrix and the total number of birds.

Q8. For $L=\begin{bmatrix}0 & 1 & 5\\ 0.5 & 0 & 0\\ 0 & 0.4 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}200\\ 80\\ 40\end{bmatrix}$, find the state matrix after two years and its total.

Q9. For $L=\begin{bmatrix}0 & 2 & 2\\ 0.4 & 0 & 0\\ 0 & 0.5 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}500\\ 200\\ 100\end{bmatrix}$, use $S_3=L^3S_0$ to find the state matrix after three years and its total.

Q10. In a Leslie model of an aphid population (eggs, nymphs, adults), each adult produces 50 eggs, 2% of eggs reach the nymph stage, 50% of nymphs reach adulthood, and adults do not survive to the next cycle. A population starts with 5000 eggs, 80 nymphs and 20 adults. (a) Find the state matrix after one cycle, after two cycles and after three cycles. (b) Compare the third state matrix with the starting one, and hence explain why the ratio of successive totals for this population never settles down to a single number.

Q11. A species with two age classes has $L=\begin{bmatrix}0.5 & 3\\ 0.4 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\ 50\end{bmatrix}$. (a) Find S_1, S_2 and S_3, and the total each year. (b) Estimate the constant factor by which the total grows each cycle in the long run.

Q12. A population has $L=\begin{bmatrix}0 & 2 & 3\\ 0.5 & 0 & 0\\ 0 & 0.4 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\ 50\\ 25\end{bmatrix}$. Using technology, project the population for several years and hence estimate, to two decimal places, the constant factor by which the total population grows each year in the long run.

Q13. A population is modelled by $L=\begin{bmatrix}0 & 3 & 2\\ 0.6 & 0 & 0\\ 0 & 0.25 & 0\end{bmatrix}$. (a) How many age classes does the model have? (b) What does the entry 0.6 tell you? (c) What does the entry 2 in the top row tell you? (d) Which age class does not reproduce, and how can you tell from L? (e) Draw the life-cycle diagram for this model, labelling every arrow.

Q14. A population with three age classes, in which only the oldest class reproduces, has this year's numbers 400, 150, 60 and next year's numbers 600, 200, 60. Assuming a Leslie model, find the fecundity of the oldest class and the two survival proportions, and predict the numbers the following year.

Q15. A population has $L=\begin{bmatrix}0 & 1 & 2\\ 0.6 & 0 & 0\\ 0 & 0.5 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}100\\ 50\\ 25\end{bmatrix}$. Using technology, iterate for many years and state (a) the approximate long-run growth factor, and (b) the approximate percentage of the population in each age class once the age distribution has stabilised.

Q16. A population has $L=\begin{bmatrix}0 & 0.5 & 1\\ 0.4 & 0 & 0\\ 0 & 0.3 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}400\\ 200\\ 100\end{bmatrix}$. (a) Find the total population for each of the next three years. (b) State whether the population is growing or declining. (c) Estimate the constant factor by which the total changes each year in the long run.

Q17. A population with three age classes has $S_0=\begin{bmatrix}500\\ 150\\ 60\end{bmatrix}$ and a Leslie matrix with top row $\begin{bmatrix}0 & 1 & 3\end{bmatrix}$, survival proportion s_1 from class 1 to class 2, and survival proportion $s_2=0.4$ from class 2 to class 3. (a) If $s_1=0.3$, find next year's total population. (b) A conservation program raises s_1 to 0.5. Find next year's total now. (c) Comment on the effect of improving the survival of the youngest class.

Q18. A fish species has four age classes with Leslie matrix and starting state

$$L=\begin{bmatrix}0 & 2 & 3 & 1\\ 0.5 & 0 & 0 & 0\\ 0 & 0.4 & 0 & 0\\ 0 & 0 & 0.3 & 0\end{bmatrix}, S_0=\begin{bmatrix}100\\ 60\\ 40\\ 20\end{bmatrix}.$$

Find next year's state matrix and the total population.

Q19. Using technology, the total population of a species over six years is found to be 800, 900, 940, 1040, 1112, 1208. (a) Find the ratio of each year's total to the previous year's, correct to three decimal places. (b) Estimate the constant factor that the total is approaching. (c) Explain why the early ratios are not all equal.

Q20. A fish population has three age classes. Each year age-1 fish produce no young, age-2 fish produce 4 young each and age-3 fish produce 6 young each; 20% of age-1 fish survive to age 2 and 30% of age-2 fish survive to age 3; age-3 fish do not survive further. There are currently 2000 age-1, 500 age-2 and 200 age-3 fish. (a) Write down the Leslie matrix and find the total population after one year and after two years. (b) Given that the long-run growth factor is about 1.07, comment on whether the population is sustainable, despite the change between year 1 and year 2.

