General Mathematics Units 3 & 4

Chapter 8 — Matrices

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Significant effort has been made to ensure this content is legal. Please send any areas of concern to admin@vwx.au with appropriate document/legal references so errors can be verified and changes be made.

Contents

Theory

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns** and enclosed in square brackets. The numbers in the array are called the **elements** (or entries) of the matrix. Matrices are used to store and organise information — a timetable, a price list, the results of a tournament — in a single, compact object that can later be operated on. The plural of matrix is **matrices**.

A matrix is named with a capital letter. For example, the table of goals scored by two teams over three rounds

	Round 1	Round 2	Round 3
Team A	2	1	4
Team B	3	0	5

is written as the matrix

$$A=\begin{bmatrix}2 & 1 & 4\\ 3 & 0 & 5\end{bmatrix}.$$

Reading across gives a row and reading down gives a column, exactly as in the table.

Order of a matrix. The **order** (or size) of a matrix states how many rows and how many columns it has, written

$$\text{order}=m\times n=(\text{number of rows})\times(\text{number of columns}).$$

The number of **rows** is always given first and the number of **columns** second — read it as “m by n”. The matrix A above has 2 rows and 3 columns, so its order is 2×3. A matrix of order $m\times n$ contains $m\times n$ elements in total.

Elements and their positions. A particular element is referred to by its **position** — the row and the column it lies in. The element in row i and column j of a matrix A is written

$$a_{ij},$$

using the lower-case letter that matches the name of the matrix, with the **row number first** and the **column number second**. For the matrix A above, $a_{13}=4$ (row 1, column 3) and $a_{21}=3$ (row 2, column 1). The two subscripts are read separately: a_{21} is “a-two-one”, not “a twenty-one”.

Constructing a matrix from a rule. Because each element has a known position, a matrix can be built from a **rule** for a_{ij}. For example, the 2×2 matrix with $a_{ij}=i+j$ has $a_{11}=2$, $a_{12}=3$, $a_{21}=3$ and $a_{22}=4$, giving $\begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$.

Special matrices. Some shapes and patterns occur so often that they are given names.

- A **row matrix** has a single row, so its order is $1\times n$, for example $\begin{bmatrix}4 & -1 & 7\end{bmatrix}$.
- A **column matrix** has a single column, so its order is $m\times 1$, for example $\begin{bmatrix}2\\ 5\\ 0\end{bmatrix}$.
- A **square matrix** has the same number of rows as columns, so its order is $n\times n$; we say it is a square matrix of **order** n. The elements $a_{11}, a_{22}, a_{33}, \ldots$ that run from the top-left corner to the bottom-right corner form the **leading diagonal**.
- A **zero matrix** (or null matrix), written O, has every element equal to 0; it may be any order, for example $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$.

- A **diagonal matrix** is a square matrix in which every element **off** the leading diagonal is 0, for example $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. The diagonal elements themselves may take any value.
- An **upper triangular matrix** is a square matrix in which every element **below** the leading diagonal is 0, for example $\begin{bmatrix} 4 & 7 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$. The non-zero entries lie on or above the diagonal, forming a triangle in the upper-right.
- A **lower triangular matrix** is a square matrix in which every element **above** the leading diagonal is 0, for example $\begin{bmatrix} 6 & 0 & 0 \\ 2 & 8 & 0 \\ 9 & 4 & 5 \end{bmatrix}$. A **diagonal** matrix has zeros both above and below the leading diagonal, so it is at the same time upper triangular **and** lower triangular.
- The **identity matrix**, written I (or I_n for order n), is the diagonal matrix whose leading-diagonal elements are all 1 and whose other elements are all 0, for example $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

A matrix can carry more than one label: the identity matrix is also square, diagonal and both upper and lower triangular, and a square zero matrix is also diagonal (and triangular either way).

The transpose of a matrix. The **transpose** of a matrix A, written A^T (some texts write A^T), is obtained by **interchanging its rows and columns**: the first row of A becomes the first column of A^T, the second row becomes the second column, and so on. If A has order $m \times n$ then A^T has order $n \times m$, and the element in row i, column j of the transpose is the element in row j, column i of the original,

$$\left(A^T\right)_{ij} = a_{ji}.$$

For example, if $A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 0 & 3 \end{bmatrix}$ (order 2×3) then

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 0 \\ 1 & 3 \end{bmatrix} (\text{order } 3 \times 2).$$

Transposing twice returns the original matrix, so $\left(A^T\right)^T = A$. The transpose of a **row** matrix is a **column** matrix (and vice versa): for instance $\begin{bmatrix} 4 & -1 & 7 \end{bmatrix}^T = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$. A square matrix that equals its own transpose, so that $A^T = A$, is called a **symmetric** matrix; its entries are mirror images of one another in the leading diagonal.

Equality of matrices. Two matrices are **equal** only when they have **the same order** *and* **every pair of corresponding elements is equal**. If $A = B$ then $a_{ij} = b_{ij}$ for every position i, j. Matrices of different orders can never be equal, even if they contain the same numbers — the row matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and the column matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are not equal. When two matrices of the same order are stated to be equal, equating corresponding elements produces one equation per position, which can be solved for any unknown elements.

Using CAS. General Mathematics is a technology-active course. A matrix is entered into the CAS matrix editor by first choosing its order and then typing the elements row by row; the calculator stores it under a name such as A and can display any element or recognise standard matrices such as the identity. Learn the language of order, position and type below so that you can set up a matrix correctly and read the calculator's output with confidence.

Worked examples

Example 1 — scaffolded

A shop records the number of small, medium and large drinks it sells on each of two days in the matrix

$$A=\begin{bmatrix}3 & 5 & 2\\ 6 & 1 & 4\end{bmatrix},$$

where the rows are the two days and the columns are the three sizes. State the order of A and write down the elements a_{13}, a_{21} and a_{22}.

Working	Comment
A has 2 rows and 3 columns.	Count the rows first, then the columns.
Order $=2\times 3$.	Written as (rows) × (columns), read "2 by 3".
$a_{13}=2$.	Row 1, column 3 — the first subscript is the row.
$a_{21}=6$.	Row 2, column 1.
$a_{22}=1$.	Row 2, column 2.

Example 2 — scaffolded

For each matrix below, state its order and name its type (row, column, square, zero, diagonal or identity).

$$P=\begin{bmatrix}5 & 2 & -3\end{bmatrix}, Q=\begin{bmatrix}7\\ 1\\ 4\end{bmatrix}, R=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, S=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, D=\begin{bmatrix}6 & 0\\ 0 & -4\end{bmatrix}.$$

Working	Comment
P: 1 row, 3 columns ⇒ order 1×3, a **row** matrix.	A single row of elements.
Q: 3 rows, 1 column ⇒ order 3×1, a **column** matrix.	A single column of elements.
R: order 2×2; diagonal elements 1, others 0 ⇒ **identity** matrix.	Also square and diagonal.
S: order 2×2; every element 0 ⇒ **zero** matrix.	A square zero matrix is also diagonal.
D: order 2×2; off-diagonal elements 0 ⇒ **diagonal** matrix.	The diagonal entries 6 and -4 may be any values.

Example 3 — unscaffolded

Find the values of x and y given that

$$\begin{bmatrix}2x & 5\\ 7 & y+1\end{bmatrix}=\begin{bmatrix}8 & 5\\ 7 & -2\end{bmatrix}.$$

Both matrices have order 2×2, so they can be equal. Equal matrices have equal corresponding elements, so we equate the entries in matching positions.

The top-left entries give $2x=8$, so $x=4$. The bottom-right entries give $y+1=-2$, so $y=-3$. (The remaining positions, $5=5$ and $7=7$, are already consistent.)

$\therefore x=4$ and $y=-3$.

Example 4 — unscaffolded

Construct the 2×3 matrix A whose element in row i, column j is given by the rule $a_{ij}=2i+j$.

The matrix has 2 rows and 3 columns, so i runs through $1, 2$ and j runs through $1, 2, 3$. Substituting each position into $a_{ij} = 2i + j$:

$$a_{11} = 3, a_{12} = 4, a_{13} = 5, a_{21} = 5, a_{22} = 6, a_{23} = 7.$$

Placing each value in its position gives

$$\therefore A = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}.$$

Example 5 — scaffolded

For each matrix below, state whether it is upper triangular, lower triangular, both (that is, diagonal) or neither.

$$U = \begin{bmatrix} 2 & 6 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix}, L = \begin{bmatrix} 5 & 0 \\ 8 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}, N = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Working	Comment
U: every entry below the leading diagonal is $0 \Rightarrow$ **upper triangular**.	The zeros sit in the lower-left corner.
L: every entry above the leading diagonal is $0 \Rightarrow$ **lower triangular**.	The single zero is in the upper-right.
D: entries both above and below the diagonal are $0 \Rightarrow$ **both** (diagonal).	A diagonal matrix is upper *and* lower triangular.
N: the entry 2 (above) and 3 (below) are non-zero $\Rightarrow$ **neither**.	Neither triangle is entirely zero.

Example 6 — scaffolded

Write the transpose of $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ and state the order of both A and A^T.

Working	Comment
A has 3 rows and 2 columns, so its order is 3×2.	Rows first, then columns.
Row 1 $\begin{bmatrix} 1 & 4 \end{bmatrix}$ becomes column 1 of A^T.	Each row of A becomes a column of A^T.
Row 2 $\rightarrow$ column 2, row 3 $\rightarrow$ column 3.	Working through every row in turn.
$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.	Rows and columns interchanged.
A^T has order 2×3.	The order reverses: $(m \times n) \rightarrow (n \times m)$.

Practice questions

Scaffolded

Q1. State the order of each matrix.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 0 \end{bmatrix}, B = \begin{bmatrix} 7 & -2 & 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

Q2. For the matrix $M=\begin{bmatrix}8 & 3 & 5\\ 2 & 7 & 4\end{bmatrix}$: (a) state the order; (b) write down m_{12}; (c) write down m_{23}; (d) the element 2 lies in which position m_{ij}?

Q3. Classify each matrix as a row, column, square, zero, diagonal or identity matrix (more than one label may apply).

$$P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, Q=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}, R=\begin{bmatrix}5 & 0\\ 0 & 9\end{bmatrix}, S=\begin{bmatrix}4 & 1 & 6\end{bmatrix}.$$

Q4. The table shows the number of adult and child tickets sold at three sessions of a film.

	Adult	Child
Session 1	20	12
Session 2	18	15
Session 3	25	10

(a) Write the information as a matrix T with the sessions as rows.
(b) State the order of T.
(c) State the value of the element in row 2, column 1 and explain what it represents.

Q5. Given that $\begin{bmatrix}x & 6\\ -1 & y\end{bmatrix}=\begin{bmatrix}4 & 6\\ -1 & 9\end{bmatrix}$: (a) explain why the two matrices are able to be equal; (b) find x; (c) find y.

Q6. Construct the 2×2 matrix A with $a_{ij}=i+2j$ by completing the steps. (a) Find a_{11}. (b) Find a_{12}. (c) Find a_{21}. (d) Find a_{22}. (e) Write out the matrix A.

Q7. Classify each matrix as upper triangular, lower triangular, both (diagonal) or neither.

$$A=\begin{bmatrix}3 & 5\\ 0 & 2\end{bmatrix}, B=\begin{bmatrix}4 & 0 & 0\\ 7 & 1 & 0\\ 2 & 6 & 9\end{bmatrix}, C=\begin{bmatrix}8 & 0\\ 0 & 5\end{bmatrix}, E=\begin{bmatrix}5 & 0 & 2\\ 0 & 3 & 0\\ 4 & 0 & 1\end{bmatrix}.$$

Q8. For the matrix $A=\begin{bmatrix}2 & 7 & 4\\ 1 & 3 & 6\end{bmatrix}$: (a) state the order of A; (b) write down the transpose A^T; (c) state the order of A^T.

Unscaffolded

Q9. State the order of each matrix.

$$\begin{bmatrix}3 & 1\\ 0 & 2\\ 5 & 4\\ 6 & 7\end{bmatrix}, \begin{bmatrix}9 & 8 & 7\end{bmatrix}, \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}.$$

Q10. For $A=\begin{bmatrix}5 & 2 & 9\\ 4 & 7 & 1\\ 8 & 3 & 6\end{bmatrix}$, write down a_{11}, a_{23} and a_{32}, and state the position a_{ij} of the element equal to 9.

Q11. Classify each matrix, giving every label that applies.

$$E=\begin{bmatrix}7 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{bmatrix}, F=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}, G=\begin{bmatrix}2 & 5\\ 7 & 1\end{bmatrix}.$$

Q12. Write out each of the following matrices in full. (a) The 3×3 identity matrix. (b) A 2×4 zero matrix. (c) The 3×3 diagonal matrix whose leading-diagonal elements are 2, 5 and 8.

Q13. A canteen records the number of pies and rolls it sells on three days in the table below.

	Pies	Rolls
Monday	14	9
Tuesday	20	11
Wednesday	16	7

(a) Write the information as a matrix C with the days as rows.

(b) State the order of C.

(c) State the value and meaning of the element c_{31}.

Q14. Find a and b given that $\begin{bmatrix} a & 3 \\ 5 & b \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 5 & 8 \end{bmatrix}$.

Q15. Find x and y given that $\begin{bmatrix} 3x & 7 \\ y-4 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 1 & 2 \end{bmatrix}$.

Q16. Construct the 3×3 matrix A with $a_{ij} = i - j$.

Q17. A matrix A has order 4×5. (a) How many rows does it have? (b) How many columns does it have? (c) How many elements does it have in total? (d) In which row and column is the element a_{34}?

Q18. Explain in one sentence why $\begin{bmatrix} 3 & 8 & 5 \\ 6 & 2 & 9 \end{bmatrix}$ and $\begin{bmatrix} 3 & 8 \\ 5 & 6 \\ 2 & 9 \end{bmatrix}$ are not equal, even though they contain the same numbers.

Q19. A square matrix A has order $n \times n$. (a) How many elements lie on its leading diagonal? (b) For the identity matrix of order n, state the value of every leading-diagonal element and of every other element.

Q20. Find p, q and r given that $\begin{bmatrix} p & 0 & 6 \\ q & 5 & r \end{bmatrix} = \begin{bmatrix} 9 & 0 & 6 \\ -3 & 5 & 8 \end{bmatrix}$.

Q21. The diagonal matrix $D = \begin{bmatrix} k & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & m \end{bmatrix}$ is equal to $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. (a) State the values of k and m. (b) Is D the identity matrix? Give a reason.

Q22. The distances (in km) between three towns A, B and C are: A to B, 40; A to C, 25; B to C, 30. A matrix M is formed so that m_{ij} is the distance between town i and town j, with the rows and columns in the order A, B, C. (a) Write out the matrix M. (b) State its order and its type. (c) Explain why $m_{ij} = m_{ji}$ for every i and j. (d) State the value and meaning of each leading-diagonal element.

Q23. Classify each matrix, giving every label that applies (upper triangular, lower triangular or diagonal).

$$P = \begin{bmatrix} 6 & 0 & 0 \\ 3 & 9 & 0 \\ 1 & 4 & 2 \end{bmatrix}, Q = \begin{bmatrix} 5 & 2 \\ 0 & 8 \end{bmatrix}, R = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

Q24. Write out the 3×3 upper triangular matrix whose leading-diagonal elements are 2, 4 and 6 and whose entries above the leading diagonal are all 1.

Q25. Consider $A=\begin{bmatrix} 4 & -2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$ and the row matrix $R=\begin{bmatrix} 6 & 9 & 2 \end{bmatrix}$. (a) Write down A^T and state its order. (b) Write down R^T and state its order and type. (c) Explain why $S=\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ is a symmetric matrix.

Answers

Q1. 3×2; 1×4; 2×1. **Q2.** (a) 2×3; (b) $m_{12}=3$; (c) $m_{23}=4$; (d) m_{21}. **Q3.** P identity (also square, diagonal); Q column (also zero); R diagonal (also square); S row. **Q4.** (a) $T=\begin{bmatrix}20 & 12\\ 18 & 15\\ 25 & 10\end{bmatrix}$; (b) 3×2; (c) 18 — the adult tickets sold at session 2. **Q5.** (a) Both are 2×2, so equality is possible; (b) $x=4$; (c) $y=9$. **Q6.** (a) 3; (b) 5; (c) 4; (d) 6; (e) $A=\begin{bmatrix}3 & 5\\ 4 & 6\end{bmatrix}$. **Q7.** A upper triangular; B lower triangular; C both (diagonal); E neither. **Q8.** (a) 2×3; (b) $A^T=\begin{bmatrix}2 & 1\\ 7 & 3\\ 4 & 6\end{bmatrix}$; (c) 3×2. **Q9.** 4×2; 1×3; 2×2. **Q10.** $a_{11}=5$, $a_{23}=1$, $a_{32}=3$; the element 9 is a_{13}. **Q11.** E diagonal (also square); F row (also zero); G square only — no other label applies. **Q12.** (a) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$; (b) $\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$; (c) $\begin{bmatrix}2 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 8\end{bmatrix}$. **Q13.** (a) $C=\begin{bmatrix}14 & 9\\ 20 & 11\\ 16 & 7\end{bmatrix}$; (b) 3×2; (c) $c_{31}=16$ — the pies sold on Wednesday. **Q14.** $a=-2$, $b=8$. **Q15.** $x=4$, $y=5$. **Q16.** $A=\begin{bmatrix}0 & -1 & -2\\ 1 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$. **Q17.** (a) 4; (b) 5; (c) 20; (d) row 3, column 4. **Q18.** They have different orders (2×3 and 3×2), and equal matrices must have the same order. **Q19.** (a) n; (b) every leading-diagonal element is 1 and every other element is 0. **Q20.** $p=9$, $q=-3$, $r=8$. **Q21.** (a) $k=5$, $m=5$; (b) no — its leading-diagonal elements are $5,3,5$, not all 1. **Q22.** (a) $M=\begin{bmatrix}0 & 40 & 25\\ 40 & 0 & 30\\ 25 & 30 & 0\end{bmatrix}$; (b) 3×3, square; (c) the distance from town i to town j equals the distance from j to i; (d) each is 0 — the distance from a town to itself. **Q23.** P lower triangular; Q upper triangular; R diagonal (also both upper and lower triangular). **Q24.** $\begin{bmatrix}2 & 1 & 1\\ 0 & 4 & 1\\ 0 & 0 & 6\end{bmatrix}$. **Q25.** (a) $A^T=\begin{bmatrix}4 & 1\\ -2 & 0\\ 5 & 3\end{bmatrix}$, order 3×2; (b) $R^T=\begin{bmatrix}6\\ 9\\ 2\end{bmatrix}$, order 3×1, a column matrix; (c) $S^T=S$, so S is symmetric.

Fully-worked solutions

Q1. Count rows then columns. A has 3 rows and 2 columns, so its order is 3×2; B has 1 row and 4 columns, so 1×4; C has 2 rows and 1 column, so 2×1.

Q2. (a) M has 2 rows and 3 columns, so the order is 2×3. (b) The elements of M are written with the matching lower-case letter, so m_{12} is the entry in row 1, column 2, which is 3. (c) m_{23} is the entry in row 2, column 3, which is 4. (d) The entry 2 is in row 2, column 1, so its position is m_{21}.

Q3. P is a 3×3 square matrix with 1s on the leading diagonal and 0s elsewhere, so it is the **identity** matrix (and hence also square and diagonal). Q is a single column, so it is a **column** matrix, and as every entry is 0 it is also a **zero** matrix. R is a 2×2 square matrix with 0s off the leading diagonal, so it is a **diagonal** matrix (and square). S is a single row, so it is a **row** matrix.