Answers

Q1. (a) $\begin{bmatrix}140\\40\\16\end{bmatrix}$; (b) 196. **Q2.** (a) $\begin{bmatrix}200\\80\\50\end{bmatrix}$, total 330; (b) $\begin{bmatrix}180\\80\\40\end{bmatrix}$, total 300. **Q3.** (a) $L=\begin{bmatrix}0&0&20\\0.1&0&0\\0&0.3&0\end{bmatrix}$, $S_0=\begin{bmatrix}1000\\200\\80\end{bmatrix}$; (b) $\begin{bmatrix}1600\\100\\60\end{bmatrix}$ (total 1760). **Q4.** (a) $\begin{bmatrix}140\\50\\8\end{bmatrix}$, total 198; (b) $\begin{bmatrix}158\\70\\10\end{bmatrix}$, total 238; (c) yes, the total rose from 160 to 238. **Q5.** (a) each juvenile produces 1 offspring; each adult produces 4; 0.5 (half) of juveniles survive to become adults. (b) $S_1=\begin{bmatrix}140\\30\end{bmatrix}$, $S_2=\begin{bmatrix}260\\70\end{bmatrix}$. (c) 170, then 330. **Q6.** (a) $\begin{bmatrix}100\\40\\20\end{bmatrix}$; (b) $S_1=S_0$; (c) $S_2=\begin{bmatrix}100\\40\\20\end{bmatrix}$ — the population is stationary (in equilibrium), staying at 160 with growth factor 1. **Q7.** $L=\begin{bmatrix}0&2&4\\0.3&0&0\\0&0.5&0\end{bmatrix}$; $S_1=\begin{bmatrix}340\\90\\45\end{bmatrix}$, total 475. **Q8.** $\begin{bmatrix}260\\140\\40\end{bmatrix}$, total 440. **Q9.** $\begin{bmatrix}680\\240\\120\end{bmatrix}$, total 1040. **Q10.** (a) $\begin{bmatrix}1000\\100\\40\end{bmatrix}$, $\begin{bmatrix}2000\\20\\50\end{bmatrix}$, $\begin{bmatrix}2500\\40\\10\end{bmatrix}$; (b) $S_3=0.5\,S_0$ — only adults breed, so the age structure repeats every three cycles and the ratio of successive totals cycles $(0.224, 1.816, 1.232, 0.224, \ldots)$ instead of settling. **Q11.** (a) $S_1=\begin{bmatrix}200\\40\end{bmatrix}$ (total 240), $S_2=\begin{bmatrix}220\\80\end{bmatrix}$ (total 300), $S_3=\begin{bmatrix}350\\88\end{bmatrix}$ (total 438); (b) about 1.37. **Q12.** About 1.22. **Q13.** (a) 3; (b) 60 % of class 1 survive into class 2; (c) each class-3 member produces on average 2 offspring; (d) class 1, since its top-row entry is 0; (e) see the diagram in the solutions. **Q14.** Fecundity 10; $s_1=0.5$, $s_2=0.4$; following year $\begin{bmatrix}600\\300\\80\end{bmatrix}$ (total 980). **Q15.** (a) about 1.08; (b) roughly 55 % in class 1, 31 % in class 2 and 14 % in class 3. **Q16.** (a) 420, 268, 168; (b) declining; (c) about 0.63. **Q17.** (a) 540; (b) 640; (c) raising the youngest class's survival increases the population — an extra 100 animals next year. **Q18.** $\begin{bmatrix}260\\50\\24\\12\end{bmatrix}$, total 346. **Q19.** (a) 1.125, 1.044, 1.106, 1.069, 1.086; (b) about 1.08; (c) the age distribution has not yet settled to its stable shape, so the total is not yet multiplied by a constant factor. **Q20.** (a) $L=\begin{bmatrix}0&4&6\\0.2&0&0\\0&0.3&0\end{bmatrix}$; totals 3750 then 3260; (b) the long-run factor $1.07>1$, so the population grows in the long run — the dip in year 2 is a short-term effect of the starting age structure, not long-run decline.

Fully-worked solutions

Q1. $S_1=L\,S_0=\begin{bmatrix}0&2&3\\0.5&0&0\\0&0.4&0\end{bmatrix}\begin{bmatrix}80\\40\\20\end{bmatrix}$. Row 1: $2(40)+3(20)=140$; row 2: $0.5(80)=40$; row 3: $0.4(40)=16$. So $S_1=\begin{bmatrix}140\\40\\16\end{bmatrix}$, and the total is $140+40+16=196$.

Q2. (a) Row 1: $1(100)+2(50)=200$; row 2: $0.4(200)=80$; row 3: $0.5(100)=50$, so $S_1=\begin{bmatrix}200\\80\\50\end{bmatrix}$, total 330. (b) Multiply L by S_1: row 1 $1(80)+2(50)=180$; row 2 $0.4(200)=80$; row 3 $0.5(80)=40$, so $S_2=\begin{bmatrix}180\\80\\40\end{bmatrix}$, total 300.

Q3. (a) Only adults reproduce and each lays 20 eggs, so the top row is $\begin{bmatrix}0 & 0 & 20\end{bmatrix}$; the survival proportions are $s_1=0.1$ (eggs to larvae) and $s_2=0.3$ (larvae to adults): $L=\begin{bmatrix}0 & 0 & 20\\0.1 & 0 & 0\\0 & 0.3 & 0\end{bmatrix}$, $S_0=\begin{bmatrix}1000\\200\\80\end{bmatrix}$. (b) $S_1=\begin{bmatrix}20(80)\\0.1(1000)\\0.3(200)\end{bmatrix}=\begin{bmatrix}1600\\100\\60\end{bmatrix}$, a total of 1760.

Q4. (a) Row 1: $3(40)+1(20)=140$; row 2: $0.5(100)=50$; row 3: $0.2(40)=8$, so $S_1=\begin{bmatrix}140\\50\\8\end{bmatrix}$, total 198. (b) Applying L to S_1: row 1 $3(50)+1(8)=158$; row 2 $0.5(140)=70$; row 3 $0.2(50)=10$, so $S_2=\begin{bmatrix}158\\70\\10\end{bmatrix}$, total 238.

(c) The total rose from 160 (in S_0) to 198 and then 238, so the population grew.

Q5. (a) In $L=\begin{bmatrix}1 & 4\\0.5 & 0\end{bmatrix}$ the top row gives the birth rates: each juvenile produces on average 1 offspring and each adult 4. The entry 0.5 is the survival proportion — half of the juveniles survive to become adults. (The bottom-right 0 means no adult survives another year.) (b) $S_1=\begin{bmatrix}1(60)+4(20)\\0.5(60)\end{bmatrix}=\begin{bmatrix}140\\30\end{bmatrix}$; $S_2=\begin{bmatrix}1(140)+4(30)\\0.5(140)\end{bmatrix}=\begin{bmatrix}260\\70\end{bmatrix}$. (c) Totals: $140+30=170$ after one year and $260+70=330$ after two years.

Q6. (a) Row 1: $1(40)+3(20)=100$; row 2: $0.4(100)=40$; row 3: $0.5(40)=20$, so $S_1=\begin{bmatrix}100\\40\\20\end{bmatrix}$. (b) S_1 is identical to S_0: the age structure has not changed. (c) Because $S_1=S_0$, applying L again gives $S_2=L\,S_1=\begin{bmatrix}100\\40\\20\end{bmatrix}$ as well, and the state stays at $\begin{bmatrix}100\\40\\20\end{bmatrix}$ forever. The population is **stationary** (in equilibrium): the total remains 160 and the growth factor is exactly 1.

Q7. Read the matrix off the diagram. The red arrows back to class 1 give the top row: there is no arrow from class 1, so $b_1=0$; class 2 contributes 2 and class 3 contributes 4. The blue arrows give the sub-diagonal: $s_1=0.3$ from class 1 to class 2 and $s_2=0.5$ from class 2 to class 3. Hence $L=\begin{bmatrix}0 & 2 & 4\\0.3 & 0 & 0\\0 & 0.5 & 0\end{bmatrix}$ and $S_0=\begin{bmatrix}300\\90\\40\end{bmatrix}$. Then $S_1=\begin{bmatrix}2(90)+4(40)\\0.3(300)\\0.5(90)\end{bmatrix}=\begin{bmatrix}340\\90\\45\end{bmatrix}$, a total of $340+90+45=475$ birds.

Q8. First year: $S_1 = \begin{bmatrix} 1(80)+5(40) \\ 0.5(200) \\ 0.4(80) \end{bmatrix} = \begin{bmatrix} 280 \\ 100 \\ 32 \end{bmatrix}$. Second year: $S_2 = L\,S_1 = \begin{bmatrix} 1(100)+5(32) \\ 0.5(280) \\ 0.4(100) \end{bmatrix} = \begin{bmatrix} 260 \\ 140 \\ 40 \end{bmatrix}$, a total of 440.

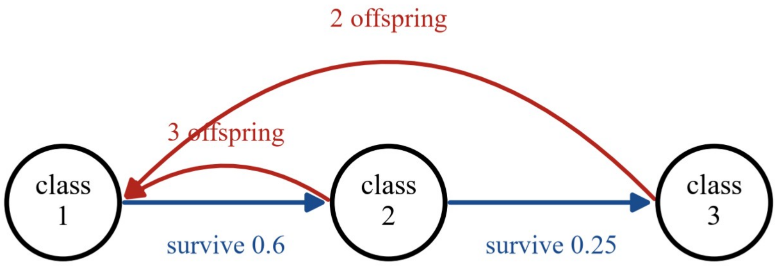
Q9. Stepping through, $S_1 = \begin{bmatrix} 2(200)+2(100) \\ 0.4(500) \\ 0.5(200) \end{bmatrix} = \begin{bmatrix} 600 \\ 200 \\ 100 \end{bmatrix}$; $S_2 = \begin{bmatrix} 2(200)+2(100) \\ 0.4(600) \\ 0.5(200) \end{bmatrix} = \begin{bmatrix} 600 \\ 240 \\ 100 \end{bmatrix}$;
$S_3 = \begin{bmatrix} 2(240)+2(100) \\ 0.4(600) \\ 0.5(240) \end{bmatrix} = \begin{bmatrix} 680 \\ 240 \\ 120 \end{bmatrix}$, which is $L^3 S_0$. Its total is $680+240+120=1040$.