Q4. (a) Reading each session across into a row gives $T=\begin{bmatrix}20 & 12\\18 & 15\\25 & 10\end{bmatrix}$. (b) There are 3 rows and 2 columns, so the order is 3×2. (c) Row 2, column 1 is 18, the number of **adult** tickets sold at **session 2**.

Q5. (a) Both matrices are 2×2, so they have the same order and equality is possible. (b) Equating corresponding entries, the top-left gives $x=4$. (c) The bottom-right gives $y=9$. The remaining entries, $6=6$ and $-1=-1$, are already equal.

Q6. Substitute each position into $a_{ij}=i+2j$. (a) $a_{11}=1+2(1)=3$. (b) $a_{12}=1+2(2)=5$. (c) $a_{21}=2+2(1)=4$. (d) $a_{22}=2+2(2)=6$. (e) Placing each value in its position, $A=\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix}$.

Q7. Check each triangle against the leading diagonal. In A the only entry below the diagonal is 0, so A is **upper triangular**. In B the entries above the diagonal are all 0, so B is **lower triangular**. In C the entries both above and below the diagonal are 0, so C is **both** — that is, a diagonal matrix. In E the entry 2 in row 1, column 3 lies above the leading diagonal and the entry 4 in row 3, column 1 lies below it, and both are non-zero, so neither triangle is entirely 0 and E is **neither** — the other zeros do not make it diagonal.

Q8. (a) A has 2 rows and 3 columns, so its order is 2×3. (b) The transpose is formed by turning each row of A into a column: row 1 $\begin{bmatrix}2 & 7 & 4\end{bmatrix}$ becomes the first column and row 2 $\begin{bmatrix}1 & 3 & 6\end{bmatrix}$ the second, giving $A^T=\begin{bmatrix}2 & 1\\7 & 3\\4 & 6\end{bmatrix}$. (c) The order reverses, so A^T is 3×2.

Q9. The first matrix has 4 rows and 2 columns, so 4×2; the second has 1 row and 3 columns, so 1×3; the third has 2 rows and 2 columns, so 2×2.

Q10. a_{11} (row 1, column 1) is 5; a_{23} (row 2, column 3) is 1; a_{32} (row 3, column 2) is 3. The entry 9 sits in row 1, column 3, so its position is a_{13}.

Q11. E is a 3×3 square matrix whose off-diagonal entries are all 0, so it is a **diagonal** matrix (and square); its diagonal entries are 7, not 1, so it is not the identity. F is a single row of entries, so it is a **row** matrix, and as every entry is 0 it is also a **zero** matrix. G has 2 rows and 2 columns, so it is a **square** matrix; its off-diagonal entries 5 and 7 are not 0, so it is not diagonal and therefore not the identity, and having more than one row and more than one column it is neither a row nor a column matrix. **Square** is the only label that applies — being square does not by itself make a matrix special.

Q12. (a) The identity matrix of order 3 has 1s on the leading diagonal and 0s elsewhere: $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. (b) A 2×4 zero matrix has 2 rows and 4 columns, all zero: $\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$. (c) A diagonal matrix keeps the given values on the leading diagonal and puts 0 elsewhere: $\begin{bmatrix}2 & 0 & 0\\0 & 5 & 0\\0 & 0 & 8\end{bmatrix}$.

Q13. (a) Reading each day across into a row gives $C=\begin{bmatrix}14 & 9\\20 & 11\\16 & 7\end{bmatrix}$. (b) There are 3 rows and 2 columns, so the order is 3×2. (c) c_{31} is the entry in row 3, column 1, which is 16 — the number of **pies** sold on **Wednesday**.

Q14. Both matrices are 2×2, so equate corresponding entries. The top-left gives $a=-2$ and the bottom-right gives $b=8$; the entries 3 and 5 already match.

Q15. Equate corresponding entries. The top-left gives $3x = 12$, so $x = 4$. The bottom-left gives $y - 4 = 1$, so $y = 5$. The entries 7 and 2 already match.

Q16. Substitute each position into $a_{ij} = i - j$. Row 1 ($i = 1$): $0, -1, -2$; row 2 ($i = 2$): $1, 0, -1$; row 3 ($i = 3$): $2, 1, 0$. Hence $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$.

Q17. For order 4×5: (a) the first number is the rows, so 4 rows; (b) the second is the columns, so 5 columns; (c) the total number of elements is $4 \times 5 = 20$; (d) a_{34} is in row 3, column 4.

Q18. The first has 2 rows and 3 columns, so its order is 2×3, while the second has 3 rows and 2 columns, so its order is 3×2. Equal matrices must have the same order, so the two cannot be equal despite containing the same six numbers.

Q19. (a) The leading diagonal runs from a_{11} to a_{nn}, one element in each of the n rows, so there are n elements. (b) By definition the identity matrix has every leading-diagonal element equal to 1 and every other element equal to 0.

Q20. Both matrices are 2×3, so equate corresponding entries: the top-left gives $p = 9$, the bottom-left gives $q = -3$, and the bottom-right gives $r = 8$. The entries 0, 6 and 5 already match.

Q21. (a) Equating corresponding entries, the top-left gives $k = 5$ and the bottom-right gives $m = 5$; the remaining entries already match. (b) No. Although D is a diagonal matrix, its leading-diagonal elements are $5, 3, 5$, and the identity matrix requires every leading-diagonal element to equal 1.

Q22. (a) Entry m_{ij} is the distance between town i and town j. The diagonal entries (a town to itself) are 0, and the given distances fill the off-diagonal positions: $M = \begin{bmatrix} 0 & 40 & 25 \\ 40 & 0 & 30 \\ 25 & 30 & 0 \end{bmatrix}$. (b) It has 3 rows and 3 columns, so the order is 3×3 and it is a square matrix. (c) The distance from town i to town j is the same as the distance from town j to town i, so $m_{ij} = m_{ji}$. (d) Each leading-diagonal element is 0, representing the distance from a town to itself.

Q23. P is square with every entry **above** the leading diagonal equal to 0, so it is **lower triangular**. Q is square with the one entry **below** the diagonal equal to 0, so it is **upper triangular**. R has 0s both above and below the diagonal, so it is **diagonal**, and being diagonal it is at the same time both upper and lower triangular.

Q24. Place the given values 2, 4, 6 on the leading diagonal, put 1 in each of the three positions above the diagonal, and 0 in every position below it: $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$.

Q25. (a) Turning each row of A into a column gives $A^T = \begin{bmatrix} 4 & 1 \\ -2 & 0 \\ 5 & 3 \end{bmatrix}$; A is 2×3, so its transpose is 3×2. (b) The row matrix R is 1×3, so its transpose is the column matrix $R^T = \begin{bmatrix} 6 \\ 9 \\ 2 \end{bmatrix}$ of order 3×1. (c) Interchanging the rows and columns of S gives $S^T = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$, which is the same as S; because $S^T = S$ (its off-diagonal entries s_{12} and s_{21} are both 3), S is a symmetric matrix.

Theory

A **matrix** is a rectangular array of numbers arranged in rows and columns and enclosed in square brackets. Matrices give a compact, ordered way to store the numerical information that would otherwise be set out in a table, so that it can be read, compared and — later in this chapter — combined using matrix arithmetic.

Order and elements. The **order** (or size) of a matrix is written $\text{rows} \times \text{columns}$. A matrix with m rows and n columns has order $m \times n$ and mn entries. Each entry is an **element**. The element in row i and column j is written a_{ij}; the first subscript is always the row, the second the column. For

$$A = \begin{bmatrix} 12 & 18 & 7 \\ 9 & 14 & 5 \end{bmatrix},$$

the order is 2×3, and $a_{13} = 7$ (row 1, column 3) while $a_{21} = 9$ (row 2, column 1).

Representing information. To store a table of numbers as a matrix you must decide what the **rows** represent and what the **columns** represent — the two dimensions of the data (for example category and location, or item and time period). Once that choice is fixed, every element has a definite meaning: a_{ij} is the value for the ith row category and the jth column category. Reading an element therefore always means naming its row and its column, so it is good practice to label the rows and columns beside the matrix so that the meaning of each element is not lost.

The same table can be written two ways — for instance with locations as rows and products as columns, or the other way round. One is the **transpose** of the other; both hold exactly the same information, and either is acceptable provided the meaning of the rows and columns is stated.

Row and column matrices. A matrix with a single row (order $1 \times n$) is a **row matrix**; a matrix with a single column (order $m \times 1$) is a **column matrix**. These are the natural way to record a simple list such as a **price list** or a set of quantities. For example, a café charging \$4, \$3 and \$5 for its three drinks has the price list

$$P = \begin{bmatrix} 4 & 3 & 5 \end{bmatrix} (1 \times 3),$$

or, equally, the same list written as the column matrix (its transpose)

$$P^T = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} (3 \times 1).$$

Scalar multiplication. Multiplying a matrix by a single number (a **scalar**) multiplies **every** element by that number. This is the simplest matrix operation and is often all that is needed to scale information up or down: to raise every price by 10% multiply the price list by 1.1, to apply a 10% discount multiply by 0.9, and to find the supplies for 5 identical classes multiply one class's list by 5. For the price list above,

$$1.1P = \begin{bmatrix} 4.4 & 3.3 & 5.5 \end{bmatrix},$$

the new prices \$4.40, \$3.30 and \$5.50.

Summing matrices. A row or column of 1s is called a **summing matrix**, because multiplying by it adds up entries. Multiplying a matrix on the right by a **column of 1s** adds along each row, giving the **row totals**; multiplying on the left by a **row of 1s** adds down each column, giving the **column totals**. For

$$S = \begin{bmatrix} 8 & 5 & 3 \\ 6 & 4 & 7 \\ 5 & 9 & 2 \end{bmatrix},$$

the row totals are

$$S\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}16\\17\\16\end{bmatrix},$$

and the column totals are

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}S=\begin{bmatrix}19 & 18 & 12\end{bmatrix}.$$

Two cautions carry over to all matrix multiplication. The orders must be **conformable** — the number of columns of the left matrix must equal the number of rows of the right — so the summing matrix must have exactly as many 1s as there are entries to add. And the order in which the two matrices are written matters: S times a column of 1s is not the same as a row of 1s times S; one gives row totals, the other column totals. The general rules for adding, subtracting and multiplying matrices are developed later in this chapter; here only scalar multiplication and these summing matrices are needed.

Worked examples

Example 1 — scaffolded

A clothing shop records the number of black and white T-shirts sold in each of three sizes.

	Small	Medium	Large
Black	12	18	7
White	9	14	5

Represent this information as a matrix with the colours as rows, state its order, and write down and interpret the element a_{23}.

Working	Comment
Rows = colours (black, white); columns = sizes (small, medium, large).	Decide what each dimension represents before writing any numbers.
$A=\begin{bmatrix}12 & 18 & 7\\9 & 14 & 5\end{bmatrix}$	Read the table straight into the array, keeping the row and column order.
Order $=2\times 3$.	2 rows (colours) and 3 columns (sizes).
$a_{23}=5$.	Row 2 (white), column 3 (large).
Meaning: 5 large white T-shirts were sold.	Always read an element back through its row and column labels.

Example 2 — scaffolded

A café charges \$4 for a coffee, \$3 for a tea and \$5 for a hot chocolate. (a) Write the price list as a row matrix and state its order. (b) All prices are increased by $10\,\%$. Use scalar multiplication to write the new price list. (c) Write the original prices as a column matrix instead.

Working	Comment
(a) $P=\begin{bmatrix}4 & 3 & 5\end{bmatrix}$, order 1×3.	A single row of three prices.
(b) A $10\,\%$ rise multiplies every price by 1.1.	Scalar multiplication scales every element equally.
$1.1\,P=\begin{bmatrix}4.4 & 3.3 & 5.5\end{bmatrix}$.	$1.1\times 4=4.4$, $1.1\times 3=3.3$, $1.1\times 5=5.5$. New prices \$4.40, \$3.30, \$5.50.
(c) $\begin{bmatrix}4\\3\\5\end{bmatrix}$, order 3×1.	The transpose: the same list written as a column.

Example 3 — unscaffolded

A poultry farm has three sheds. The number of dozens of eggs collected from each shed in week 1 and then week 2 was: shed 1 — 18 then 22; shed 2 — 25 then 20; shed 3 — 14 then 19. Represent this in a matrix with the sheds as rows, state the order, interpret the element in row 2, column 1, and determine which shed produced the most eggs in week 2.

Taking the rows as the three sheds and the columns as the two weeks,

$$E=\begin{bmatrix}18 & 22\\ 25 & 20\\ 14 & 19\end{bmatrix},$$

which has order 3×2 (three sheds, two weeks). The element in row 2, column 1 is $e_{21}=25$: shed 2 produced 25 dozen eggs in week 1. Week 2 is the second column, whose entries (read down) are 22, 20 and 19, and the largest of these is 22, in row 1.

∴ shed 1 produced the most eggs in week 2, with 22 dozen.

Example 4 — unscaffolded

A hardware chain stores hammers, saws and drills at its North, East and West branches, as the matrix

$$S=\begin{bmatrix}8 & 5 & 3\\ 6 & 4 & 7\\ 5 & 9 & 2\end{bmatrix}$$

(rows = branches in the order North, East, West; columns = tools in the order hammers, saws, drills). Use summing matrices to find the total number of tools at each branch and the total number of each tool, and hence state which branch is best stocked and which tool is most plentiful.

The total for each branch is a row total, found by multiplying on the right by a column of three 1s:

$$S\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}8+5+3\\ 6+4+7\\ 5+9+2\end{bmatrix}=\begin{bmatrix}16\\ 17\\ 16\end{bmatrix}.$$

The total of each tool is a column total, found by multiplying on the left by a row of three 1s:

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}S=\begin{bmatrix}19 & 18 & 12\end{bmatrix}.$$

The branch totals are 16, 17 and 16, so the East branch (17) holds the most tools; the tool totals are 19 hammers, 18 saws and 12 drills.

∴ the East branch is best stocked, with 17 tools, and hammers are the most plentiful tool, with 19 in total.

Practice questions

Scaffolded

Q1. A fruit stall records the number of pieces of fruit sold at two stalls.

	Apples	Bananas	Oranges
Stall A	8	5	6
Stall B	4	9	7

(a) Write the information as a matrix with the stalls as rows.
(b) State the order of the matrix.
(c) State the value of a_{13} and explain what it represents.

(d) Which stall sold more bananas?

Q2. A bakery charges \$4 for a loaf of bread, \$6 for a pie and \$12 for a cake. (a) Write the price list as a row matrix. (b) State its order. (c) All prices are increased by $50\,\%$. Use scalar multiplication to write the new price list. (d) Write the original prices as a column matrix.

Q3. The number of two products, P and Q, held in stock at three shops X, Y and Z is

$$\begin{bmatrix} 10 & 4 \\ 7 & 8 \\ 5 & 6 \end{bmatrix},$$

with rows for the shops (X, Y, Z) and columns for the products (P, Q). (a) State the order of the matrix. (b) State the value of a_{22} and explain what it represents. (c) Multiply the matrix by a column of two 1s to find the total number of items at each shop. (d) Which shop holds the most items?

Q4. A classroom kit contains 25 pencils, 25 erasers and 10 glue sticks, written as the row matrix $K=\begin{bmatrix} 25 & 25 & 10 \end{bmatrix}$. (a) State the order of K. (b) Use scalar multiplication to find the total supplies needed for 5 identical classes. (c) Use scalar multiplication to find the total supplies needed for 3 identical classes.

Q5. The number of goals scored by two teams in each of three rounds is

$$G=\begin{bmatrix} 3 & 5 & 2 \\ 4 & 1 & 6 \end{bmatrix},$$

with rows for the teams and columns for the rounds. (a) State the order of G. (b) State the value of g_{13} and explain what it represents. (c) Multiply G on the left by a row of two 1s to find the total goals scored in each round. (d) In which round were the most goals scored in total?

Q6. Two suppliers quote the following prices (in dollars) for three building materials.

	Material 1	Material 2	Material 3
Supplier 1	12	8	20
Supplier 2	15	6	18

(a) Write the prices as a matrix with the suppliers as rows, and state its order.

(b) State the value of a_{23} and explain what it represents.

(c) Which supplier is cheaper for material 1?

Unscaffolded

Q7. Three car dealers sold the following numbers of cars in January and then February: dealer 1 — 14 then 18; dealer 2 — 9 then 12; dealer 3 — 20 then 15. Write this as a matrix with the dealers as rows, and state its order.

Q8. The matrix $A=\begin{bmatrix} 6 & 9 & 4 \\ 3 & 7 & 5 \end{bmatrix}$ records the goals scored by two teams (rows) in three games (columns). State the value and meaning of a_{13} and of a_{21}.

Q9. A restaurant's three main courses cost \$10, \$15 and \$20, written as the column matrix $\begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix}$. All prices are increased by $20\,\%$. Use scalar multiplication to write the new prices as a column matrix.

Q10. The stock of a product at three shops over three weeks is

$$\begin{bmatrix} 5 & 2 & 3 \\ 4 & 4 & 4 \\ 6 & 1 & 2 \end{bmatrix}$$

(rows = shops, columns = weeks). Use a column of three 1s to find each shop's three-week total, and state which shop had the greatest total.

Q11. Three stores sold the following units of four products:

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 0 & 2 & 3 \\ 1 & 6 & 2 & 0 \end{bmatrix}$$

(rows = stores, columns = products). Use a row of three 1s to find the total units sold of each product, and state which product sold the most.

Q12. A biscuit recipe uses 200 g of flour, 3 eggs and 150 g of sugar for one batch, written as the row matrix $R = \begin{bmatrix} 200 & 3 & 150 \end{bmatrix}$. Use scalar multiplication to find the amounts needed for 4 batches.

Q13. A gym counted attendances at its morning and evening classes on Saturday and Sunday. On Saturday 40 attended in the morning and 25 in the evening; on Sunday 30 attended in the morning and 20 in the evening. Write this as a 2×2 matrix with the days as rows, state what the rows and columns represent, and interpret the element a_{12}.

Q14. A shop's stock of shirts, pants and jackets in three sizes is

$$\begin{bmatrix} 12 & 15 & 8 \\ 10 & 9 & 6 \\ 5 & 7 & 4 \end{bmatrix}$$

(rows = garments in the order shirts, pants, jackets; columns = sizes small, medium, large). (a) State the value of a_{31} and explain what it represents. (b) Use a column of three 1s to find the total number of shirts. (c) Use a row of three 1s to find the total number of small garments.

Q15. A store's three items are priced at \$50, \$80 and \$120, written as the row matrix $\begin{bmatrix} 50 & 80 & 120 \end{bmatrix}$. A 10% discount is applied to every item. Use scalar multiplication to write the discounted price list.

Q16. For the matrix $M = \begin{bmatrix} 3 & 5 \\ 2 & 6 \\ 4 & 1 \end{bmatrix}$, use summing matrices to find (a) the row totals, (b) the column totals, and (c) the grand total of all the entries.

Q17. Two phone plans have the following monthly charges (in dollars) for calls, data and texts:

$$\begin{bmatrix} 30 & 20 & 10 \\ 25 & 25 & 5 \end{bmatrix}$$

(rows = plans, columns = calls, data, texts). Use a column of three 1s to find the total monthly cost of each plan, and state which plan is cheaper.

Q18. A cinema charges \$20 for an adult ticket and \$12 for a child ticket, written as the row matrix $\begin{bmatrix} 20 & 12 \end{bmatrix}$. On a discount day all prices are halved. (a) Use scalar multiplication to write the discount-day price list. (b) Write the discount-day prices as a column matrix.