Q10. (a) $L = \begin{bmatrix} 0 & 0 & 50 \\ 0.02 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$ (only adults reproduce, laying 50 eggs; $2\% = 0.02$ of eggs survive, $50\% = 0.5$ of nymphs survive). Then $S_1 = \begin{bmatrix} 50(20) \\ 0.02(5000) \\ 0.5(80) \end{bmatrix} = \begin{bmatrix} 1000 \\ 100 \\ 40 \end{bmatrix}$, $S_2 = \begin{bmatrix} 50(40) \\ 0.02(1000) \\ 0.5(100) \end{bmatrix} = \begin{bmatrix} 2000 \\ 20 \\ 50 \end{bmatrix}$ and $S_3 = \begin{bmatrix} 50(50) \\ 0.02(2000) \\ 0.5(20) \end{bmatrix} = \begin{bmatrix} 2500 \\ 40 \\ 10 \end{bmatrix}$
. (b) Every entry of S_3 is exactly half the matching entry of S_0: $S_3 = 0.5\,S_0$, since $b_3 s_1 s_2 = 50 \times 0.02 \times 0.5 = 0.5$. Only the oldest class breeds, so the age structure returns to its starting shape every three cycles rather than settling to a stable one. The totals are 5100, 1140, 2070, 2550, 570, …, so the ratio of successive totals repeats 0.224, 1.816, 1.232, 0.224, … and never settles down to a single long-run factor.

Q11. (a) $S_1 = \begin{bmatrix} 0.5(100)+3(50) \\ 0.4(100) \end{bmatrix} = \begin{bmatrix} 200 \\ 40 \end{bmatrix}$ (total 240); $S_2 = \begin{bmatrix} 0.5(200)+3(40) \\ 0.4(200) \end{bmatrix} = \begin{bmatrix} 220 \\ 80 \end{bmatrix}$ (total 300);
$S_3 = \begin{bmatrix} 0.5(220)+3(80) \\ 0.4(220) \end{bmatrix} = \begin{bmatrix} 350 \\ 88 \end{bmatrix}$ (total 438). (b) Continuing the recurrence with technology, the ratio of successive totals settles to about 1.37, so in the long run the total is multiplied by about 1.37 each cycle.

Q12. Iterating $S_{n+1} = L\,S_n$ with a calculator, the totals are 175, 245, 267.5, 350, …; the ratio of one total to the previous total wanders at first and then settles. After enough steps the ratio steadies at about 1.22, so the long-run growth factor is 1.22 (to two decimal places) — a growth of roughly 22% per year.

Q13. (a) L is 3×3, so there are 3 age classes. (b) The 0.6 is on the sub-diagonal in the class-1 column, so 60% of class 1 survive to become class 2. (c) The 2 is in the top row, in the class-3 column, so each class-3 member produces on average 2 class-1 offspring per step. (d) Class 1 does not reproduce, because its top-row entry (the first entry of the top row) is 0. (e) Draw the three classes in a row. The sub-diagonal gives the survival arrows class 1 → class 2 labelled 0.6 and class 2 → class 3 labelled 0.25; the top row gives the birth arrows back to class 1, one from class 2 labelled 3 and one from class 3 labelled 2, with no arrow from class 1 because $b_1 = 0$.



blue = survival proportions (sub-diagonal); red = birth rates to class 1 (top row)

Q14. With only the oldest class breeding, $L=\begin{bmatrix}0&0&b_3\\s_1&0&0\\0&s_2&0\end{bmatrix}$ and $S_1=L\,S_0$ gives: top row $b_3\times 60=600\Rightarrow b_3=10$; second row $s_1\times 400=200\Rightarrow s_1=0.5$; third row $s_2\times 150=60\Rightarrow s_2=0.4$. So $L=\begin{bmatrix}0&0&10\\0.5&0&0\\0&0.4&0\end{bmatrix}$, and the following year is $S_2=L\,S_1=\begin{bmatrix}10(60)\\0.5(600)\\0.4(200)\end{bmatrix}=\begin{bmatrix}600\\300\\80\end{bmatrix}$, a total of 980.

Q15. Iterating $S_{n+1}=L\,S_n$ for many steps with technology, successive totals settle to a fixed ratio of about 1.08, so the long-run growth factor is approximately 1.08 (growth of about $8\,\%$ per year). The age distribution stabilises at proportions of about $0.55:0.31:0.14$, i.e. roughly $55\,\%$ of the population in class 1, $31\,\%$ in class 2 and $14\,\%$ in class 3.

Q16. (a) $S_1=\begin{bmatrix}0.5(200)+1(100)\\0.4(400)\\0.3(200)\end{bmatrix}=\begin{bmatrix}200\\160\\60\end{bmatrix}$ (total 420); $S_2=\begin{bmatrix}0.5(160)+1(60)\\0.4(200)\\0.3(160)\end{bmatrix}=\begin{bmatrix}140\\80\\48\end{bmatrix}$ (total 268); $S_3=\begin{bmatrix}0.5(80)+1(48)\\0.4(140)\\0.3(80)\end{bmatrix}=\begin{bmatrix}88\\56\\24\end{bmatrix}$ (total 168). (b) The totals $700,420,268,168$ fall each year, so the population is declining. (c) Continuing the iteration, the ratio of successive totals settles to about 0.63, so in the long run the total is multiplied by about 0.63 each year.

Q17. The Leslie matrix is $L=\begin{bmatrix}0&1&3\\s_1&0&0\\0&0.4&0\end{bmatrix}$. (a) With $s_1=0.3$: $S_1=\begin{bmatrix}1(150)+3(60)\\0.3(500)\\0.4(150)\end{bmatrix}=\begin{bmatrix}330\\150\\60\end{bmatrix}$, total 540. (b) With $s_1=0.5$ only the second row changes: $0.5(500)=250$, giving $S_1=\begin{bmatrix}330\\250\\60\end{bmatrix}$, total 640. (c) Raising the survival of the youngest class from 0.3 to 0.5 lifts next year's population from 540 to 640 — an extra 100 animals — so improving juvenile survival grows the population.

Q18. $S_1=L\,S_0$: row 1 $2(60)+3(40)+1(20)=120+120+20=260$; row 2 $0.5(100)=50$; row 3 $0.4(60)=24$; row 4 $0.3(40)=12$. So $S_1=\begin{bmatrix}260\\50\\24\\12\end{bmatrix}$, a total of $260+50+24+12=346$.

Q19. (a) Dividing each total by the previous one: $\frac{900}{800}=1.125$, $\frac{940}{900}\approx 1.044$, $\frac{1040}{940}\approx 1.106$, $\frac{1112}{1040}\approx 1.069$, $\frac{1208}{1112}\approx 1.086$. (b) The ratios are closing in on a value near 1.08, so the long-run growth factor is about 1.08. (c) Early on, the population is not yet in its stable age distribution, so the different age classes grow by different amounts and the total is not multiplied by a constant factor; only once the age distribution settles does the ratio become steady.

Q20. (a) Age-1 fish do not breed, so the top row is $\begin{bmatrix}0&4&6\end{bmatrix}$; the survival proportions are $s_1=0.2$ and $s_2=0.3$, giving $L=\begin{bmatrix}0&4&6\\0.2&0&0\\0&0.3&0\end{bmatrix}$, $S_0=\begin{bmatrix}2000\\500\\200\end{bmatrix}$. Then $S_1=\begin{bmatrix}4(500)+6(200)\\0.2(2000)\\0.3(500)\end{bmatrix}=\begin{bmatrix}3200\\400\\150\end{bmatrix}$ (total 3750), and

$S_2 = \begin{bmatrix} 4(400)+6(150) \\ 0.2(3200) \\ 0.3(400) \end{bmatrix} = \begin{bmatrix} 2500 \\ 640 \\ 120 \end{bmatrix}$ (total 3260). (b) The total rises to 3750 then dips to 3260, but this reflects the uneven starting age structure working its way through. In the long run the total is multiplied by about 1.07 each year, and since $1.07 > 1$ the population grows steadily — so the population is sustainable.

Topic 3 Test A — Multiple-choice

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 3 minutes. Writing time: 20 minutes. 12 multiple-choice questions, 12 marks. Choose the response that is ***correct*** *for the question.*

Question 1

Consider the matrix $A=\begin{bmatrix} 4 & -2 & 7 \\ 0 & 5 & -1 \end{bmatrix}$.

The order of A and the value of the element a_{23} are respectively

A. 3×2 and -1

B. 2×3 and -1

C. 2×3 and 7

D. 3×2 and 5

Question 2

If $A=\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 0 \\ -2 & 6 \end{bmatrix}$, then $2A-B$ is equal to

A. $\begin{bmatrix} 1 & -2 \\ 6 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 11 & -2 \\ 2 & 14 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 2 \\ -6 & -2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

Question 3

The matrix P has order 3×2 and the matrix Q has order 2×5.

Which one of the following products is defined, and what is its order?

A. QP, order 5×3

B. PQ, order 2×2

C. PQ, order 3×5

D. QP, order 3×5

Question 4

On Monday and Tuesday a bakery sells pies and rolls. The numbers sold are recorded in $\begin{bmatrix} 12 & 20 \\ 15 & 10 \end{bmatrix}$, where the rows are Monday and Tuesday and the columns are pies and rolls. A pie sells for $\$5$ and a roll for $\$3$.