Q19. A courier company delivers parcels from each of 3 depots to each of 4 regions. (a) State the order of the matrix that would represent the number of parcels sent from each depot to each region, with depots as rows. (b) Explain what the element a_{23} would represent.

Q20. A supermarket records the weekly sales (in units) of three brands of milk at two stores. Store A sold 120, 90 and 60; store B sold 80, 110 and 70. (a) Write the information as a matrix with the stores as rows, and state its order. (b) State the value of a_{13} and explain what it represents. (c) Use a summing matrix to find the total units of milk sold at store A. (d) Use a summing matrix to find the total units of brand 2 sold across both stores. (e) Sales are expected to double next week. Use scalar multiplication to write next week's matrix.

Answers

Q1. (a) $\begin{bmatrix}8 & 5 & 6\\4 & 9 & 7\end{bmatrix}$; (b) 2×3; (c) $a_{13}=6$, the 6 oranges sold at stall A; (d) stall B (9 bananas). **Q2.** (a) $\begin{bmatrix}4 & 6 & 12\end{bmatrix}$; (b) 1×3; (c) $\begin{bmatrix}6 & 9 & 18\end{bmatrix}$; (d) $\begin{bmatrix}4\\6\\12\end{bmatrix}$. **Q3.** (a) 3×2; (b) $a_{22}=8$, the 8 units of product Q at shop Y; (c) $\begin{bmatrix}14\\15\\11\end{bmatrix}$; (d) shop Y (15). **Q4.** (a) 1×3; (b) $\begin{bmatrix}125 & 125 & 50\end{bmatrix}$; (c) $\begin{bmatrix}75 & 75 & 30\end{bmatrix}$. **Q5.** (a) 2×3; (b) $g_{13}=2$, the 2 goals team 1 scored in round 3; (c) $\begin{bmatrix}7 & 6 & 8\end{bmatrix}$; (d) round 3 (8 goals). **Q6.** (a) $\begin{bmatrix}12 & 8 & 20\\15 & 6 & 18\end{bmatrix}$, 2×3; (b) $a_{23}=18$, supplier 2's price of \$18 for material 3; (c) supplier 1 (\$12 vs \$15). **Q7.** $\begin{bmatrix}14 & 18\\9 & 12\\20 & 15\end{bmatrix}$, order 3×2. **Q8.** $a_{13}=4$: team 1 scored 4 goals in game 3. $a_{21}=3$: team 2 scored 3 goals in game 1. **Q9.** $\begin{bmatrix}12\\18\\24\end{bmatrix}$. **Q10.** Row totals $\begin{bmatrix}10\\12\\9\end{bmatrix}$; shop 2 had the greatest total (12). **Q11.** Column totals $\begin{bmatrix}8 & 9 & 5 & 7\end{bmatrix}$; product 2 sold the most (9). **Q12.** $\begin{bmatrix}800 & 12 & 600\end{bmatrix}$. **Q13.** $\begin{bmatrix}40 & 25\\30 & 20\end{bmatrix}$; rows = days (Saturday, Sunday), columns = class times (morning, evening); $a_{12}=25$ is the 25 attendances at the Saturday evening class. **Q14.** (a) $a_{31}=5$, the 5 small jackets; (b) $12+15+8=35$ shirts; (c) $12+10+5=27$ small garments. **Q15.** $\begin{bmatrix}45 & 72 & 108\end{bmatrix}$. **Q16.** (a) $\begin{bmatrix}8\\8\\5\end{bmatrix}$; (b) $\begin{bmatrix}9 & 12\end{bmatrix}$; (c) 21. **Q17.** Plan totals $\begin{bmatrix}60\\55\end{bmatrix}$; plan 2 is cheaper (\$55). **Q18.** (a) $\begin{bmatrix}10 & 6\end{bmatrix}$; (b) $\begin{bmatrix}10\\6\end{bmatrix}$. **Q19.** (a) 3×4; (b) the number of parcels sent from depot 2 to region 3. **Q20.** (a) $\begin{bmatrix}120 & 90 & 60\\80 & 110 & 70\end{bmatrix}$, 2×3; (b) $a_{13}=60$, the 60 units of brand 3 sold at store A; (c) 270; (d) 200; (e) $\begin{bmatrix}240 & 180 & 120\\160 & 220 & 140\end{bmatrix}$.

Fully-worked solutions

Q1. Take the rows as the stalls and the columns as the fruit (apples, bananas, oranges), reading each row of the table into a row of the matrix:

$$\begin{bmatrix}8 & 5 & 6\\4 & 9 & 7\end{bmatrix}.$$

There are 2 rows and 3 columns, so the order is 2×3. The element a_{13} is in row 1 (stall A), column 3 (oranges), so $a_{13}=6$: stall A sold 6 oranges. Comparing the bananas column, stall A sold 5 and stall B sold 9, so stall B sold more.

Q2. (a) A price list of three items is a single row: $\begin{bmatrix}4 & 6 & 12\end{bmatrix}$. (b) One row and three columns give order 1×3. (c) A 50% increase multiplies every price by 1.5, so

$$1.5\begin{bmatrix}4 & 6 & 12\end{bmatrix}=\begin{bmatrix}6 & 9 & 18\end{bmatrix}.$$

(d) The same list written down the page is the column matrix $\begin{bmatrix}4\\6\\12\end{bmatrix}$.

Q3. (a) The array has 3 rows and 2 columns, so the order is 3×2. (b) a_{22} is in row 2 (shop Y), column 2 (product Q), so $a_{22}=8$: shop Y holds 8 units of product Q. (c) Multiplying on the right by a column of two 1s adds along each row:

$$\begin{bmatrix} 10 & 4 \\ 7 & 8 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+4 \\ 7+8 \\ 5+6 \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \\ 11 \end{bmatrix}.$$

(d) The totals are 14, 15 and 11, so shop Y holds the most (15 items).

Q4. (a) K is a single row of three numbers, so its order is 1×3. (b) Five identical classes need $5K = \begin{bmatrix} 125 & 125 & 50 \end{bmatrix}$. (c) Three identical classes need $3K = \begin{bmatrix} 75 & 75 & 30 \end{bmatrix}$.

Q5. (a) G has 2 rows and 3 columns, so its order is 2×3. (b) g_{13} is in row 1 (team 1), column 3 (round 3), so $g_{13} = 2$: team 1 scored 2 goals in round 3. (c) Multiplying on the left by a row of two 1s adds down each column:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3+4 & 5+1 & 2+6 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 8 \end{bmatrix}.$$

(d) The round totals are 7, 6 and 8, so the most goals (8) were scored in round 3.

Q6. (a) With the suppliers as rows and the three materials as columns,

$$\begin{bmatrix} 12 & 8 & 20 \\ 15 & 6 & 18 \end{bmatrix},$$

which has order 2×3. (b) a_{23} is in row 2 (supplier 2), column 3 (material 3), so $a_{23} = 18$: supplier 2 charges \$18 for material 3. (c) For material 1 (column 1) supplier 1 charges \$12 and supplier 2 charges \$15, so supplier 1 is cheaper.

Q7. Take the rows as the three dealers and the columns as the two months (January, February):

$$\begin{bmatrix} 14 & 18 \\ 9 & 12 \\ 20 & 15 \end{bmatrix}.$$

There are 3 rows and 2 columns, so the order is 3×2.

Q8. a_{13} is in row 1 (team 1), column 3 (game 3), so $a_{13} = 4$: team 1 scored 4 goals in game 3. a_{21} is in row 2 (team 2), column 1 (game 1), so $a_{21} = 3$: team 2 scored 3 goals in game 1.

Q9. A 20% increase multiplies every price by 1.2:

$$1.2 \begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ 24 \end{bmatrix}.$$

Q10. Multiplying on the right by a column of three 1s adds along each row:

$$\begin{bmatrix} 5 & 2 & 3 \\ 4 & 4 & 4 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5+2+3 \\ 4+4+4 \\ 6+1+2 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 9 \end{bmatrix}.$$

The shop totals are 10, 12 and 9, so shop 2 had the greatest three-week total (12).

Q11. Multiplying on the left by a row of three 1s adds down each column:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 0 & 2 & 3 \\ 1 & 6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 5 & 7 \end{bmatrix}.$$

The product totals are 8, 9, 5 and 7, so product 2 sold the most (9 units).

Q12. Four batches need $4R=4\begin{bmatrix}200 & 3 & 150\end{bmatrix}=\begin{bmatrix}800 & 12 & 600\end{bmatrix}$, that is, 800 g of flour, 12 eggs and 600 g of sugar.

Q13. Taking the rows as the days (Saturday, Sunday) and the columns as the class times (morning, evening),

$$\begin{bmatrix}40 & 25\\30 & 20\end{bmatrix}.$$

The element a_{12} is in row 1 (Saturday), column 2 (evening), so $a_{12}=25$: 25 people attended the Saturday evening class.

Q14. (a) a_{31} is in row 3 (jackets), column 1 (small), so $a_{31}=5$: there are 5 small jackets. (b) The shirts are row 1, so a column of three 1s gives their total, $12+15+8=35$ shirts. (c) The small garments are column 1, so a row of three 1s gives their total, $12+10+5=27$ small garments.

Q15. A 10% discount leaves 90% of each price, so multiply by 0.9:

$$0.9\begin{bmatrix}50 & 80 & 120\end{bmatrix}=\begin{bmatrix}45 & 72 & 108\end{bmatrix}.$$

Q16. (a) A column of two 1s gives the row totals:

$$\begin{bmatrix}3 & 5\\2 & 6\\4 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}8\\8\\5\end{bmatrix}.$$

(b) A row of three 1s gives the column totals:

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}3 & 5\\2 & 6\\4 & 1\end{bmatrix}=\begin{bmatrix}9 & 12\end{bmatrix}.$$

(c) The grand total is the sum of either set of totals: $8+8+5=21$ (equally $9+12=21$).

Q17. A column of three 1s adds each plan's charges along its row:

$$\begin{bmatrix}30 & 20 & 10\\25 & 25 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}60\\55\end{bmatrix}.$$

Plan 1 costs \$60 and plan 2 costs \$55 per month, so plan 2 is cheaper.

Q18. (a) Halving multiplies by 0.5: $0.5\begin{bmatrix}20 & 12\end{bmatrix}=\begin{bmatrix}10 & 6\end{bmatrix}$, that is, \$10 for an adult and \$6 for a child. (b) The same list as a column is $\begin{bmatrix}10\\6\end{bmatrix}$.

Q19. (a) One row for each depot and one column for each region gives a 3×4 matrix. (b) The element a_{23} is in row 2 (depot 2), column 3 (region 3), so it represents the number of parcels sent from depot 2 to region 3.

Q20. (a) With the stores as rows and the three brands as columns,

$$\begin{bmatrix}120 & 90 & 60\\80 & 110 & 70\end{bmatrix},$$

of order 2×3. (b) a_{13} is in row 1 (store A), column 3 (brand 3), so $a_{13}=60$: store A sold 60 units of brand 3. (c) The store A total is the row 1 sum, $120+90+60=270$ units (a column of three 1s). (d) The brand 2 total is the column 2 sum, $90+110=200$ units (a row of two 1s). (e) Doubling multiplies every entry by 2:

$$2\begin{bmatrix} 120 & 90 & 60 \\ 80 & 110 & 70 \end{bmatrix} = \begin{bmatrix} 240 & 180 & 120 \\ 160 & 220 & 140 \end{bmatrix}.$$

Theory

A **matrix** is a rectangular array of numbers set out in rows and columns and enclosed in square brackets. Its **order** is written $\text{rows} \times \text{columns}$, and the entry in row i, column j is the **element** a_{ij}. Matrices were introduced in an earlier section; this section develops the first of the matrix operations — **addition**, **subtraction** and **multiplication by a scalar**. (Multiplication of one matrix by another is a separate operation, developed later in this chapter.)

When are two matrices equal? Two matrices are **equal** only when they have the *same order* and every pair of corresponding elements is equal. This idea of "corresponding elements" is the key to the operations below.

Addition and subtraction.

Two matrices can be **added** (or **subtracted**) only when they have the **same order**. You then add (or subtract) **corresponding elements**, and the answer has that same order:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix},$$

and subtraction works the same way with a minus sign. If the two matrices have **different orders**, their sum and difference are **undefined** — there is no way to pair up the elements, so the operation simply does not exist. For example, a 2×2 matrix cannot be added to a 2×3 matrix.

Scalar multiplication.

A **scalar** is an ordinary number. To multiply a matrix by a scalar k, multiply **every** element by k; the order is unchanged:

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$$

Two special cases are worth naming. Taking $k = -1$ gives the **negative** of a matrix, $-A = (-1)A$, in which every element changes sign. Subtraction can then be viewed as *adding the negative*: $A - B = A + (-1)B$.

Combining the operations.

Expressions such as $2A - 3B$ are handled by scalar-multiplying each matrix first and then adding or subtracting corresponding elements. As with any sum or difference, the matrices being combined must have the same order.

The zero matrix. A **zero matrix** O has every element 0. It behaves like the number 0: for any matrix A of the same order, $A + O = A$, and $A - A = O$.

Properties. For matrices of the same order, addition behaves much like ordinary addition of numbers:

Property	Statement
Commutative (addition)	$A + B = B + A$
Associative (addition)	$(A+B)+C = A+(B+C)$
Additive identity	$A + O = A$
Scalar over a sum	$k(A+B) = kA + kB$
Scalars added	$(k+l)A = kA + lA$

Subtraction, however, is **not** commutative: in general $A - B \neq B - A$. In fact the two results are negatives of each other, $B - A = -(A - B)$.

Using matrices to organise data. Because a matrix is just a labelled table of numbers, these operations model everyday situations directly. Adding two stock matrices for consecutive weeks gives the total held over the fortnight; subtracting them measures the change; and multiplying by a scalar rescales every entry at once — for instance, a forecast **10% increase** multiplies a stock matrix by 1.1, and a **10% discount** multiplies a price matrix by 0.9. The row and column labels must match before you add, which is exactly the “same order” requirement in disguise.

Using CAS. General Mathematics is a technology-active, open-book course. In practice you enter each matrix into your CAS calculator and let it evaluate sums, differences and scalar multiples, and it will report an error (or leave the expression unevaluated) when the orders do not match. Learn the by-hand method below so that you can set up the matrices correctly and check that the CAS output is reasonable.

Worked examples

Example 1 — scaffolded

For $A=\begin{bmatrix}3 & -1\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}4 & 6\\ -3 & 1\end{bmatrix}$, find $A+B$ and $A-B$.

Working	Comment
A and B are both 2×2.	Same order, so both operations are defined.
$A+B=\begin{bmatrix}3+4 & -1+6\\ 2+(-3) & 5+1\end{bmatrix}=\begin{bmatrix}7 & 5\\ -1 & 6\end{bmatrix}$.	Add corresponding elements.
$A-B=\begin{bmatrix}3-4 & -1-6\\ 2-(-3) & 5-1\end{bmatrix}=\begin{bmatrix}-1 & -7\\ 5 & 4\end{bmatrix}$.	Subtract corresponding elements; watch the signs.
A 2×3 matrix could **not** be added to A.	Addition needs matching orders — otherwise it is undefined.

Example 2 — scaffolded

For $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 5 & 2\end{bmatrix}$, find (a) $3A$ and (b) $2A-3B$.

Working	Comment
(a) $3A=\begin{bmatrix}3\times 1 & 3\times 2\\ 3\times 3 & 3\times 4\end{bmatrix}=\begin{bmatrix}3 & 6\\ 9 & 12\end{bmatrix}$.	Multiply every element by the scalar 3.
(b) $2A=\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}$ and $3B=\begin{bmatrix}0 & -3\\ 15 & 6\end{bmatrix}$.	Scalar-multiply each matrix first.
$2A-3B=\begin{bmatrix}2-0 & 4-(-3)\\ 6-15 & 8-6\end{bmatrix}=\begin{bmatrix}2 & 7\\ -9 & 2\end{bmatrix}$.	Then subtract corresponding elements.

Example 3 — unscaffolded

A clothing retailer with two branches, Eastside and Westside, records the number of caps, shirts and bags sold each week. The rows are the two branches and the columns are the three items. In week 1 and week 2 the sales were

$$W_1=\begin{bmatrix}40 & 30 & 20\\ 50 & 20 & 10\end{bmatrix},\ W_2=\begin{bmatrix}60 & 50 & 40\\ 30 & 60 & 50\end{bmatrix}.$$

Find the total number of each item sold over the fortnight, and hence forecast the next fortnight’s sales assuming a $10\,\%$ increase.

Both matrices are 2×3, so they can be added. The fortnight total is

$$W_1+W_2=\begin{bmatrix}40+60 & 30+50 & 20+40\\ 50+30 & 20+60 & 10+50\end{bmatrix}=\begin{bmatrix}100 & 80 & 60\\ 80 & 80 & 60\end{bmatrix}.$$

A 10% increase multiplies every entry by 1.1, which is a scalar multiplication:

$$1.1\begin{bmatrix}100 & 80 & 60\\ 80 & 80 & 60\end{bmatrix}=\begin{bmatrix}110 & 88 & 66\\ 88 & 88 & 66\end{bmatrix}.$$

$\therefore$ over the fortnight the branches sold the amounts in $\begin{bmatrix}100 & 80 & 60\\ 80 & 80 & 60\end{bmatrix}$, and the forecast for the next fortnight is $\begin{bmatrix}110 & 88 & 66\\ 88 & 88 & 66\end{bmatrix}$.

Example 4 — unscaffolded

Find the matrix X that satisfies $2X+A=B$, where $A=\begin{bmatrix}3 & 5\\ -2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}7 & 1\\ 6 & 0\end{bmatrix}$.

The equation is solved just like an ordinary linear equation, keeping every step as a matrix operation. Subtracting A from both sides gives $2X=B-A$, so

$$2X=\begin{bmatrix}7-3 & 1-5\\ 6-(-2) & 0-4\end{bmatrix}=\begin{bmatrix}4 & -4\\ 8 & -4\end{bmatrix}.$$

Multiplying both sides by the scalar $1/2$ (which halves every element) gives

$$X=1/2\begin{bmatrix}4 & -4\\ 8 & -4\end{bmatrix}=\begin{bmatrix}2 & -2\\ 4 & -2\end{bmatrix}.$$

Checking: $2X+A=\begin{bmatrix}4 & -4\\ 8 & -4\end{bmatrix}+\begin{bmatrix}3 & 5\\ -2 & 4\end{bmatrix}=\begin{bmatrix}7 & 1\\ 6 & 0\end{bmatrix}=B$.

$\therefore X=\begin{bmatrix}2 & -2\\ 4 & -2\end{bmatrix}$.

Practice questions

Scaffolded

Q1. Let $A=\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix}$ and $B=\begin{bmatrix}5 & -2\\ 0 & 6\end{bmatrix}$. (a) Find $A+B$. (b) Find $B+A$. (c) Find $A-B$. (d) Find $B-A$, and state how it is related to $A-B$.

Q2. Let $C=\begin{bmatrix}3 & -1 & 2\\ 0 & 4 & 5\end{bmatrix}$. (a) Find $2C$. (b) Find $5C$. (c) Find $-C$.

Q3. Let $A=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}4 & 1\\ -1 & 2\end{bmatrix}$. (a) Find $2A$. (b) Find $3B$. (c) Find $2A+3B$. (d) Find $2A-3B$.