Tuesday's takings are

A. $\$135$

B. $\$90$

C. $\$120$

D. $\$105$

Question 5

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then A^3 is equal to

A. $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Question 6

The matrix $\begin{bmatrix} k & 3 \\ 2 & k-1 \end{bmatrix}$ does **not** have an inverse when

A. $k = -3$ or $k = 2$

B. $k = 3$ or $k = -2$

C. $k = 6$ or $k = 1$

D. $k = 3$ only

Question 7

The inverse of $A=\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$ is

A. $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$

B. $\frac{1}{7}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

C. $\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

D. $\begin{bmatrix}-2 & 1\\3 & -2\end{bmatrix}$

Question 8

The batting order of four cricketers E, F, G and H is recorded in the column matrix $\begin{bmatrix}E\\F\\G\\H\end{bmatrix}$. For the next match a new batting order is formed by the product $P\begin{bmatrix}E\\F\\G\\H\end{bmatrix}$, where P is the permutation matrix

$$P=\begin{bmatrix}0 & 0 & 0 & 1\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

The new batting order, from first to last, is

A. E, F, G, H

B. H, F, E, G

C. G, F, H, E

D. H, G, F, E

Question 9

In a round-robin tournament between four teams W, X, Y, Z, the dominance matrix D below records one-step dominances (a 1 in row i, column j means team i defeated team j). The teams are in the order W, X, Y, Z.

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The teams are ranked using the sum of their one-step and two-step dominances, $D + D^2$. From strongest to weakest, the ranking is

A. W, X, Z, Y

B. W, X, Y, Z

C. X, W, Z, Y

D. W, Z, X, Y

Question 10

A transition matrix M models the yearly movement of residents between a town (T) and a nearby city (C):

$$M = \begin{bmatrix} 0.85 & 0.10 \\ 0.15 & 0.90 \end{bmatrix}$$

where the rows and columns are in the order T, C and $S_{n+1} = M S_n$.

Which one of the following statements is correct?

A. Each year, 10% of town residents move to the city.

B. Each year, 15% of city residents move to the town.

C. Each year, 90% of town residents remain in the town.

D. Each year, 15% of town residents move to the city.

Question 11

A population of 500 is distributed between two states according to the regular transition matrix $T = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$, with $S_{n+1} = T S_n$.

In the long term, the number in the first state approaches

A. 300

B. 250

C. 100

D. 400

Question 12

A population with three age classes has Leslie matrix $L=\begin{bmatrix} 0 & 3 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ and initial state $S_0=\begin{bmatrix} 40 \\ 30 \\ 20 \end{bmatrix}$.

The total population after one time period is

A. 90

B. 142

C. 122

D. 130

End of Topic 3 Test A

Test A — answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12
Ans	B	A	C	D	A	B	C	B	A	D	C	B

Q1. B — the order is (rows × columns) $=2\times3$; a_{23} is the entry in row 2, column 3, which is -1. **Q2.** A — $2A=\begin{bmatrix}6 & -2\\4 & 8\end{bmatrix}$, so $2A-B=\begin{bmatrix}1 & -2\\6 & 2\end{bmatrix}$. **Q3.** C — PQ is $(3\times2)(2\times5)$: the inner dimensions match, so it is defined with order 3×5; QP is $(2\times5)(3\times2)$ and is undefined. **Q4.** D — Tuesday's row $\begin{bmatrix}15 & 10\end{bmatrix}$ times the price column gives $15(5)+10(3)=75+30=\$105$. **Q5.** A — $A^2=\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}$, so $A^3=A^2A=\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 6\\0 & 1\end{bmatrix}$ (or read A^3 straight from CAS). **Q6.** B — there is no inverse when the determinant is zero: $k(k-1)-3(2)=k^2-k-6=(k-3)(k+2)=0$, so $k=3$ or $k=-2$. **Q7.** C — $det(A)=2(2)-1(3)=1$, so $A^{-1}=\frac{1}{1}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}=\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$. **Q8.** B — row 1 of P has its 1 in column 4, so the new first batter is the old fourth, H; rows 2, 3 and 4 have their 1s in columns 2, 1 and 3, giving F, E, G. The new order is H, F, E, G (option C reorders by the columns of P, that is by P^T). **Q9.** A — the row totals of $D+D^2$ are $W=5$, $X=4$, $Z=3$, $Y=2$ (one-step alone ties W and X on 2). **Q10.** D — the entry 0.15 in column T, row C, is the proportion of town residents who move to the city each year. **Q11.** C — T has no zero entries, so it is regular and the state matrices settle to the same distribution whatever the starting split. Iterating $S_{n+1}=TS_n$ on CAS from any S_0 totalling 500 gives no noticeable change once $S_n=\begin{bmatrix}100\\400\end{bmatrix}$, so the first state approaches 100. **Q12.** B — $S_1=LS_0=\begin{bmatrix}3(30)+1(20)\\0.5(40)\\0.4(30)\end{bmatrix}=\begin{bmatrix}110\\20\\12\end{bmatrix}$; total $=110+20+12=142$.