Q4. Two cafés belonging to one owner record the number of coffees and teas sold. The rows are the two cafés and the columns are coffee and tea. On Monday and Tuesday the sales were $M=\begin{bmatrix}20 & 15\\ 18 & 12\end{bmatrix}$ and $T=\begin{bmatrix}25 & 10\\ 22 & 18\end{bmatrix}$. (a) Find $M+T$, the total for the two days. (b) On Wednesday each café sells exactly double its Monday figures. Find $2M$. (c) Find $T-M$, and say what it represents.

Q5. Let $A=\begin{bmatrix}6 & 2\\4 & 8\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\5 & 0\end{bmatrix}$. Find the matrix X if (a) $A+X=B$; (b) $X-B=A$; (c) $2X=A$.

Q6. Consider the matrices $P=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $Q=\begin{bmatrix}0 & 5\\6 & 7\end{bmatrix}$, $R=\begin{bmatrix}1 & 0 & 2\\3 & 1 & 4\end{bmatrix}$ and $S=\begin{bmatrix}1 & 2\\0 & 3\\4 & 5\end{bmatrix}$. For each of the following, state whether it is defined; if it is, evaluate it. (a) $P+Q$ (b) $P+R$ (c) $R-S$ (d) $2R$

Unscaffolded

Q7. For $A=\begin{bmatrix}8 & -3\\5 & 2\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 4\\7 & -6\end{bmatrix}$, find $A+B$ and $A-B$.

Q8. Evaluate $4\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}-\begin{bmatrix}3 & 5\\-2 & 1\end{bmatrix}$.

Q9. For the row matrices $a=\begin{bmatrix}3 & -2 & 5\end{bmatrix}$ and $b=\begin{bmatrix}1 & 4 & -3\end{bmatrix}$, find $2a+b$.

Q10. For the column matrices $C=\begin{bmatrix}2\\-5\\3\end{bmatrix}$ and $D=\begin{bmatrix}-1\\4\\0\end{bmatrix}$, find $3C-2D$.

Q11. Let $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}$ and $C=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$. By evaluating both sides, verify that $(A+B)+C=A+(B+C)$.

Q12. Let $A=\begin{bmatrix}2 & -1\\3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\-2 & 5\end{bmatrix}$. By evaluating both sides, verify that $2(A+B)=2A+2B$.

Q13. Find the matrix X if $3X+\begin{bmatrix}1 & 2\\0 & -1\end{bmatrix}=\begin{bmatrix}7 & -4\\9 & 5\end{bmatrix}$.

Q14. Find the values of x, y and z if $\begin{bmatrix}x & 3\\y & -2\end{bmatrix}+\begin{bmatrix}4 & z\\1 & 6\end{bmatrix}=\begin{bmatrix}9 & 7\\-3 & 4\end{bmatrix}$.

Q15. A hardware store with two branches stocks three items. The rows are the branches and the columns are the items. In two successive weeks the numbers sold were $W_1=\begin{bmatrix}30 & 45 & 25\\20 & 50 & 35\end{bmatrix}$ and $W_2=\begin{bmatrix}20 & 15 & 15\\30 & 10 & 25\end{bmatrix}$. (a) Find the total sold over the two weeks. (b) The manager forecasts a 20% increase on that total for the following fortnight. Find the forecast matrix.

Q16. A shop lists the price (in dollars) of four products as $P=\begin{bmatrix}20 & 50\\80 & 120\end{bmatrix}$. (a) In a sale, every price is cut by 10%. Find the sale-price matrix. (b) Find the matrix of savings, and confirm it equals P minus the sale-price matrix.

Q17. Let $A=\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 5\\0 & 2\end{bmatrix}$. (a) Find $A-B$ and $B-A$. (b) State the relationship between your two answers, and explain why it holds for any two matrices of the same order.

Q18. Let $A=\begin{bmatrix}4 & -2\\7 & 1\end{bmatrix}$ and let O be the 2×2 zero matrix. (a) Find $A+O$. (b) Find $A-A$. (c) State what each result illustrates.

Q19. For $A=\begin{bmatrix}1 & -1\\2 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 3\\1 & -2\end{bmatrix}$ and $C=\begin{bmatrix}2 & 1\\-1 & 4\end{bmatrix}$, evaluate $2A-B+3C$.

Q20. A netball club records goals in a 3×2 matrix (three players down the rows, two quarters across the columns). In round 1 and round 2 the goals scored were $G_1 = \begin{bmatrix} 5 & 3 \\ 4 & 6 \\ 2 & 7 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 6 & 2 \\ 3 & 5 \\ 8 & 1 \end{bmatrix}$. (a) A rival club's data is a 2×3 matrix. Explain why it cannot be added to G_1. (b) Find $G_1 + G_2$, the total goals over the two rounds.

Answers

Q1. (a) $\begin{bmatrix}7 & -1\\4 & 9\end{bmatrix}$; (b) $\begin{bmatrix}7 & -1\\4 & 9\end{bmatrix}$ (equal to (a), since addition is commutative); (c) $\begin{bmatrix}-3 & 3\\4 & -3\end{bmatrix}$; (d) $\begin{bmatrix}3 & -3\\-4 & 3\end{bmatrix}=-(A-B)$. **Q2.** (a) $\begin{bmatrix}6 & -2 & 4\\0 & 8 & 10\end{bmatrix}$; (b) $\begin{bmatrix}15 & -5 & 10\\0 & 20 & 25\end{bmatrix}$; (c) $\begin{bmatrix}-3 & 1 & -2\\0 & -4 & -5\end{bmatrix}$. **Q3.** (a) $\begin{bmatrix}2 & 0\\4 & 6\end{bmatrix}$; (b) $\begin{bmatrix}12 & 3\\-3 & 6\end{bmatrix}$; (c) $\begin{bmatrix}14 & 3\\1 & 12\end{bmatrix}$; (d) $\begin{bmatrix}-10 & -3\\7 & 0\end{bmatrix}$. **Q4.** (a) $\begin{bmatrix}45 & 25\\40 & 30\end{bmatrix}$; (b) $\begin{bmatrix}40 & 30\\36 & 24\end{bmatrix}$; (c) $\begin{bmatrix}5 & -5\\4 & 6\end{bmatrix}$, the change in sales from Monday to Tuesday. **Q5.** (a) $\begin{bmatrix}-5 & 1\\1 & -8\end{bmatrix}$; (b) $\begin{bmatrix}7 & 5\\9 & 8\end{bmatrix}$; (c) $\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$. **Q6.** (a) defined, $\begin{bmatrix}1 & 7\\9 & 11\end{bmatrix}$; (b) undefined (2×2 and 2×3); (c) undefined (2×3 and 3×2); (d) defined, $\begin{bmatrix}2 & 0 & 4\\6 & 2 & 8\end{bmatrix}$. **Q7.** $A+B=\begin{bmatrix}7 & 1\\12 & -4\end{bmatrix}$; $A-B=\begin{bmatrix}9 & -7\\-2 & 8\end{bmatrix}$. **Q8.** $\begin{bmatrix}5 & -9\\2 & 11\end{bmatrix}$. **Q9.** $\begin{bmatrix}7 & 0 & 7\end{bmatrix}$. **Q10.** $\begin{bmatrix}8\\-23\\9\end{bmatrix}$. **Q11.** Both sides equal $\begin{bmatrix}3 & 3\\6 & 12\end{bmatrix}$. **Q12.** Both sides equal $\begin{bmatrix}6 & 6\\2 & 10\end{bmatrix}$. **Q13.** $X=\begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}$. **Q14.** $x=5,\ y=-4,\ z=4$. **Q15.** (a) $\begin{bmatrix}50 & 60 & 40\\50 & 60 & 60\end{bmatrix}$; (b) $\begin{bmatrix}60 & 72 & 48\\60 & 72 & 72\end{bmatrix}$. **Q16.** (a) $\begin{bmatrix}18 & 45\\72 & 108\end{bmatrix}$; (b) savings $\begin{bmatrix}2 & 5\\8 & 12\end{bmatrix}$, which equals P minus the sale prices. **Q17.** (a) $A-B=\begin{bmatrix}2 & -4\\2 & 2\end{bmatrix}$, $B-A=\begin{bmatrix}-2 & 4\\-2 & -2\end{bmatrix}$; (b) $B-A=-(A-B)$, because subtracting in the reverse order negates every element. **Q18.** (a) $\begin{bmatrix}4 & -2\\7 & 1\end{bmatrix}$; (b) $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$; (c) O is the additive identity ($A+O=A$) and $A-A=O$. **Q19.** $\begin{bmatrix}8 & -2\\0 & 14\end{bmatrix}$. **Q20.** (a) a 2×3 matrix has a different order from the 3×2 matrix G_1, so no corresponding elements can be paired and the sum is undefined; (b) $\begin{bmatrix}11 & 5\\7 & 11\\10 & 8\end{bmatrix}$.

Fully-worked solutions

Q1. Both are 2×2, so add and subtract corresponding elements. $A+B=\begin{bmatrix}2+5 & 1+(-2)\\4+0 & 3+6\end{bmatrix}=\begin{bmatrix}7 & -1\\4 & 9\end{bmatrix}$. $B+A$ gives the same matrix, illustrating that addition is commutative. $A-B=\begin{bmatrix}2-5 & 1-(-2)\\4-0 & 3-6\end{bmatrix}=\begin{bmatrix}-3 & 3\\4 & -3\end{bmatrix}$, and $B-A=\begin{bmatrix}3 & -3\\-4 & 3\end{bmatrix}$. Each element of $B-A$ is the negative of the corresponding element of $A-B$, so $B-A=-(A-B)$.

Q2. Multiply every element by the scalar. $2C=\begin{bmatrix}6 & -2 & 4\\0 & 8 & 10\end{bmatrix}$, $5C=\begin{bmatrix}15 & -5 & 10\\0 & 20 & 25\end{bmatrix}$, $-C=(-1)C=\begin{bmatrix}-3 & 1 & -2\\0 & -4 & -5\end{bmatrix}$.

Q3. $2A=\begin{bmatrix}2 & 0\\4 & 6\end{bmatrix}$ and $3B=\begin{bmatrix}12 & 3\\-3 & 6\end{bmatrix}$. Then $2A+3B=\begin{bmatrix}2+12 & 0+3\\4+(-3) & 6+6\end{bmatrix}=\begin{bmatrix}14 & 3\\1 & 12\end{bmatrix}$ and $2A-3B=\begin{bmatrix}2-12 & 0-3\\4-(-3) & 6-6\end{bmatrix}=\begin{bmatrix}-10 & -3\\7 & 0\end{bmatrix}$.

Q4. (a) $M+T=\begin{bmatrix}20+25 & 15+10\\18+22 & 12+18\end{bmatrix}=\begin{bmatrix}45 & 25\\40 & 30\end{bmatrix}$. (b) $2M=\begin{bmatrix}40 & 30\\36 & 24\end{bmatrix}$. (c) $T-M=\begin{bmatrix}25-20 & 10-15\\22-18 & 18-12\end{bmatrix}=\begin{bmatrix}5 & -5\\4 & 6\end{bmatrix}$; this is the change in each café's sales from Monday to Tuesday (a negative entry means a fall).

Q5. (a) $A+X=B\Rightarrow X=B-A=\begin{bmatrix}1-6 & 3-2\\5-4 & 0-8\end{bmatrix}=\begin{bmatrix}-5 & 1\\1 & -8\end{bmatrix}$. (b) $X-B=A\Rightarrow X=A+B=\begin{bmatrix}6+1 & 2+3\\4+5 & 8+0\end{bmatrix}=\begin{bmatrix}7 & 5\\9 & 8\end{bmatrix}$. (c) $2X=A\Rightarrow X=1/2\,A=\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$.

Q6. Addition and subtraction need matching orders. (a) P and Q are both 2×2: $P+Q=\begin{bmatrix}1 & 7\\9 & 11\end{bmatrix}$. (b) P is 2×2 and R is 2×3: different orders, so $P+R$ is **undefined**. (c) R is 2×3 and S is 3×2: different orders, so $R-S$ is **undefined**. (d) $2R$ is a scalar multiple, always defined: $2R=\begin{bmatrix}2 & 0 & 4\\6 & 2 & 8\end{bmatrix}$.

Q7. $A+B=\begin{bmatrix}8+(-1) & -3+4\\5+7 & 2+(-6)\end{bmatrix}=\begin{bmatrix}7 & 1\\12 & -4\end{bmatrix}$ and $A-B=\begin{bmatrix}8-(-1) & -3-4\\5-7 & 2-(-6)\end{bmatrix}=\begin{bmatrix}9 & -7\\-2 & 8\end{bmatrix}$.

Q8. First $4\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}=\begin{bmatrix}8 & -4\\0 & 12\end{bmatrix}$, then subtract: $\begin{bmatrix}8-3 & -4-5\\0-(-2) & 12-1\end{bmatrix}=\begin{bmatrix}5 & -9\\2 & 11\end{bmatrix}$.

Q9. $2a=\begin{bmatrix}6 & -4 & 10\end{bmatrix}$, so $2a+b=\begin{bmatrix}6+1 & -4+4 & 10+(-3)\end{bmatrix}=\begin{bmatrix}7 & 0 & 7\end{bmatrix}$.

Q10. $3C=\begin{bmatrix}6\\-15\\9\end{bmatrix}$ and $2D=\begin{bmatrix}-2\\8\\0\end{bmatrix}$, so $3C-2D=\begin{bmatrix}6-(-2)\\-15-8\\9-0\end{bmatrix}=\begin{bmatrix}8\\-23\\9\end{bmatrix}$.

Q11. Left side: $A+B=\begin{bmatrix}3 & 2\\4 & 9\end{bmatrix}$, then $(A+B)+C=\begin{bmatrix}3 & 3\\6 & 12\end{bmatrix}$. Right side: $B+C=\begin{bmatrix}2 & 1\\3 & 8\end{bmatrix}$, then $A+(B+C)=\begin{bmatrix}3 & 3\\6 & 12\end{bmatrix}$. The two sides agree, confirming that addition is associative.

Q12. Left side: $A+B=\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}$, so $2(A+B)=\begin{bmatrix}6 & 6\\2 & 10\end{bmatrix}$. Right side: $2A=\begin{bmatrix}4 & -2\\6 & 0\end{bmatrix}$ and $2B=\begin{bmatrix}2 & 8\\-4 & 10\end{bmatrix}$, so $2A+2B=\begin{bmatrix}6 & 6\\2 & 10\end{bmatrix}$. The two sides agree, confirming that a scalar distributes over a sum.

Q13. Subtract the constant matrix from both sides: $3X=\begin{bmatrix}7 & -4\\9 & 5\end{bmatrix}-\begin{bmatrix}1 & 2\\0 & -1\end{bmatrix}=\begin{bmatrix}6 & -6\\9 & 6\end{bmatrix}$. Then $X=1/3\begin{bmatrix}6 & -6\\9 & 6\end{bmatrix}=\begin{bmatrix}2 & -2\\3 & 2\end{bmatrix}$.

Q14. Adding the left side gives $\begin{bmatrix}x+4 & 3+z\\y+1 & -2+6\end{bmatrix}$. Equating corresponding elements with the right-hand matrix: $x+4=9\Rightarrow x=5$; $3+z=7\Rightarrow z=4$; $y+1=-3\Rightarrow y=-4$; and the last entry checks, since $-2+6=4$.

Q15. (a) Both are 2×3, so $W_1+W_2=\begin{bmatrix}30+20 & 45+15 & 25+15\\20+30 & 50+10 & 35+25\end{bmatrix}=\begin{bmatrix}50 & 60 & 40\\50 & 60 & 60\end{bmatrix}$. (b) A 20% increase multiplies the total by 1.2: $1.2\begin{bmatrix}50 & 60 & 40\\50 & 60 & 60\end{bmatrix}=\begin{bmatrix}60 & 72 & 48\\60 & 72 & 72\end{bmatrix}$.

Q16. (a) Cutting every price by 10% leaves 90%, so the sale prices are $0.9P = 0.9\begin{bmatrix} 20 & 50 \\ 80 & 120 \end{bmatrix} = \begin{bmatrix} 18 & 45 \\ 72 & 108 \end{bmatrix}$. (b) The saving is 10% of each price, $0.1P = \begin{bmatrix} 2 & 5 \\ 8 & 12 \end{bmatrix}$. This equals $P - 0.9P = \begin{bmatrix} 20-18 & 50-45 \\ 80-72 & 120-108 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 8 & 12 \end{bmatrix}$, as expected.

Q17. (a) $A - B = \begin{bmatrix} 3-1 & 1-5 \\ 2-0 & 4-2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & 2 \end{bmatrix}$ and $B - A = \begin{bmatrix} 1-3 & 5-1 \\ 0-2 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & -2 \end{bmatrix}$. (b) $B - A = -(A - B)$: reversing the order of a subtraction changes the sign of every element, because $b_{ij} - a_{ij} = -(a_{ij} - b_{ij})$ for each element. Subtraction is therefore not commutative.

Q18. (a) Adding the zero matrix changes nothing: $A + O = \begin{bmatrix} 4 & -2 \\ 7 & 1 \end{bmatrix}$. (b) $A - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$. (c) Part (a) shows that O is the additive identity ($A + O = A$), and part (b) shows that subtracting a matrix from itself gives the zero matrix.

Q19. Scalar-multiply first: $2A = \begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$ and $3C = \begin{bmatrix} 6 & 3 \\ -3 & 12 \end{bmatrix}$. Then combine element by element: $2A - B + 3C = \begin{bmatrix} 2-0+6 & -2-3+3 \\ 4-1-3 & 0-(-2)+12 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 0 & 14 \end{bmatrix}$.

Q20. (a) G_1 has order 3×2, whereas the rival's data has order 2×3. Because the orders differ, the elements cannot be paired up and the sum is undefined. (b) G_1 and G_2 are both 3×2, so $G_1 + G_2 = \begin{bmatrix} 5+6 & 3+2 \\ 4+3 & 6+5 \\ 2+8 & 7+1 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 7 & 11 \\ 10 & 8 \end{bmatrix}$.

Theory

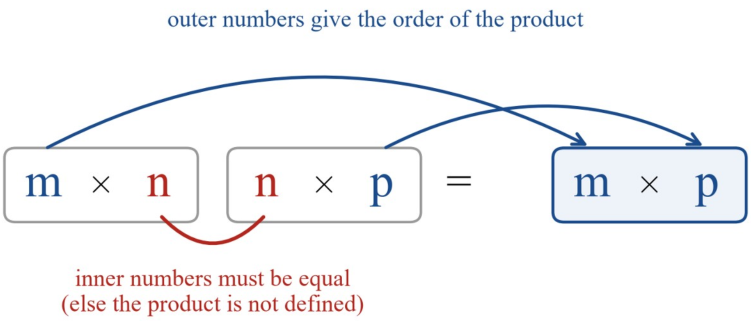
An earlier section combined matrices by adding, subtracting and multiplying by a scalar. **Matrix multiplication** — forming the **product** of two matrices — is a different and more involved operation. It does not multiply matching entries; instead it combines the **rows of the first matrix** with the **columns of the second** in a "row-by-column" calculation. This one rule is the engine behind transition matrices, communication and dominance matrices, and many cost-and- quantity applications later in this chapter.

When is a product defined? (conformability).

Two matrices can be multiplied only when their orders fit together. To form the product AB, the number of **columns of** A must equal the number of **rows of** B. If

$$A \text{ is } m \times n \text{ and } B \text{ is } n \times p,$$

then the product AB exists and has order $m \times p$. We say A and B are **conformable** for the product AB. Write the two orders side by side: the **inner** numbers must match (they are the n that is "used up" in the multiplication), and the **outer** numbers give the order of the product.



If the inner numbers are not equal, the product is **not defined**. For example, a 2×3 matrix times a 3×4 matrix gives a 2×4 matrix, but a 3×4 matrix times a 2×3 matrix is not defined ($4 \neq 2$).

The row-by-column rule.

The entry in **row** i**, column** j of the product AB is found from **row** i **of** A and **column** j **of** B: multiply their matching entries in order and add the results. For a 2×2 product,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}.$$

The top-left entry $ae+bg$ uses the **1st row** of the left matrix and the **1st column** of the right matrix; the bottom-right entry $cf+dh$ uses the **2nd row** and the **2nd column**; and so on. In general the ij-entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj},$$

the sum of the products of row i of A with column j of B. There are exactly n products to add, one for each of the n columns of A (= rows of B).

For a worked instance,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Matrix multiplication is *not* commutative.

For ordinary numbers $xy = yx$, but for matrices AB and BA are usually **different**. In fact:

- BA may **not even be defined** — a 2×3 times a 3×2 gives a 2×2, while the reverse gives a 3×3;
- even when both products exist and have the same order, they are generally **unequal**. Using the two matrices above,

$$\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}23 & 34\\31 & 46\end{bmatrix}\neq\begin{bmatrix}19 & 22\\43 & 50\end{bmatrix}.$$

So the **order of multiplication matters**: AB means "A times B", and this is not the same as BA. (Matrix multiplication *is* associative, $(AB)C = A(BC)$, and it distributes over addition, $A(B+C) = AB + AC$ — but it is not commutative.)

The identity matrix.

The **identity matrix** I is the square matrix with 1s down the leading diagonal and 0s everywhere else:

$$I_2=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

It behaves for matrices exactly as the number 1 does for ordinary multiplication: for any matrix A of a conformable order,

$$AI = IA = A.$$

For instance $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}a & b\\c & d\end{bmatrix}$. Multiplying by I leaves a matrix unchanged. (Unlike most products, this one *does* commute: $AI = IA$ for every square A.)

Powers of a square matrix.

Because a matrix must be conformable with itself to be squared, **only square matrices have powers**. For a square matrix A,

$$A^2 = A\times A, A^3 = A\times A\times A, A^n = A\times A\times\cdots\times A\,(n\,\text{factors}).$$

Take $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$. Then

$$A^2=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}7 & 10\\15 & 22\end{bmatrix},$$

and $A^3 = A^2\times A$, and so on. Because multiplication is associative it does not matter how the factors are grouped, so A^n is unambiguous, and a matrix always commutes with its own powers ($A^2A = AA^2 = A^3$). We take $A^1 = A$.

Applications — quantities times prices.

A common use of the product is to combine a **quantity matrix** with a **price matrix** to get a **cost** (or revenue). If the quantities of several items are written as a **row** and their unit prices as a matching **column**, the row-by- column product is the single total. For example, buying 3 pens at \$2, 2 books at \$8 and 1 folder at \$3 costs

$$\begin{bmatrix}3 & 2 & 1\end{bmatrix}\begin{bmatrix}2\\8\\3\end{bmatrix}=\left[3(2)+2(8)+1(3)\right]=[25].$$

Stacking several quantity rows into one matrix produces one total per row in a single multiplication — a very efficient way to price many orders at once.

Using CAS. General Mathematics is technology-active and open-book, so in practice you enter the two matrices into your CAS calculator and read off the product; the calculator also computes powers such as A^{10} instantly. Learn the row-by-column rule by hand so that you can set up the matrices in the right order, predict the order of the answer, and check that the CAS output is reasonable.

Worked examples

Example 1 — scaffolded

Given $A=\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}4 & -1\\ 2 & 5\end{bmatrix}$, find the product AB.

Working	Comment
A is 2×2 and B is 2×2; inner numbers $2=2$, so AB is 2×2.	Check conformability and the order of the answer first.
$(1,1)$: row 1 $[2\quad 1]$, column 1 $\begin{bmatrix}4\\ 2\end{bmatrix}$: $2(4)+1(2)=10$.	Row 1 of A with column 1 of B.
$(1,2)$: row 1 with column 2 $\begin{bmatrix}-1\\ 5\end{bmatrix}$: $2(-1)+1(5)=3$.	Row 1 of A with column 2 of B.
$(2,1)$: row 2 $[0\quad 3]$ with column 1: $0(4)+3(2)=6$.	Row 2 of A with column 1 of B.
$(2,2)$: row 2 with column 2: $0(-1)+3(5)=15$.	Row 2 of A with column 2 of B.
$AB=\begin{bmatrix}10 & 3\\ 6 & 15\end{bmatrix}$.	Assemble the four entries.

Example 2 — scaffolded

At a café, two customers order from a menu of coffee, muffins and juice. Their orders and the prices are

$$Q=\begin{bmatrix}2 & 1 & 3\\ 1 & 2 & 0\end{bmatrix},\ P=\begin{bmatrix}5\\ 4\\ 6\end{bmatrix},$$

where the rows of Q are the two customers, the columns are coffee, muffins and juice, and P lists the prices ($5, $4, $6). Find QP and interpret it.

Working	Comment
Q is 2×3 and P is 3×1; inner numbers $3=3$, so QP is 2×1.	Columns of Q (items) match rows of P (prices).
Customer 1: $2(5)+1(4)+3(6)=10+4+18=32$.	Row 1 of Q with the price column.
Customer 2: $1(5)+2(4)+0(6)=5+8+0=13$.	Row 2 of Q with the price column.
$QP=\begin{bmatrix}32\\ 13\end{bmatrix}$.	A 2×1 column, one total per customer.

$\therefore$ customer 1 spends $32 and customer 2 spends $13.

Example 3 — unscaffolded

For $A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 1 & 4\end{bmatrix}$, find AB and BA, and state what they show.

Multiplying row-by-column,

$$AB=\begin{bmatrix}1&2\\3&0\end{bmatrix}\begin{bmatrix}2&1\\1&4\end{bmatrix}=\begin{bmatrix}1(2)+2(1)&1(1)+2(4)\\3(2)+0(1)&3(1)+0(4)\end{bmatrix}=\begin{bmatrix}4&9\\6&3\end{bmatrix},$$

while reversing the order gives

$$BA=\begin{bmatrix}2&1\\1&4\end{bmatrix}\begin{bmatrix}1&2\\3&0\end{bmatrix}=\begin{bmatrix}2(1)+1(3)&2(2)+1(0)\\1(1)+4(3)&1(2)+4(0)\end{bmatrix}=\begin{bmatrix}5&4\\13&2\end{bmatrix}.$$

$\therefore\ AB\neq BA$: matrix multiplication is not commutative, so the order of the factors matters.

Example 4 — unscaffolded

Let $M=\begin{bmatrix}3&1\\2&4\end{bmatrix}$ and let I be the 2×2 identity matrix. Verify that $MI=M$, and find M^2 and M^3.

Multiplying by the identity,

$$MI=\begin{bmatrix}3&1\\2&4\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}3(1)+1(0)&3(0)+1(1)\\2(1)+4(0)&2(0)+4(1)\end{bmatrix}=\begin{bmatrix}3&1\\2&4\end{bmatrix}=M,$$

as expected. Squaring,

$$M^2=M\times M=\begin{bmatrix}3&1\\2&4\end{bmatrix}\begin{bmatrix}3&1\\2&4\end{bmatrix}=\begin{bmatrix}9+2&3+4\\6+8&2+16\end{bmatrix}=\begin{bmatrix}11&7\\14&18\end{bmatrix},$$

and then

$$M^3=M^2\times M=\begin{bmatrix}11&7\\14&18\end{bmatrix}\begin{bmatrix}3&1\\2&4\end{bmatrix}=\begin{bmatrix}33+14&11+28\\42+36&14+72\end{bmatrix}=\begin{bmatrix}47&39\\78&86\end{bmatrix}.$$

$\therefore\ MI=M$, $M^2=\begin{bmatrix}11&7\\14&18\end{bmatrix}$ and $M^3=\begin{bmatrix}47&39\\78&86\end{bmatrix}$.

Example 5 — unscaffolded

A bakery records the number of pies, rolls and drinks sold on Monday and Tuesday in the matrix Q, and charges each item at a **standard** price and a **member** price given by the columns of P:

$$Q=\begin{bmatrix}8&5&10\\6&9&7\end{bmatrix},P=\begin{bmatrix}4&3\\3&2\\2&2\end{bmatrix}.$$

Find QP and explain what each entry represents.

Q is 2×3 and P is 3×2 (inner numbers $3=3$), so QP is 2×2. Computing each entry row-by-column,

$$QP=\begin{bmatrix}8(4)+5(3)+10(2)&8(3)+5(2)+10(2)\\6(4)+9(3)+7(2)&6(3)+9(2)+7(2)\end{bmatrix}=\begin{bmatrix}67&54\\65&50\end{bmatrix}.$$

The rows are the two days and the columns are the two pricing schemes. So at standard prices the bakery would take \$67 on Monday and \$65 on Tuesday; at member prices it would take \$54 on Monday and \$50 on Tuesday.

$\therefore\ QP=\begin{bmatrix}67&54\\65&50\end{bmatrix}$, giving each day’s takings under each pricing scheme.

Practice questions

Scaffolded

Q1. Let $A=\begin{bmatrix}1 & 3\\2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}4 & 2\\1 & 5\end{bmatrix}$. (a) State the order of the product AB. (b) Find the $(1,1)$ entry, using row 1 of A and column 1 of B. (c) Find the $(1,2)$ entry. (d) Find the $(2,1)$ entry. (e) Find the $(2,2)$ entry, and hence write down AB.

Q2. Four matrices have orders A: 2×3, B: 3×4, C: 3×2, D: 2×2. For each product below, state its order or write "not defined". (a) AB (b) BA (c) AC (d) CA (e) D^2 (f) DC

Q3. Let $r=\begin{bmatrix}2 & 1 & 3\end{bmatrix}$ and $c=\begin{bmatrix}4\\0\\5\end{bmatrix}$. (a) State the order of rc. (b) Find rc. (c) State the order of cr. (d) Find cr. (e) Comment on what parts (b) and (d) show about the order of multiplication.

Q4. Let $A=\begin{bmatrix}5 & 2\\3 & 7\end{bmatrix}$ and let $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. (a) Find AI. (b) Find IA. (c) What do your answers illustrate about the identity matrix?

Q5. Let $A=\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$. (a) Find $A^2=A\times A$. (b) Find $A^3=A^2\times A$.

Q6. A shop sells three items. Two customers buy the quantities in the rows of Q, and the item prices ($6, $5, $2) are the column P:

$$Q=\begin{bmatrix}3 & 2 & 1\\1 & 0 & 4\end{bmatrix},\ P=\begin{bmatrix}6\\5\\2\end{bmatrix}.$$

(a) Explain why QP is defined, and state its order. (b) Find the first entry of QP (customer 1's total). (c) Find the second entry (customer 2's total). (d) Write down QP and interpret it.

Unscaffolded

Q7. Find AB, where $A=\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\5 & 0\end{bmatrix}$.

Q8. For $A=\begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\2 & 1\end{bmatrix}$, find AB and BA, and state whether $AB=BA$.

Q9. Find the product $\begin{bmatrix}1 & 2 & 0\\3 & 1 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 3\\1 & 2\end{bmatrix}$.

Q10. Find the product $\begin{bmatrix}2 & 1\\0 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\3 & 1 & 4\end{bmatrix}$, and explain why it cannot possibly equal the product in the previous question.

Q11. For $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, find A^2 and A^3, and predict the form of A^n.

Q12. Let $A=\begin{bmatrix}4 & -1 & 2\\0 & 3 & 5\end{bmatrix}$. Let I_3 be the 3×3 identity and I_2 the 2×2 identity. (a) Find AI_3. (b) Find I_2A. (c) What do the results show?

Q13. Two students buy pens, notebooks and folders. Their quantities and the prices ($2, $8, $3) are

$$Q=\begin{bmatrix}5 & 3 & 2\\ 4 & 1 & 6\end{bmatrix}, P=\begin{bmatrix}2\\ 8\\ 3\end{bmatrix}.$$

Find $Q\,P$ and state how much each student spends.

Q14. A matrix A has order 2×3 and a matrix B has order 3×2. (a) State the order of $A\,B$ and the order of $B\,A$. (b) Explain why $A\,B$ and $B\,A$ can never be equal, even though both are defined.

Q15. Let $A=\begin{bmatrix}x & 1\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix}$. (a) Find $A\,B$ in terms of x. (b) Find the value of x for which the $(1,1)$ entry of $A\,B$ equals 9.

Q16. Two sports teams order shirts, shorts and socks. The quantities and the prices ($20, $12, $4) are

$$Q=\begin{bmatrix}11 & 11 & 22\\ 15 & 15 & 30\end{bmatrix}, P=\begin{bmatrix}20\\ 12\\ 4\end{bmatrix}.$$

Find the total cost for each team.

Q17. Let $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. (a) Find A^2. (b) Find A^4. (c) What do you notice about A^4?

Q18. Let $A=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$. (a) Find A^2 and A^3. (b) Describe the pattern in the top-left entry as the power increases.

Q19. A theatre sells adult, concession and child tickets at $40, $25 and $15. One night it sells the numbers in the row $N=\begin{bmatrix}120 & 45 & 30\end{bmatrix}$, and the prices form the column $P=\begin{bmatrix}40\\ 25\\ 15\end{bmatrix}$. (a) Find $N\,P$ and interpret it. (b) State the order of the product $P\,N$, and explain why it does not give the night's total takings.

Q20. A workshop makes chairs and tables. Each chair uses 3 kg of wood and 8 screws; each table uses 10 kg of wood and 20 screws. Wood costs $5 per kg and screws cost $1 each:

$$M=\begin{bmatrix}3 & 8\\ 10 & 20\end{bmatrix}, C=\begin{bmatrix}5\\ 1\end{bmatrix}.$$

(a) Find $M\,C$ and state the material cost of one chair and of one table. (b) The workshop makes 4 chairs and 2 tables. Using the quantity row $\begin{bmatrix}4 & 2\end{bmatrix}$, find the total material cost.

Answers

Q1. (a) 2×2; (b) 7; (c) 17; (d) 8; (e) $AB=\begin{bmatrix}7 & 17\\ 8 & 4\end{bmatrix}$. **Q2.** (a) 2×4; (b) not defined; (c) 2×2; (d) 3×3; (e) 2×2; (f) not defined. **Q3.** (a) 1×1; (b) $[23]$; (c) 3×3; (d) $\begin{bmatrix}8 & 4 & 12\\ 0 & 0 & 0\\ 10 & 5 & 15\end{bmatrix}$; (e) $rc \neq cr$ — the order of multiplication changes even the order of the answer. **Q4.** (a) $\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}$; (b) $\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}$; (c) $AI=IA=A$; the identity leaves a matrix unchanged. **Q5.** (a) $\begin{bmatrix}4 & 5\\ 0 & 9\end{bmatrix}$; (b) $\begin{bmatrix}8 & 19\\ 0 & 27\end{bmatrix}$. **Q6.** (a) columns of Q (3) match rows of P (3), order 2×1; (b) 30; (c) 14; (d) $QP=\begin{bmatrix}30\\ 14\end{bmatrix}$: customer 1 spends \$30, customer 2 spends \$14. **Q7.** $\begin{bmatrix}8 & 6\\ 22 & 4\end{bmatrix}$. **Q8.** $AB=\begin{bmatrix}2 & 8\\ 7 & 7\end{bmatrix}$, $BA=\begin{bmatrix}6 & 12\\ 5 & 3\end{bmatrix}$; $AB \neq BA$. **Q9.** $\begin{bmatrix}2 & 7\\ 10 & 14\end{bmatrix}$. **Q10.** $\begin{bmatrix}5 & 5 & 4\\ 9 & 3 & 12\\ 7 & 4 & 8\end{bmatrix}$; it is 3×3 whereas the earlier product is 2×2, so they differ in order and cannot be equal. **Q11.** $A^2=\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$, $A^3=\begin{bmatrix}1 & 6\\ 0 & 1\end{bmatrix}$; $A^n=\begin{bmatrix}1 & 2n\\ 0 & 1\end{bmatrix}$. **Q12.** (a) $\begin{bmatrix}4 & -1 & 2\\ 0 & 3 & 5\end{bmatrix}$; (b) $\begin{bmatrix}4 & -1 & 2\\ 0 & 3 & 5\end{bmatrix}$; (c) $AI_3=I_2A=A$ — the correctly sized identity leaves A unchanged on either side. **Q13.** $QP=\begin{bmatrix}40\\ 34\end{bmatrix}$: student 1 spends \$40, student 2 spends \$34. **Q14.** (a) AB is 2×2, BA is 3×3; (b) they have different orders, so they cannot be the same matrix. **Q15.** (a) $\begin{bmatrix}2x+1 & 4\\ 7 & 12\end{bmatrix}$; (b) $x=4$. **Q16.** $QP=\begin{bmatrix}440\\ 600\end{bmatrix}$: team 1 pays \$440, team 2 pays \$600. **Q17.** (a) $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$; (b) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; (c) $A^4=I$, the identity matrix. **Q18.** (a) $A^2=\begin{bmatrix}4 & 0\\ 3 & 1\end{bmatrix}$, $A^3=\begin{bmatrix}8 & 0\\ 7 & 1\end{bmatrix}$; (b) the top-left entry is 2^n $(2,4,8,\ldots)$. **Q19.** (a) $NP=[6375]$, the night's total takings of \$6375; (b) PN is 3×3 — it multiplies every price by every ticket count, so it does not represent a single total. **Q20.** (a) $MC=\begin{bmatrix}23\\ 70\end{bmatrix}$: a chair costs \$23 and a table \$70; (b) \$232.

Fully-worked solutions

Q1. A is 2×2 and B is 2×2, and the inner numbers match, so AB is 2×2. The entries are $(1,1)=1(4)+3(1)=7$, $(1,2)=1(2)+3(5)=17$, $(2,1)=2(4)+0(1)=8$ and $(2,2)=2(2)+0(5)=4$, so

$$AB=\begin{bmatrix}7 & 17\\ 8 & 4\end{bmatrix}.$$

Q2. Write each pair of orders side by side and check the inner numbers. (a) $(2\times3)(3\times4)$: inner $3=3$, product 2×4. (b) $(3\times4)(2\times3)$: inner $4\neq2$, **not defined**. (c) $(2\times3)(3\times2)$: inner $3=3$, product 2×2. (d) $(3\times2)(2\times3)$: inner $2=2$, product 3×3. (e) $D^2=(2\times2)(2\times2)$: inner $2=2$, product 2×2. (f) $(2\times2)(3\times2)$: inner $2\neq3$, **not defined**.

Q3. (a) r is 1×3 and c is 3×1; inner $3=3$, so rc is 1×1. (b) $rc=2(4)+1(0)+3(5)=8+0+15=23$, i.e. $[23]$. (c) c is 3×1 and r is 1×3; inner $1=1$, so cr is 3×3. (d) Each entry is (a row of c) times (a column of r), a single product:

$$cr=\begin{bmatrix}4(2) & 4(1) & 4(3)\\ 0(2) & 0(1) & 0(3)\\ 5(2) & 5(1) & 5(3)\end{bmatrix}=\begin{bmatrix}8 & 4 & 12\\ 0 & 0 & 0\\ 10 & 5 & 15\end{bmatrix}.$$

(e) $r\,c$ is a single number while $c\,r$ is a 3×3 matrix: reversing the order of multiplication changes even the order of the result, so matrix multiplication is not commutative.