Topic 3 Test B — Short-answer

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 5 minutes. Writing time: 40 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

Two market stalls, North and South, sell apples, bananas and cherries. Last week's sales, in kilograms, are recorded in the matrix

$$S = \begin{bmatrix} 30 & 45 & 20 \\ 25 & 50 & 15 \end{bmatrix},$$

where the rows are the North and South stalls and the columns are apples, bananas and cherries.

a. State the order of S and the value of the element s_{23}, and explain what s_{23} represents. 2 marks

b. This week each stall's sales, in kilograms, are 20% higher than last week. Write this week's sales as a scalar multiple of S and evaluate it. 2 marks

c. Apples sell for $\$4$/kg, bananas for $\$3$/kg and cherries for $\$8$/kg. Write these prices as a suitable matrix P, evaluate the product SP, and interpret each entry of the result. 3 marks

Question 2 (7 marks)

Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

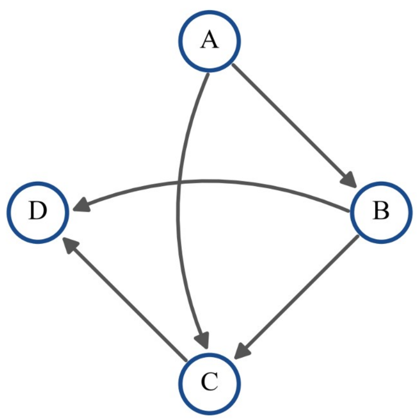
a. Find $det(A)$ and hence write down A^{-1}. 3 marks

b. Hence solve the matrix equation $AX = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ for the column matrix X. 2 marks

c. Hence find the row matrix Y for which $YA = \begin{bmatrix} 1 & 1 \end{bmatrix}$. 2 marks

Question 3 (6 marks)

The directed network below shows which members of a four-person study group can send a text message directly to which other members. An arrow drawn from one person to another means the first person can text the second directly.



an arrow points from the sender to the receiver

a. Write down the communication matrix C for this network, using the order A, B, C, D for both the rows and the columns. 2 marks

b. Calculate C^2. 2 marks

c. State what the entry in row A, column D of C^2 represents and give its value. Which member can only receive messages, and never send one directly? 2 marks

Question 4 (8 marks)

Each working day, 900 commuters travel to the city centre either by train (T) or by car (C). Their day-to-day choice is modelled by the transition matrix

$$M=\begin{bmatrix}0.9 & 0.2\\ 0.1 & 0.8\end{bmatrix},$$

where the rows and columns are in the order T, C. Today 500 travel by train and 400 by car, so $S_0=\begin{bmatrix}500\\ 400\end{bmatrix}$.

a. Interpret the entry 0.2 in the matrix M in the context of this problem. 1 mark

b. Find the state matrices S_1 and S_2, and hence the number travelling by train after two days. 3 marks

c. Write down a matrix recurrence relation, of the form $S_{n+1}=M\,S_n$, that generates the sequence of state matrices, and state S_0. 1 mark

d. Find the steady-state distribution of commuters and hence the number travelling by train in the long term. 3 marks

Question 5 (6 marks)

A lake is managed as two zones, the Shallows (S) and the Deep (D). Each month the fish numbers, in hundreds, are modelled by $S_{n+1}=T\,S_n+B$, where

$$T=\begin{bmatrix}0.7 & 0.2\\ 0.3 & 0.8\end{bmatrix}, B=\begin{bmatrix}100\\ -50\end{bmatrix},$$

the rows and columns are in the order S, D, and the current numbers are $S_0=\begin{bmatrix}400\\ 500\end{bmatrix}$.

a. Explain what the matrix B represents in the context of this problem. 2 marks

b. Find the state matrix S_1. 2 marks

c. Find the state matrix S_2. 2 marks

Question 6 (6 marks)

An insect population is divided into three one-month age classes. Its Leslie matrix and this month's state matrix are

$$L=\begin{bmatrix}0 & 4 & 3\\ 0.4 & 0 & 0\\ 0 & 0.25 & 0\end{bmatrix}, S_0=\begin{bmatrix}200\\ 100\\ 40\end{bmatrix}.$$

a. Interpret the entries 4 (row 1, column 2) and 0.4 (row 2, column 1) of L in the context of this problem. 2 marks

b. Find the state matrix after one month and the total population at that time. 2 marks

c. Find the number of insects in the youngest age class after two months. 2 marks

End of Topic 3 Test B

Test B — answers and solutions

Question 1.

a. S has 2 rows and 3 columns, so its order is 2×3. The element s_{23} is in row 2, column 3, so $s_{23} = 15$: the South stall sold 15 kg of cherries last week.

b. A $20\,\%$ increase multiplies every entry by 1.2, so this week's sales are

$$1.2\,S = 1.2\begin{bmatrix} 30 & 45 & 20 \\ 25 & 50 & 15 \end{bmatrix} = \begin{bmatrix} 36 & 54 & 24 \\ 30 & 60 & 18 \end{bmatrix}\text{kg}.$$

c. The prices must form a 3×1 column so that $S\,P$ is defined:

$$P = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}, S\,P = \begin{bmatrix} 30 & 45 & 20 \\ 25 & 50 & 15 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 30(4)+45(3)+20(8) \\ 25(4)+50(3)+15(8) \end{bmatrix} = \begin{bmatrix} 415 \\ 370 \end{bmatrix}.$$

The North stall took $\$415$ and the South stall took $\$370$ from these sales last week.