Q4. (a) $A\,I=\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5(1)+2(0) & 5(0)+2(1)\\ 3(1)+7(0) & 3(0)+7(1)\end{bmatrix}=\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}$. (b) $I\,A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}=\begin{bmatrix}5 & 2\\ 3 & 7\end{bmatrix}$. (c) Both products equal A, so $A\,I=I\,A=A$: multiplying by the identity leaves a matrix unchanged, just as multiplying a number by 1 does.

Q5. (a) $A^2=\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}=\begin{bmatrix}2(2)+1(0) & 2(1)+1(3)\\ 0(2)+3(0) & 0(1)+3(3)\end{bmatrix}=\begin{bmatrix}4 & 5\\ 0 & 9\end{bmatrix}$. (b) $A^3=A^2\,A=\begin{bmatrix}4 & 5\\ 0 & 9\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}=\begin{bmatrix}4(2)+5(0) & 4(1)+5(3)\\ 0(2)+9(0) & 0(1)+9(3)\end{bmatrix}=\begin{bmatrix}8 & 19\\ 0 & 27\end{bmatrix}$.

Q6. (a) The columns of Q (the three items) match the rows of P (the three prices): $3=3$, so $Q\,P$ is defined and is 2×1. (b) Customer 1: $3(6)+2(5)+1(2)=18+10+2=30$. (c) Customer 2: $1(6)+0(5)+4(2)=6+0+8=14$. (d) $Q\,P=\begin{bmatrix}30\\ 14\end{bmatrix}$; customer 1 spends \$30 and customer 2 spends \$14.

Q7. $A\,B=\begin{bmatrix}3 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 5 & 0\end{bmatrix}=\begin{bmatrix}3(1)+1(5) & 3(2)+1(0)\\ 2(1)+4(5) & 2(2)+4(0)\end{bmatrix}=\begin{bmatrix}8 & 6\\ 22 & 4\end{bmatrix}$.

Q8. $A\,B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2(1)+0(2) & 2(4)+0(1)\\ 1(1)+3(2) & 1(4)+3(1)\end{bmatrix}=\begin{bmatrix}2 & 8\\ 7 & 7\end{bmatrix}$, while $B\,A=\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}=\begin{bmatrix}1(2)+4(1) & 1(0)+4(3)\\ 2(2)+1(1) & 2(0)+1(3)\end{bmatrix}=\begin{bmatrix}6 & 12\\ 5 & 3\end{bmatrix}$. Since the two results differ, $A\,B\neq B\,A$.

Q9. The left matrix is 2×3 and the right is 3×2, so the product is 2×2:

$$\begin{bmatrix}1 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1(2)+2(0)+0(1) & 1(1)+2(3)+0(2)\\ 3(2)+1(0)+4(1) & 3(1)+1(3)+4(2)\end{bmatrix}=\begin{bmatrix}2 & 7\\ 10 & 14\end{bmatrix}.$$

Q10. Now the left matrix is 3×2 and the right is 2×3, so the product is 3×3:

$$\begin{bmatrix}2 & 1\\ 0 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}=\begin{bmatrix}2(1)+1(3) & 2(2)+1(1) & 2(0)+1(4)\\ 0(1)+3(3) & 0(2)+3(1) & 0(0)+3(4)\\ 1(1)+2(3) & 1(2)+2(1) & 1(0)+2(4)\end{bmatrix}=\begin{bmatrix}5 & 5 & 4\\ 9 & 3 & 12\\ 7 & 4 & 8\end{bmatrix}.$$

This product is 3×3 while the previous one is 2×2; matrices of different orders can never be equal, another illustration that reversing the order of a matrix product changes the result.

Q11. $A^2=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1(1)+2(0) & 1(2)+2(1)\\ 0(1)+1(0) & 0(2)+1(1)\end{bmatrix}=\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$, and $A^3=A^2\,A=\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 6\\ 0 & 1\end{bmatrix}$. The off-diagonal entry increases by 2 each time, so $A^n=\begin{bmatrix}1 & 2n\\ 0 & 1\end{bmatrix}$.

Q12. (a) With A of order 2×3, the conformable identity on the right is I_3 (3×3), and $A\,I_3=\begin{bmatrix}4 & -1 & 2\\ 0 & 3 & 5\end{bmatrix}$ unchanged. (b) On the left the conformable identity is I_2 (2×2), and $I_2\,A=\begin{bmatrix}4 & -1 & 2\\ 0 & 3 & 5\end{bmatrix}$ unchanged. (c) $A\,I_3=I_2\,A=A$: an identity matrix of the correct order leaves A unchanged whether it multiplies from the left or the right, even though A is not square.

Q13. Q is 2×3 and P is 3×1, so $Q\,P$ is 2×1:

$$QP=\begin{bmatrix}5&3&2\\4&1&6\end{bmatrix}\begin{bmatrix}2\\8\\3\end{bmatrix}=\begin{bmatrix}5(2)+3(8)+2(3)\\4(2)+1(8)+6(3)\end{bmatrix}=\begin{bmatrix}40\\34\end{bmatrix}.$$

Student 1 spends $40 and student 2 spends $34.

Q14. (a) AB is $(2\times3)(3\times2)=2\times2$, and BA is $(3\times2)(2\times3)=3\times3$. (b) AB is a 2×2 matrix and BA is a 3×3 matrix; two matrices of different orders can never be equal, so $AB\neq BA$.

Q15. (a) $AB=\begin{bmatrix}x&1\\2&3\end{bmatrix}\begin{bmatrix}2&0\\1&4\end{bmatrix}=\begin{bmatrix}x(2)+1(1)&x(0)+1(4)\\2(2)+3(1)&2(0)+3(4)\end{bmatrix}=\begin{bmatrix}2x+1&4\\7&12\end{bmatrix}$. (b) The $(1,1)$ entry is $2x+1$; setting $2x+1=9$ gives $2x=8$, so $x=4$.

Q16. Q is 2×3 and P is 3×1, so QP is 2×1:

$$QP=\begin{bmatrix}11&11&22\\15&15&30\end{bmatrix}\begin{bmatrix}20\\12\\4\end{bmatrix}=\begin{bmatrix}11(20)+11(12)+22(4)\\15(20)+15(12)+30(4)\end{bmatrix}=\begin{bmatrix}440\\600\end{bmatrix}.$$

Team 1 pays $440 and team 2 pays $600.

Q17. (a) $A^2=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0(0)+(-1)(1)&0(-1)+(-1)(0)\\1(0)+0(1)&1(-1)+0(0)\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$. (b) $A^4=A^2A^2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. (c) $A^4=I$, the identity matrix, so the powers of A cycle and repeat.

Q18. (a) $A^2=\begin{bmatrix}2&0\\1&1\end{bmatrix}\begin{bmatrix}2&0\\1&1\end{bmatrix}=\begin{bmatrix}2(2)+0(1)&2(0)+0(1)\\1(2)+1(1)&1(0)+1(1)\end{bmatrix}=\begin{bmatrix}4&0\\3&1\end{bmatrix}$, and $A^3=A^2A=\begin{bmatrix}4&0\\3&1\end{bmatrix}\begin{bmatrix}2&0\\1&1\end{bmatrix}=\begin{bmatrix}8&0\\7&1\end{bmatrix}$. (b) The top-left entries are $2,4,8$ for A^1,A^2,A^3, doubling each time: the top-left entry of A^n is 2^n.

Q19. (a) $NP=\begin{bmatrix}120&45&30\end{bmatrix}\begin{bmatrix}40\\25\\15\end{bmatrix}=\begin{bmatrix}120(40)+45(25)+30(15)\end{bmatrix}=\begin{bmatrix}4800+1125+450\end{bmatrix}=\begin{bmatrix}6375\end{bmatrix}$, the total takings of $6375. (b) PN is $(3\times1)(1\times3)=3\times3$; it multiplies every price by every ticket count and produces a 3×3 table, not a single total, so it is NP (the row of quantities times the column of prices) that gives the takings.

Q20. (a) $MC=\begin{bmatrix}3&8\\10&20\end{bmatrix}\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}3(5)+8(1)\\10(5)+20(1)\end{bmatrix}=\begin{bmatrix}23\\70\end{bmatrix}$, so one chair costs $23 in materials and one table costs $70.

(b) The total is the quantity row times the cost column MC: $\begin{bmatrix}4&2\end{bmatrix}\begin{bmatrix}23\\70\end{bmatrix}=4(23)+2(70)=92+140=232$.

(Equivalently $\begin{bmatrix}4&2\end{bmatrix}M=\begin{bmatrix}32&72\end{bmatrix}$, and then $\begin{bmatrix}32&72\end{bmatrix}C=232$, using associativity.) The total material cost is $232.

Theory

For ordinary numbers, dividing by a number is the same as multiplying by its reciprocal, and every number except 0 has a reciprocal. Matrices have a parallel idea. There is no operation of matrix division, but a square matrix may have an **inverse** that plays the role of a reciprocal, and this lets us solve matrix equations. This section develops the inverse of a 2×2 matrix, the **determinant** that decides whether the inverse exists, and the use of the inverse to solve matrix equations.

The identity matrix.

The 2×2 **identity matrix** is

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

It behaves like the number 1 under multiplication: for every 2×2 matrix A,

$$A\,I=I\,A=A.$$

The identity is the matrix version of "multiplying by 1 changes nothing".

The determinant.

For the square matrix

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix},$$

the **determinant** is the single number

$$det(A)=a\,d-b\,c.$$

It is also written with vertical bars,

$$det(A)=/\begin{matrix}a & b\\ c & d\end{matrix}/=a\,d-b\,c,$$

that is, the product of the **leading diagonal** (a and d) minus the product of the **other diagonal** (b and c). The determinant may be positive, negative or zero, and — as we are about to see — its value decides whether A can be inverted.

The inverse of a 2×2 matrix.

The **inverse** of A is the matrix, written A^{-1}, for which

$$A\,A^{-1}=A^{-1}A=I.$$

For a 2×2 matrix there is a formula:

$$A^{-1}=\frac{1}{a\,d-b\,c}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\frac{1}{det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$$

To build A^{-1} from A:

- **swap** the two entries on the leading diagonal ($a\leftrightarrow d$);
- **change the sign** of the other two entries (b and c);
- **divide** every entry by the determinant $a\,d-b\,c$.

The scalar $\frac{1}{det(A)}$ in front may be kept out the front or multiplied through each entry; both are correct. It is always worth **checking** an inverse by confirming that $AA^{-1}=I$.

The condition for an inverse — singular matrices.

The formula divides by $det(A)=ad-bc$. Division by 0 is undefined, so:

- if $det(A)\neq 0$, the matrix A has an inverse and is called **non-singular** (or invertible);
- if $det(A)=0$, the matrix A has **no inverse** and is called **singular**.

Checking the determinant is therefore the first thing to do: a zero determinant means "stop — there is no inverse".

Using CAS. The determinant and inverse of a 2×2 matrix are quick to find by hand, and you should be fluent with the formulas above. For a 3×3 or larger matrix, or when the entries are awkward, enter the matrix into your CAS and read off $det(A)$ and A^{-1} directly. A CAS will report an error (or a "singular matrix" message) precisely when the determinant is zero.

Matrix equations.

A **matrix equation** has an unknown matrix. The standard form is

$$AX=B,$$

where A and B are known matrices and X is unknown. Provided A is non-singular, multiply **both sides on the left** by A^{-1}:

$$A^{-1}(AX)=A^{-1}B\Rightarrow(A^{-1}A)X=A^{-1}B\Rightarrow IX=A^{-1}B,$$

so

$$X=A^{-1}B.$$

Because matrix multiplication is **not commutative**, the side matters: A^{-1} must go on the **same side** of each product, here the left (**pre-multiplying**). Writing BA^{-1} would generally give a different — and wrong — answer. (For the different equation $XA=B$ you would instead **post-multiply**, giving $X=BA^{-1}$.) The order of B also fixes the order of X: if A is 2×2 and B is 2×1, then $X=A^{-1}B$ is 2×1.

Rearranging first, and when the method fails.

If an equation is not already in the form $AX=B$, rearrange it before any inverse is written down. For example, $AX+C=D$ becomes $AX=D-C$ — that step is ordinary matrix subtraction, and only then is A^{-1} used.

Everything above depends on A^{-1} existing. If $det(A)=0$ then A is singular, A^{-1} is undefined, and a matrix equation with that coefficient matrix cannot be solved by this method at all. So **check the determinant first**: it tells you immediately whether there is any point building an inverse. Once X is found, it is always worth substituting it back to confirm that AX really does equal B.

Worked examples

Example 1 — scaffolded

Find the determinant of each matrix and state whether it has an inverse.

$$A=\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}6 & 3\\ 4 & 2\end{bmatrix}, C=\begin{bmatrix}7 & -2\\ 1 & 3\end{bmatrix}.$$

Working	Comment
$det(A) = 5 \times 4 - 2 \times 3 = 20 - 6 = 14.$	$ad - bc$: leading diagonal minus other diagonal.
$det(A) = 14 \neq 0$, so A has an inverse.	A non-zero determinant means non-singular.
$det(B) = 6 \times 2 - 3 \times 4 = 12 - 12 = 0.$	Careful with the products before subtracting.
$det(B) = 0$, so B has **no** inverse (B is singular).	A zero determinant means singular.
$det(C) = 7 \times 3 - (-2) \times 1 = 21 - (-2) = 23.$	Subtracting a negative adds: $21 + 2$.
$det(C) = 23 \neq 0$, so C has an inverse.	

Example 2 — scaffolded

Find the inverse of $A = \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}$ and verify that $A A^{-1} = I$.

Working	Comment
$det(A) = 3 \times 2 - 2 \times 4 = 6 - 8 = -2.$	Find the determinant first.
$det(A) = -2 \neq 0$, so the inverse exists.	Non-singular.
Swap the leading diagonal and negate the other: $\begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix}$.	$a \leftrightarrow d$; change the sign of b and c.
$A^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix}.$	Divide each entry by $det(A) = -2$.
$A A^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix} = \begin{bmatrix} -3+4 & 3-3 \\ -4+4 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.	The product is I, confirming the inverse.

Example 3 — unscaffolded

Solve the matrix equation $A X = B$ for X, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Since A is square, first test whether it can be inverted:

$$det(A) = 2 \times 3 - 1 \times 1 = 6 - 1 = 5 \neq 0,$$

so A^{-1} exists. Building it,

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Pre-multiplying $A X = B$ by A^{-1} gives $X = A^{-1} B$, so

$$X = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3(5) + (-1)(10) \\ (-1)(5) + 2(10) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

$\therefore X = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

Example 4 — unscaffolded

Solve the matrix equation $X A = B$ for X, where $A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \end{bmatrix}$.

Here the unknown appears on the **left** of A, so A^{-1} must be applied on the right of each side (post-multiplying):

$$(XA)A^{-1}=BA^{-1}\Rightarrow X(AA^{-1})=BA^{-1}\Rightarrow X=BA^{-1}.$$

Testing A first,

$$det(A)=4\times4-3\times5=16-15=1\neq0,$$

so A^{-1} exists and

$$A^{-1}=\frac{1}{1}\begin{bmatrix}4 & -3\\ -5 & 4\end{bmatrix}=\begin{bmatrix}4 & -3\\ -5 & 4\end{bmatrix}.$$

Since B is 1×2 and A^{-1} is 2×2, the product BA^{-1} is 1×2:

$$X=BA^{-1}=\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}4 & -3\\ -5 & 4\end{bmatrix}=\begin{bmatrix}8-5 & -6+4\end{bmatrix}=\begin{bmatrix}3 & -2\end{bmatrix}.$$

Checking: $XA=\begin{bmatrix}3 & -2\end{bmatrix}\begin{bmatrix}4 & 3\\ 5 & 4\end{bmatrix}=\begin{bmatrix}12-10 & 9-8\end{bmatrix}=\begin{bmatrix}2 & 1\end{bmatrix}.$ ✓

$\therefore X=\begin{bmatrix}3 & -2\end{bmatrix}.$

Example 5 — unscaffolded

For

$$A=\begin{bmatrix}2 & 1 & 0\\ 1 & 3 & 1\\ 0 & 2 & 1\end{bmatrix},$$

use CAS to find $det(A)$ and A^{-1}, confirm that $AA^{-1}=I$, and hence solve $AX=B$, where $B=\begin{bmatrix}4\\ 10\\ 7\end{bmatrix}$.

The matrix is 3×3, so the by-hand formula does not apply. Entering A into CAS and asking for its determinant and inverse gives

$$det(A)=1,\ A^{-1}=\begin{bmatrix}1 & -1 & 1\\ -1 & 2 & -2\\ 2 & -4 & 5\end{bmatrix}.$$

The determinant is not zero, so A is non-singular and the inverse is genuine. Checking the first row of the product by hand, row 1 of A against the three columns of A^{-1} gives $2(1)+1(-1)+0(2)=1$, then $2(-1)+1(2)+0(-4)=0$, then $2(1)+1(-2)+0(5)=0$; the remaining rows behave the same way, so

$$AA^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I.$$

Pre-multiplying $AX=B$ by A^{-1} gives $X=A^{-1}B$, and since B is 3×1 the answer is 3×1:

$$X=\begin{bmatrix}1 & -1 & 1\\ -1 & 2 & -2\\ 2 & -4 & 5\end{bmatrix}\begin{bmatrix}4\\ 10\\ 7\end{bmatrix}=\begin{bmatrix}4-10+7\\ -4+20-14\\ 8-40+35\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}.$$

$\therefore det(A)=1$ and $X=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}.$

Practice questions

Scaffolded

Q1. For each matrix, find the determinant and state whether the matrix has an inverse. (a) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$

Q2. Find the inverse of $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ by completing the steps. (a) Find $det(A)$. (b) Swap the leading diagonal and negate the other diagonal. (c) Divide by the determinant to write A^{-1}.

Q3. Let $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$. (a) Find $det(A)$. (b) Write down A^{-1}. (c) Multiply out $A A^{-1}$ to confirm that it equals I.

Q4. The matrix $\begin{bmatrix} 2 & k \\ 3 & 6 \end{bmatrix}$ is singular. (a) Write an expression for its determinant in terms of k. (b) Set the determinant equal to 0 and solve for k.

Q5. Solve the matrix equation $A X = B$, where $A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 24 \\ 7 \end{bmatrix}$. (a) Find $det(A)$. (b) Write down A^{-1}. (c) Evaluate $X = A^{-1} B$.

Q6. The matrix X satisfies $X A = C$, where $A = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$. (a) Explain why X must be a 1×2 row matrix. (b) Find $det(A)$ and write down A^{-1}. (c) Explain why $X = C A^{-1}$ rather than $A^{-1} C$. (d) Evaluate X.

Unscaffolded

Q7. Find the determinant of each matrix. (a) $\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

Q8. Find the inverse of $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

Q9. Find the inverse of $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.

Q10. Show that $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ is singular, and explain what this tells you about A^{-1}.

Q11. Find the value(s) of k for which $\begin{bmatrix} k & 4 \\ 1 & k \end{bmatrix}$ has no inverse.

Q12. Find the value of p for which $\begin{bmatrix} p & 3 \\ 4 & 6 \end{bmatrix}$ is singular.

Q13. Solve $A X = B$ for X, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$.

Q14. Solve $A X = B$ for X, where $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$.

Q15. Solve the matrix equation $AX = B$, where $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$. (Here X is a 2×2 matrix.)

Q16. Given $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ and $AX = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$, find the column matrix X.

Q17. Solve $XA = B$ for the row matrix X, where $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \end{bmatrix}$.

Q18. Solve $AX + C = D$ for the column matrix X, where $A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$.

Q19. Show that the matrix equation $AX = B$, where $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$, cannot be solved by the method $X = A^{-1}B$, and explain why.

Q20. Use CAS to find the determinant and the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix},$$

and state whether A is singular or non-singular.

Answers

Q1. (a) 10, inverse exists; (b) 0, singular (no inverse); (c) 14, inverse exists; (d) -5, inverse exists. **Q2.** (a) 2; (b) $\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$; (c) $A^{-1}=\begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}$. **Q3.** (a) 1; (b) $A^{-1}=\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$; (c) $AA^{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$. **Q4.** (a) $12-3k$; (b) $k=4$. **Q5.** (a) 1; (b) $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$; (c) $X=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. **Q6.** (a) X is 1×2 so that XA is $(1\times 2)(2\times 2)=1\times 2$, matching C; (b) $det(A)=1$, $A^{-1}=\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$; (c) post-multiply, because A sits on the right of X; (d) $X=\begin{bmatrix} -5 & 9 \end{bmatrix}$. **Q7.** (a) 1; (b) -13; (c) 0. **Q8.** $\begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix}$. **Q9.** $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. **Q10.** $det(A)=12-12=0$, so A is singular and has no inverse. **Q11.** $k=2$ or $k=-2$. **Q12.** $p=2$. **Q13.** $X=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. **Q14.** $X=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. **Q15.** $X=\begin{bmatrix} -3 & 2 \\ 8 & -3 \end{bmatrix}$. **Q16.** $X=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. **Q17.** $X=\begin{bmatrix} 5 & -7 \end{bmatrix}$. **Q18.** $X=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. **Q19.** $det=2\times 3-1\times 6=0$, so A is singular, A^{-1} does not exist and the method cannot be applied. **Q20.** $det(A)=1$; $A^{-1}=\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$; non-singular.

Fully-worked solutions

Q1. Apply $det=ad-bc$ to each. (a) $3\times 4-1\times 2=12-2=10\neq 0$, so an inverse exists. (b) $2\times 3-6\times 1=6-6=0$, so the matrix is singular and has no inverse. (c) $5\times 4-(-3)(-2)=20-6=14\neq 0$, so an inverse exists. (d) $(-1)(-3)-2\times 4=3-8=-5\neq 0$, so an inverse exists.

Q2. (a) $det(A)=4\times 2-3\times 2=8-6=2$. (b) Swapping the leading diagonal and negating the other diagonal gives $\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$. (c) Dividing by $det(A)=2$,

$$A^{-1}=\frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}=\begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}.$$

Q3. (a) $det(A)=4\times 2-7\times 1=8-7=1$. (b) With determinant 1, $A^{-1}=\frac{1}{1}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}=\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$. (c) Multiplying,

$$AA^{-1}=\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}=\begin{bmatrix} 8-7 & -28+28 \\ 2-2 & -7+8 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I.$$

Q4. (a) The determinant is $2\times 6-k\times 3=12-3k$. (b) A singular matrix has determinant 0: $12-3k=0$, so $3k=12$ and $k=4$.

Q5. (a) $det(A)=7\times 1-3\times 2=7-6=1$. (b) $A^{-1}=\frac{1}{1}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}=\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$. (c) $X=A^{-1}B=\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 24 \\ 7 \end{bmatrix}=\begin{bmatrix} 24-21 \\ -48+49 \end{bmatrix}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Q6. (a) A is 2×2, so for XA to be defined X must have 2 columns, and the product then has as many rows as X. Since $C=XA$ is 1×2, the matrix X is 1×2 — a row matrix. (b) $det(A)=7\times 3-5\times 4=21-20=1$, so $A^{-1}=\frac{1}{1}\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}=\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}$. (c) In $XA=C$ the matrix A sits on the **right** of the unknown, so A^{-1} must be applied on the right of both sides: $(XA)A^{-1}=CA^{-1}$ gives $X(AA^{-1})=CA^{-1}$, that is $X=CA^{-1}$. Multiplication is not commutative, so $A^{-1}C$ is a different (here undefined) product. (d) $X = CA^{-1} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 3-8 & -5+14 \end{bmatrix} = \begin{bmatrix} -5 & 9 \end{bmatrix}$. (Check: $\begin{bmatrix} -5 & 9 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -35+36 & -25+27 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$.)

Q7. (a) $7 \times 3 - 4 \times 5 = 21 - 20 = 1$. (b) $(-2)(-1) - 5 \times 3 = 2 - 15 = -13$. (c) $6 \times 2 - (-4)(-3) = 12 - 12 = 0$.

Q8. $det = 3 \times 2 - 4 \times 1 = 6 - 4 = 2$, so

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix}.$$

Q9. $det = 3 \times 3 - (-4)(-2) = 9 - 8 = 1$, so

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Q10. $det(A) = 4 \times 3 - 6 \times 2 = 12 - 12 = 0$. Because the determinant is zero, A is **singular**: the inverse formula would require dividing by 0, so A^{-1} does not exist.

Q11. The matrix has no inverse when its determinant is zero: $k \times k - 4 \times 1 = k^2 - 4 = 0$, so $k^2 = 4$ and $k = 2$ or $k = -2$.

Q12. Singular means determinant zero: $p \times 6 - 3 \times 4 = 6p - 12 = 0$, so $6p = 12$ and $p = 2$.

Q13. $det(A) = 1 \times 5 - 2 \times 3 = 5 - 6 = -1$, so $A^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. Then

$$X = A^{-1}B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} -20+22 \\ 12-11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Q14. $det(A) = 3 \times 2 - 1 \times 5 = 6 - 5 = 1$, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$. Then

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 18-16 \\ -45+48 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Q15. $det(A) = 5 \times 3 - 2 \times 7 = 15 - 14 = 1$, so $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$. Pre-multiplying B,

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 3-6 & 12-10 \\ -7+15 & -28+25 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 8 & -3 \end{bmatrix}.$$

Q16. $det(A) = 5 \times 1 - 2 \times 2 = 5 - 4 = 1$, so $A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$. Then

$$X = A^{-1}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 9-8 \\ -18+20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Q17. The unknown sits on the left of A, so post-multiply by A^{-1}: $XA = B$ gives $X = BA^{-1}$. Here $det(A) = 5 \times 2 - 3 \times 3 = 10 - 9 = 1$, so $A^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ and

$$X = BA^{-1} = \begin{bmatrix} 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 8-3 & -12+5 \end{bmatrix} = \begin{bmatrix} 5 & -7 \end{bmatrix}.$$

(Check: $\begin{bmatrix}5 & -7\end{bmatrix}\begin{bmatrix}5 & 3\\3 & 2\end{bmatrix}=\begin{bmatrix}25-21 & 15-14\end{bmatrix}=\begin{bmatrix}4 & 1\end{bmatrix}$.)

Q18. Subtract C from both sides first:

$$AX=D-C=\begin{bmatrix}6\\15\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}5\\13\end{bmatrix}.$$

Here $det(A)=4\times3-(-1)(2)=12+2=14\neq0$, so $A^{-1}=\frac{1}{14}\begin{bmatrix}3 & 1\\-2 & 4\end{bmatrix}$ and

$$X=A^{-1}\begin{bmatrix}5\\13\end{bmatrix}=\frac{1}{14}\begin{bmatrix}3 & 1\\-2 & 4\end{bmatrix}\begin{bmatrix}5\\13\end{bmatrix}=\frac{1}{14}\begin{bmatrix}15+13\\-10+52\end{bmatrix}=\frac{1}{14}\begin{bmatrix}28\\42\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}.$$

(Check: $AX=\begin{bmatrix}8-3\\4+9\end{bmatrix}=\begin{bmatrix}5\\13\end{bmatrix}$, and adding C gives D.)

Q19. The coefficient matrix is $A=\begin{bmatrix}2 & 1\\6 & 3\end{bmatrix}$, with $det(A)=2\times3-1\times6=6-6=0$. The inverse formula divides every entry by the determinant, and division by 0 is undefined, so A is **singular** and A^{-1} does not exist. Since the method $X=A^{-1}B$ requires A^{-1}, it cannot be applied to this equation at all — no amount of arithmetic with B will help.

Q20. Entering the 3×3 matrix into CAS and reading off the determinant and the inverse gives

$$det(A)=1,\ A^{-1}=\begin{bmatrix}-24 & 18 & 5\\20 & -15 & -4\\-5 & 4 & 1\end{bmatrix}.$$

Because $det(A)=1\neq0$, the matrix A is **non-singular** — which is why the CAS returned an inverse rather than an error. (Check on the first row: $1(-24)+2(20)+3(-5)=-24+40-15=1$ and $1(18)+2(-15)+3(4)=18-30+12=0$, as required for the first row of I.)

Theory

Some matrices record nothing more than a **yes/no** relationship — present or absent, connected or not connected, on or off. Every entry is then either 0 or 1. This section looks at three such matrices and what their **products** and **powers** can tell us: **binary matrices**, **permutation matrices**, and **communication matrices**.

Binary matrices.

A **binary matrix** is a matrix in which every entry is either 0 or 1 (these are sometimes called 0–1 matrices). For example,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

is a binary matrix, whereas a matrix containing a 2, or a -1, is not. Binary matrices are the natural way to store information that is simply *true or false* for each object or for each pair of objects. The identity matrix, every permutation matrix, and every communication matrix below are binary.

Permutation matrices.

A **permutation matrix** is a **square** binary matrix with **exactly one 1 in each row and exactly one 1 in each column** (all other entries 0). Equivalently, it is the identity matrix with its rows (or columns) rearranged. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a permutation matrix, but $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not (its second column is all zeros and its first column has two 1s).

Multiplying by a permutation matrix **reorders** the entries of another matrix:

- PA (with P on the **left**) reorders the **rows** of A;
- AP (with P on the **right**) reorders the **columns** of A.

With the P above and the column matrix $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$$P\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \\ a \end{bmatrix},$$

so the list (a, b, c) is reordered to (b, c, a). The 1 in row 1 sits in column 2, so the new first entry is the old second entry, and so on. This is how permutation matrices **model a reordering** — reshuffling a list, rotating a roster, or relabelling positions.

Permutation matrices have three useful properties.

- The **product** of two permutation matrices is again a permutation matrix (doing one reordering after another is still a reordering).
- A permutation matrix is **invertible**, and its inverse is its **transpose**:
  $$P^{-1} = P^T,$$
  because reversing a reordering simply undoes it, and $PP^T = I$.

- Repeating the **same** reordering eventually returns to the start: for some whole number n, $P^n = I$. The reordering has then come full circle.

Because matrix multiplication is **not commutative**, PA and AP generally differ, and the matrices must be **conformable** for the product to exist.

Communication matrices.

A directed *who-can-reach-whom* relationship is recorded in a **communication matrix**. Label the rows and the columns with the **same** list of people, and set

$$c_{ij} = \begin{cases} 1 & \text{if person } i \text{ can send a message directly to person } j, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix is square and binary. Its diagonal entries are 0 (a person does not message themselves), and it need **not** be symmetric — person i being able to send to person j does not force j to be able to send to i. Each **row sum** is the number of people that person can contact directly (an *out* count); each **column sum** is the number of people who can contact that person directly (an *in* count). For the three people $1, 2, 3$ with the direct links $1 \to 2$, $2 \to 1$, $2 \to 3$ and $3 \to 1$,

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Two-step links and the square of the matrix.

Even with no direct link, a message can travel from i to j through one **relay** person k, along the route $i \to k \to j$. This uses two links, so it is a **two-step** (redirected) communication link. The number of two-step links from i to j is exactly the (i, j) entry of C^2:

$$\left(C^2\right)_{ij} = \sum_k c_{ik} c_{kj},$$

because the product $c_{ik} c_{kj}$ equals 1 precisely when **both** $i \to k$ and $k \to j$ exist — that is, when k is a relay — and adding over every possible relay k counts every two-step route. For the matrix above,

$$C^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and, for instance, the entry $\left(C^2\right)_{13} = 1$ counts the single two-step route $1 \to 2 \to 3$. A **diagonal** entry $\left(C^2\right)_{ii}$ counts two-step links that return to the sender ($i \to k \to i$), i.e. round trips; here $\left(C^2\right)_{11} = 1$ is the round trip $1 \to 2 \to 1$.

To count the ways a message can pass from i to j in **one step or two steps**, add the two matrices:

$$C + C^2,$$

whose (i, j) entry is (direct links) + (two-step links) from i to j. The **sum of all the entries** of $C + C^2$ is the total number of one- or two-step communication routes in the whole network. (Higher powers extend the idea: C^3 counts three-step links, but this course concentrates on one and two steps.)

Using CAS. General Mathematics is technology-active. In practice you enter C into your CAS, then read off C^2 and $C + C^2$ directly. Learn the meaning of each entry so that you can interpret the output and check that it is reasonable.

Worked examples

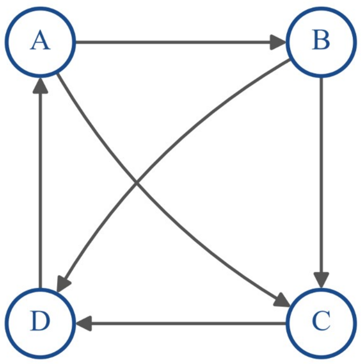
Example 1 — scaffolded

The four members of a relay team currently run in the order given by the column matrix $T=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ (runner 1 first, runner 4 last). The coach tries a new order by forming PT, where $P=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$. (a) Confirm P is a permutation matrix. (b) Find the new order PT. (c) Find P^T and verify that $P^TP=I$.

Working	Comment
Each row of P has a single 1: rows have their 1 in columns 3, 1, 4, 2.	One 1 in each **row**.
Each column has a single 1: columns have their 1 in rows 2, 4, 1, 3.	One 1 in each **column**, and every entry is 0 or 1 — so P is a permutation matrix.
$PT=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}3\\1\\4\\2\end{bmatrix}$.	Row 1 of P picks out the 3rd entry, row 2 the 1st, and so on.
New order: runner 3, then 1, then 4, then 2.	P on the left reorders the rows (the running order).
$P^T=\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix}$.	Transpose: swap rows and columns.
$P^TP=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$.	So $P^{-1}=P^T$: multiplying by P^T restores the original order.

Example 2 — scaffolded

Four friends Anna (A), Ben (B), Cara (C) and Dev (D) can send a text message directly as shown by the network below (an arrow $X\rightarrow Y$ means X can text Y directly).



(a) Write down the communication matrix C, with rows and columns in the order A, B, C, D.

(b) Find C^2 and explain what the entry in row A, column D means. (c) Find $C+C^2$ and state in how many ways Anna can get a message to Dev in one or two steps.

Working	Comment
Direct links: $A \to B$, $A \to C$, $B \to C$, $B \to D$, $C \to D$, $D \to A$.	Read the arrows off the diagram.
$C=\begin{bmatrix}0&1&1&0\\0&0&1&1\\0&0&0&1\\1&0&0&0\end{bmatrix}$.	Row = sender, column = receiver; a 1 for each arrow.
$C^2=\begin{bmatrix}0&0&1&2\\1&0&0&1\\1&0&0&0\\0&1&1&0\end{bmatrix}$.	Square the matrix (by hand or on CAS).
Row A, column D entry is 2.	There are **two** two-step links from Anna to Dev: $A \to B \to D$ and $A \to C \to D$.
$C+C^2=\begin{bmatrix}0&1&2&2\\1&0&1&2\\1&0&0&1\\1&1&1&0\end{bmatrix}$.	Add direct links to two-step links, entry by entry.
Row A, column D of $C+C^2$ is $0+2=2$.	Anna can reach Dev in **2 ways** using one or two steps (no direct link, so both are two-step).

Example 3 — unscaffolded

Four volunteers take the four shifts on a help desk. Each week the roster advances by one shift, a reordering modelled by the permutation matrix $P=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$. Find P^2, P^3 and P^4, and hence state after how many weeks the roster first returns to its starting arrangement. State P^{-1}.

Squaring and cubing the matrix gives

$$P^2=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}, P^3=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix},$$

each of which is again a permutation matrix (advancing the roster by two and by three weeks). Multiplying once more,

$$P^4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I,$$

so after four weeks the roster is back exactly where it began. Because $P^{-1}=P^T$ for any permutation matrix (and here $P^3=P^{-1}$, since $P^4=I$),

$$P^{-1}=P^T=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}.$$

$\therefore$ the roster first returns to its starting arrangement after **4 weeks**, and $P^{-1} = P^T$ (which equals P^3).

Example 4 — unscaffolded

Five staff members $1, 2, 3, 4, 5$ can pass a message directly according to the links $1 \to 2$, $1 \to 4$, $2 \to 3$, $3 \to 4$, $3 \to 5$, $4 \to 1$ and $5 \to 2$. Write the communication matrix C, find C^2, and use it to find the total number of two-step links and a route by which staff member 3 can get a message to staff member 2 in two steps. How many one- or two-step communication routes are there altogether?

With rows and columns in the order $1, 2, 3, 4, 5$,

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The total number of two-step links is the sum of all entries of C^2, which is 9. (Two of these lie on the diagonal — the round trips $1 \to 4 \to 1$ and $4 \to 1 \to 4$.) The entry in row 3, column 2 of C^2 is 1, and the relay must be a person k with $3 \to k$ and $k \to 2$; the only choice is $k = 5$, giving the route $3 \to 5 \to 2$. The number of one- or two-step routes altogether is the sum of all entries of $C + C^2$; since C has 7 direct links and C^2 has 9 two-step links, this total is $7 + 9 = 16$.

$\therefore$ there are **9** two-step links, staff member 3 reaches staff member 2 via $3 \to 5 \to 2$, and there are **16** one- or two-step communication routes in all.

Practice questions

Scaffolded

Q1. State which of the following are binary matrices. $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Q2. State which of the following are permutation matrices, giving a reason for each that is not. $N_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $N_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $N_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Q3. Let $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$. (a) Find PX. (b) Describe, in words, the reordering that P carries out. (c) Find P^T and verify that $PP^T = I$.

Q4. In an office, Wafa (W), Xin (X), Yuki (Y) and Zoe (Z) can email directly as follows: $W \to X$, $W \to Y$, $X \to Z$, $Y \to X$ and $Z \to W$. (a) Write the communication matrix C with rows and columns in the order W, X, Y, Z. (b) Find C^2. (c) Interpret the entry in row W, column X of C^2. (d) Find $C + C^2$ and hence state in how many ways a message can pass from Wafa to Xin in one or two steps.

Q5. A communication matrix for four people $1,2,3,4$ is $C=\begin{bmatrix}0&1&0&1\\1&0&1&0\\0&0&0&1\\1&1&0&0\end{bmatrix}$. (a) How many direct communication links are there in total? (b) Find C^2. (c) How many two-step links can person 2 make? (d) List the routes by which person 2 can send a message to person 4 in exactly two steps.

Q6. The morning takings (in hundreds of dollars) at three market stalls are recorded, in stall order, by the row matrix $A=\begin{bmatrix}3&5&2\end{bmatrix}$. The stalls are relabelled by forming AP, where $P=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$. (a) Find AP. (b) Explain why the entries are reordered rather than the rows. (c) Find P^2 and hence AP^2.

Unscaffolded

Q7. Classify the matrix $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ as fully as possible (binary, permutation, both or neither), giving reasons.

Q8. Find the permutation matrix P for which $P\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}4\\1\\3\\2\end{bmatrix}$, and check your answer by multiplying.

Q9. For $P=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$, find P^2 and P^3, state the smallest whole number n for which $P^n=I$, and write down P^{-1}.