Question 2.

a. $det(A) = 3(2) - 5(1) = 1$. Since the determinant is 1,

$$A^{-1} = \frac{1}{det(A)}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}.$$

b. Multiplying both sides of $A\,X = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ on the left by A^{-1} gives

$$X = A^{-1}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8-15 \\ -4+9 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$

c. Here A sits on the **right** of the unknown, so post-multiply both sides by A^{-1}: $Y\,A\,A^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} A^{-1}$ gives

$$Y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & -5+3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \end{bmatrix}.$$

(Check: $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3-2 & 5-4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ ✓.)

Question 3.

a. Reading each arrow as a 1 from the sender's row to the receiver's column ($A \to B$, $A \to C$, $B \to C$, $B \to D$, $C \to D$):

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b.

$$C^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

c. The entry in row A, column D of C^2 is 2: it is the number of two-step (one-relay) communication routes from A to D, namely $A \to B \to D$ and $A \to C \to D$. Member D has a row of zeros in C, so D can only receive messages and never sends one directly.

Question 4.

a. The entry 0.2 is in row T, column C, so it is the proportion of car users who switch to the train the next day: each day, $20\,\%$ of those who travel by car change to the train.

b.

$$S_1 = M\,S_0 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 500 \\ 400 \end{bmatrix} = \begin{bmatrix} 450+80 \\ 50+320 \end{bmatrix} = \begin{bmatrix} 530 \\ 370 \end{bmatrix},$$

$$S_2 = M\,S_1 = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 530 \\ 370 \end{bmatrix} = \begin{bmatrix} 477+74 \\ 53+296 \end{bmatrix} = \begin{bmatrix} 551 \\ 349 \end{bmatrix}.$$

After two days, 551 commuters travel by train.

c. $S_{n+1} = M\,S_n = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} S_n$, with $S_0 = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$.

d. M has no zero entries, so it is a regular transition matrix and the state matrices settle to a fixed distribution. Continuing the recurrence with technology, using $S_n = M^n\,S_0$,

$$S_{10} \approx \begin{bmatrix} 597.2 \\ 302.8 \end{bmatrix}, S_{20} \approx \begin{bmatrix} 599.9 \\ 300.1 \end{bmatrix}, S_{30} \approx \begin{bmatrix} 600.0 \\ 300.0 \end{bmatrix},$$

and from there on there is no noticeable change from one state matrix to the next. The steady state is therefore $\begin{bmatrix} 600 \\ 300 \end{bmatrix}$, which checks against $M \begin{bmatrix} 600 \\ 300 \end{bmatrix} = \begin{bmatrix} 540+60 \\ 60+240 \end{bmatrix} = \begin{bmatrix} 600 \\ 300 \end{bmatrix}$. In the long term 600 commuters travel by train.

Question 5.

a. The matrix $B = \begin{bmatrix} 100 \\ -50 \end{bmatrix}$ is added each month after the transition. It represents restocking and harvesting: 100 (hundred) fish are added to the Shallows and 50 (hundred) fish are removed from the Deep every month.

b.

$$S_1 = T\,S_0 + B = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 400 \\ 500 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 280+100 \\ 120+400 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 380 \\ 520 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 480 \\ 470 \end{bmatrix}.$$

c.

$$S_2 = T\,S_1 + B = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 480 \\ 470 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 336+94 \\ 144+376 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 430 \\ 520 \end{bmatrix} + \begin{bmatrix} 100 \\ -50 \end{bmatrix} = \begin{bmatrix} 530 \\ 470 \end{bmatrix}.$$

Question 6.

a. The entry 4 (row 1, column 2) is a birth rate: on average each class-2 insect produces 4 offspring that survive into the youngest class next month. The entry 0.4 (row 2, column 1) is a survival proportion: $40\,\%$ of the class-1 insects survive to become class-2 insects.

b.

$$S_1 = L\,S_0 = \begin{bmatrix} 0 & 4 & 3 \\ 0.4 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \\ 40 \end{bmatrix} = \begin{bmatrix} 4(100)+3(40) \\ 0.4(200) \\ 0.25(100) \end{bmatrix} = \begin{bmatrix} 520 \\ 80 \\ 25 \end{bmatrix}.$$

The total population after one month is $520+80+25=625$.

c. Apply L again: $S_2 = L\,S_1$. The youngest class (row 1) is $4(80)+3(25)=320+75=395$. There are 395 insects in the youngest age class after two months. (For reference, $S_2 = \begin{bmatrix} 395 \\ 208 \\ 20 \end{bmatrix}$.)