Q10. In a group of five people $1,2,3,4,5$, the direct messaging links are $1\to2$, $1\to3$, $1\to4$, $2\to3$, $3\to5$, $4\to5$ and $5\to1$. (a) Write the communication matrix C. (b) Which person can send a message directly to the most others? (c) Find C^2 and state the total number of two-step links. (d) Interpret the entry in row 1, column 5 of C^2. (e) Find $C+C^2$ and hence state which people person 2 cannot reach in one or two steps.

Q11. A communication matrix for four people A, B, C, D is $C=\begin{bmatrix}0&1&1&0\\0&0&1&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$. Find C^2, and hence state how many two-step links person A has and the route of each.

Q12. For the communication matrix $C=\begin{bmatrix}0&1&0&1\\0&0&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}$, find $C+C^2$ and state the total number of ways a message can be sent in one or two steps.

Q13. Four friends sit in seats numbered 1 to 4, in the order given by $S=\begin{bmatrix}10\\20\\30\\40\end{bmatrix}$ (the numbers are their player numbers).

Each round they rotate by forming $P\,S$, where $P=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$. Find the seating order after 1, 2 and 3 rounds, and state after how many rounds everyone is first back in their original seat.

Q14. Let $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ and $B=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ be permutation matrices. Find $A\,B$ and $B\,A$, confirm that each is a permutation matrix, and state whether $A\,B=B\,A$.

Q15. For $P=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{bmatrix}$, write down P^T and, by computing $P\,P^T$, verify that $P^{-1}=P^T$.

Q16. Four students form a study group in which every messaging link is **two-way** (if i can message j, then j can message i). The links are 1–2, 1–3, 2–3 and 2–4. (a) Write the communication matrix C and explain why it is symmetric. (b) Find C^2. (c) Interpret the entry in row 2, column 2 of C^2.

Q17. The daily rainfall (mm) at four gauges, in gauge order, is $A=\begin{bmatrix}12&7&9&4\end{bmatrix}$. The gauges are relabelled by forming $A\,P$, where $P=\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix}$. Find $A\,P$ and $A\,P^2$, and describe the reordering produced by $A\,P^2$.

Q18. In a small team, the direct messaging links between members A, B, C, D are $A\rightarrow B$, $A\rightarrow D$, $B\rightarrow C$, $C\rightarrow A$ and $C\rightarrow D$. (a) Write the communication matrix C. (b) Explain why member D lies in an all-zero row, and what that means. (c) Person A wants to get a message to person C but has no direct link. Find C^2 and use it to name the relay and the two-step route.

Q19. Five sensors relay a signal directly according to $C=\begin{bmatrix}0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&1\\0&0&1&0&0\end{bmatrix}$. (a) Find the total number of two-step links. (b) By finding $C+C^2$, decide whether every sensor can pass a signal to every other sensor in at most two steps. Justify your answer.

Q20. In a network of four people $1,2,3,4$, the direct messaging links are $1\rightarrow 2$, $2\rightarrow 3$, $2\rightarrow 4$, $3\rightarrow 1$, $4\rightarrow 2$ and $4\rightarrow 3$. (a) Write the communication matrix C and state the total number of direct links. (b) Find C^2 and the total number of two-step links. (c) Find $C+C^2$ and hence the total number of one- or two-step communication routes. (d) Interpret the entry in row 2, column 3 of C^2.

Answers

Q1. Binary: M_1 and M_3 (every entry 0 or 1). M_2 has a 2 and M_4 has a -1. **Q2.** Permutation: N_1 and N_3. N_2 is not (row 2 has two 1s and row 3 is all zero); N_4 is not (column 2 all zero, column 1 has two 1s). **Q3.** (a) $\begin{bmatrix}8\\5\\2\end{bmatrix}$; (b) swaps the first two entries, leaving the third; (c) $P^T=P$ and $P\,P^T=I$. **Q4.** (a) $\begin{bmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{bmatrix}$; (b) $\begin{bmatrix}0&1&0&1\\1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}$; (c) one two-step link W to X, the route $W\to Y\to X$; (d) $\begin{bmatrix}0&2&1&1\\1&0&0&1\\0&1&0&1\\1&1&1&0\end{bmatrix}$, so 2 ways ($W\to X$ and $W\to Y\to X$). **Q5.** (a) 7; (b) $\begin{bmatrix}2&1&1&0\\0&1&0&2\\1&1&0&0\\1&1&1&1\end{bmatrix}$; (c) $0+1+0+2=3$; (d) $2\to1\to4$ and $2\to3\to4$. **Q6.** (a) $\begin{bmatrix}5&2&3\end{bmatrix}$; (b) P on the right reorders columns, and A has a single row; (c) $P^2=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $A\,P^2=\begin{bmatrix}2&3&5\end{bmatrix}$. **Q7.** Both: every entry is 0 or 1 (binary), and there is exactly one 1 in each row and column (permutation). **Q8.** $P=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{bmatrix}$. **Q9.** $P^2=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $P^3=I$, smallest $n=3$, $P^{-1}=P^T=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$. **Q10.** (a) $\begin{bmatrix}0&1&1&1&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&0&1\\1&0&0&0&0\end{bmatrix}$; (b) person 1 (reaches 3 others); (c) C^2 has sum 9; (d) two two-step links $1\to3\to5$ and $1\to4\to5$; (e) person 2 cannot reach persons 1 or 4. **Q11.** $C^2=\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}$; A has 3 two-step links: $A\to C\to A$, $A\to B\to C$, $A\to B\to D$. **Q12.** $C+C^2=\begin{bmatrix}0&2&1&1\\1&0&1&1\\1&2&0&2\\0&1&1&0\end{bmatrix}$; total $=14$. **Q13.** After 1: $(20,30,40,10)$; after 2: $(30,40,10,20)$; after 3: $(40,10,20,30)$; back in original seats after 4 rounds. **Q14.** $A\,B=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, $B\,A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$; both are permutation matrices; $A\,B\neq B\,A$. **Q15.** $P^T=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix}$ and $P\,P^T=I$, so $P^{-1}=P^T$. **Q16.** (a) $\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&0\\0&1&0&0\end{bmatrix}$, symmetric because every link is two-way ($c_{ij}=c_{ji}$); (b)

$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$; (c) $(C^2)_{22} = 3$: person 2 is directly connected to 3 people (1, 3, 4), giving 3 there-and-back links. **Q17.** $A\,P = [9 \quad 12 \quad 4 \quad 7]$, $A\,P^2 = [4 \quad 9 \quad 7 \quad 12]$; $A\,P^2$ applies the reordering twice. **Q18.** (a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; (b) D's row is all zero because D sends to no one; (c) $C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, relay B, route $A \to B \to C$. **Q19.** (a) 9; (b) no — for example, row 1 of $C + C^2 = [0 \quad 1 \quad 2 \quad 0 \quad 1]$ has a 0 in column 4, so sensor 1 cannot reach sensor 4 in at most two steps. **Q20.** (a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, 6 direct links; (b) $C^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, 9 two-step links; (c) $C + C^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$, total 15; (d) one two-step link from 2 to 3, the route $2 \to 4 \to 3$.

Fully-worked solutions

Q1. A matrix is binary when every entry is 0 or 1. M_1 (all entries 0 or 1) and M_3 (all entries 0 or 1) qualify. M_2 contains a 2 and M_4 contains a -1, so neither is binary. **Binary: M_1 and M_3.**

Q2. A permutation matrix is square and binary with exactly one 1 in each row and each column. N_1: rows have their 1 in columns 2, 1, 3 and columns have their 1 in rows 2, 1, 3 — a permutation matrix. N_3: rows have their 1 in columns 3, 1, 2 and columns in rows 2, 3, 1 — a permutation matrix. N_2 fails (row 2 has two 1s and row 3 is all zero). N_4 fails (column 2 is all zero while column 1 has two 1s). **Permutation matrices: N_1 and N_3.**

Q3. (a) $P\,X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$. (b) Row 1 of P picks the 2nd entry and row 2 picks the 1st, so P **swaps the first two entries** and leaves the third unchanged. (c) $P^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P$ (the matrix is symmetric), and $P\,P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$, confirming $P^{-1} = P^T$.

Q4. (a) With rows/columns W, X, Y, Z and a 1 for each arrow, $C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. (b) $C^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. (c) The row W, column X entry is 1: there is one two-step link from Wafa to Xin, namely $W \to Y \to X$. (d)

$C+C^2=\begin{bmatrix}0&2&1&1\\1&0&0&1\\0&1&0&1\\1&1&1&0\end{bmatrix}$; the row W, column X entry is $1+1=2$, so a message can pass from Wafa to Xin in **2 ways** — directly ($W\to X$) or via Yuki ($W\to Y\to X$).

Q5. (a) The number of direct links is the sum of all entries of C, which is 7. (b) $C^2=\begin{bmatrix}2&1&1&0\\0&1&0&2\\1&1&0&0\\1&1&1&1\end{bmatrix}$. (c) Person 2's two-step links are the row-2 entries of C^2: $0+1+0+2=3$. (d) The row 2, column 4 entry is 2. The relay k must satisfy $2\to k$ and $k\to 4$; person 2 sends to 1 and 3, and both $1\to 4$ and $3\to 4$ hold, giving the routes $2\to 1\to 4$ and $2\to 3\to 4$.

Q6. (a) $AP=\begin{bmatrix}3&5&2\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}5&2&3\end{bmatrix}$. (b) With P on the **right** the columns of A are reordered; since A is a single row, this simply reorders its three entries. (c) $P^2=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, so $AP^2=\begin{bmatrix}3&5&2\end{bmatrix}P^2=\begin{bmatrix}2&3&5\end{bmatrix}$ (equivalently, applying the reordering to $\begin{bmatrix}5&2&3\end{bmatrix}$ again).

Q7. Every entry is 0 or 1, so the matrix is **binary**. There is exactly one 1 in each row (in columns 2, 3, 1) and exactly one 1 in each column (in rows 3, 1, 2), so it is also a **permutation matrix**. Hence it is **both**.

Q8. We need each row of P to pick out the required entry. The result is $(4,1,3,2)$, so row 1 must select the 4th entry, row 2 the 1st, row 3 the 3rd and row 4 the 2nd: $P=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{bmatrix}$. Check: $P\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}4\\1\\3\\2\end{bmatrix}$, as required.

Q9. $P^2=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ and $P^3=P^2P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$. The smallest whole number with $P^n=I$ is $n=3$. Since $P^3=I$, we have $P^{-1}=P^2$, and for a permutation matrix $P^{-1}=P^T=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ (which indeed equals P^2).

Q10. (a) $C=\begin{bmatrix}0&1&1&1&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&0&1\\1&0&0&0&0\end{bmatrix}$. (b) The row sums (out counts) are $3,1,1,1,1$, so **person 1** sends directly to the most others (three of them). (c) $C^2=\begin{bmatrix}0&0&1&0&2\\0&0&0&0&1\\1&0&0&0&0\\1&0&0&0&0\\0&1&1&1&0\end{bmatrix}$; the sum of its entries is 9 two-step links. (d) The row 1, column 5 entry is 2 — there are two two-step routes from person 1 to person 5: $1\to 3\to 5$ and $1\to 4\to 5$. (e)

$C+C^2=\begin{bmatrix}0&1&2&1&2\\0&0&1&0&1\\1&0&0&0&1\\1&0&0&0&1\\1&1&1&1&0\end{bmatrix}$. Row 2 is $\begin{bmatrix}0&0&1&0&1\end{bmatrix}$, with 0 in columns 1 and 4, so **person 2 cannot reach persons 1 or 4** in one or two steps.

Q11. $C^2=\begin{bmatrix}1&0&1&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}$. Row A is $\begin{bmatrix}1&0&1&1\end{bmatrix}$, summing to 3 two-step links. Reading each entry with its relay: $(C^2)_{AA}=1$ is the round trip $A\to C\to A$; $(C^2)_{AC}=1$ is $A\to B\to C$; and $(C^2)_{AD}=1$ is $A\to B\to D$. So A has **3 two-step links**: $A\to C\to A$, $A\to B\to C$ and $A\to B\to D$.

Q12. $C^2=\begin{bmatrix}0&1&1&0\\1&0&0&1\\0&2&0&1\\0&0&1&0\end{bmatrix}$, so $C+C^2=\begin{bmatrix}0&2&1&1\\1&0&1&1\\1&2&0&2\\0&1&1&0\end{bmatrix}$. The total number of one- or two-step routes is the sum of all its entries. This equals the 6 direct links plus the 8 two-step links, i.e. $6+8=14$.

Q13. $PS=\begin{bmatrix}20\\30\\40\\10\end{bmatrix}$ after round 1; $P^2S=\begin{bmatrix}30\\40\\10\\20\end{bmatrix}$ after round 2; $P^3S=\begin{bmatrix}40\\10\\20\\30\end{bmatrix}$ after round 3. Since P is a 4-cycle, $P^4=I$, so $P^4S=S$ and everyone is first back in their **original seat after 4 rounds**.

Q14. $AB=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ and $BA=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$. Each has exactly one 1 in every row and column, so both are permutation matrices — confirming that the product of two permutation matrices is a permutation matrix. However $AB\neq BA$, so multiplication of permutation matrices is **not commutative**.

Q15. $P^T=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix}$. Then $PP^T=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$, so P^T is the inverse of P; that is, $P^{-1}=P^T$.

Q16. (a) A two-way link means $c_{ij}=c_{ji}$, so $C=\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&0\\0&1&0&0\end{bmatrix}$ is **symmetric** ($C=C^T$). (b) $C^2=\begin{bmatrix}2&1&1&1\\1&3&1&0\\1&1&2&1\\1&0&1&1\end{bmatrix}$. (c) The row 2, column 2 entry is 3. Because links are two-way, each person that person 2 is joined to gives a there-and-back route $2\to k\to 2$; person 2 is directly joined to 1, 3 and 4, so $(C^2)_{22}=3$ counts those three round trips (equivalently, person 2 has three direct links).

Q17. $AP=\begin{bmatrix}12&7&9&4\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix}=\begin{bmatrix}9&12&4&7\end{bmatrix}$. Squaring, $P^2=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$, so $AP^2=\begin{bmatrix}4&9&7&12\end{bmatrix}$. Applying P twice reorders the four readings so that the 1st and 4th are swapped and the 2nd and 3rd are swapped.

Q18. (a) $C=\begin{bmatrix}0&1&0&1\\0&0&1&0\\1&0&0&1\\0&0&0&0\end{bmatrix}$. (b) D's row is all zero because D sends a message directly to nobody, so D can only ever **receive** messages, never start one. (c) $C^2=\begin{bmatrix}0&0&1&0\\1&0&0&1\\0&1&0&1\\0&0&0&0\end{bmatrix}$. The row A, column C entry is 1, and the relay k needs $A\to k$ and $k\to C$; person A sends to B and D, and only $B\to C$ holds, so the relay is **B** and the route is $A\to B\to C$.

Q19. (a) $C^2=\begin{bmatrix}0&0&2&0&0\\0&0&0&1&0\\1&0&0&0&1\\0&1&1&0&1\\0&0&0&1&0\end{bmatrix}$, whose entries sum to 9 two-step links. (b) $C+C^2=\begin{bmatrix}0&1&2&0&1\\0&0&1&1&0\\1&0&0&1&1\\1&1&1&0&2\\0&0&1&1&0\end{bmatrix}$. Several off-diagonal entries are 0 — for example the row 1, column 4 entry is 0, so **sensor 1 cannot get a signal to sensor 4 in at most two steps**. Hence not every sensor can reach every other sensor within two steps.

Q20. (a) $C=\begin{bmatrix}0&1&0&0\\0&0&1&1\\1&0&0&0\\0&1&1&0\end{bmatrix}$; the entries sum to 6 direct links. (b) $C^2=\begin{bmatrix}0&0&1&1\\1&1&1&0\\0&1&0&0\\1&0&1&1\end{bmatrix}$, with entry sum 9 two-step links. (c) $C+C^2=\begin{bmatrix}0&1&1&1\\1&1&2&1\\1&1&0&0\\1&1&2&1\end{bmatrix}$; the sum of all entries is $6+9=15$ one- or two-step routes. (d) The row 2, column 3 entry of C^2 is 1: there is one two-step link from person 2 to person 3, the route $2\to4\to3$ (person 2 already reaches person 3 directly as well, but that is a one-step link).

Theory

In a **round-robin competition** every competitor plays every other competitor exactly once. If each game produces a definite winner (no draws), the results can be recorded in a square matrix and then used to **rank** the competitors — even when the win–loss records do not, on their own, separate them.

The dominance matrix.

Suppose n competitors play a round robin. The **dominance matrix** D is the $n \times n$ matrix whose rows and columns are labelled by the competitors in the same order, with

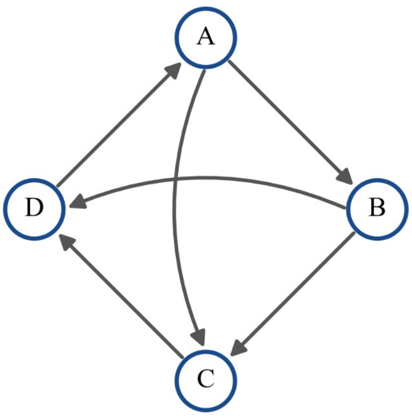
$$d_{ij} = \begin{cases} 1 & \text{if competitor } i \text{ beat competitor } j, \\ 0 & \text{otherwise.} \end{cases}$$

Because a competitor does not play itself, every entry on the leading diagonal is 0; and because each game has exactly one winner, for any two different competitors exactly one of d_{ij}, d_{ji} equals 1. So D is a **binary** (0–1) matrix, it is **square**, its diagonal is all zeros, and it is generally **not symmetric**.

Consider four teams A, B, C, D in which

- A beat B and C;
- B beat C and D;
- C beat D;
- D beat A.

The “who-beat-whom” network below records the same results — each arrow points **from the winner to the loser**.



an arrow points from the winner to the loser

Reading the wins out of each row (in the order A, B, C, D) gives the dominance matrix

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

One-step dominance.

The **one-step dominance** of a competitor is the number of opponents it beat directly. This is simply the **row sum** of D (each 1 in the row is one direct win):

$$\text{one-step} = A:2, B:2, C:1, D:1.$$

Ranking by one-step dominance alone would place A and B equal first, and C and D equal third — the win–loss records do not separate them. It also treats every win as equally valuable, which is unfair: beating a **strong** opponent should count for more than beating a weak one.

Two-step dominance.

To reward the *quality* of a win, we also count **two-step dominance**: the number of ways a competitor beat someone who in turn beat someone. These two-step wins are counted by the matrix D^2. The entry (i, j) of D^2 is

$$(D^2)_{ij} = \sum_k d_{ik} d_{kj},$$

which counts the competitors k that i beat **and** who beat j — that is, the number of two-step chains $i \to k \to j$. For our example,

$$D^2 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

For instance $(D^2)_{AD} = 2$: team A dominates D in two steps in **two** different ways — A beat B and B beat D ($A \to B \to D$), and A beat C and C beat D ($A \to C \to D$). The **two-step dominance** of each competitor is the row sum of D^2:

$$\text{two-step} = A:3, B:2, C:1, D:2.$$

A useful check: the two-step dominance of a competitor equals the **sum of the one-step scores of the teams it beat**. Team D beat only A, and A has one-step score 2, so D's two-step score is 2 — its single win was over a strong team, so it counts well.

Total dominance and the ranking.

The **total dominance matrix** combines direct wins and two-step wins:

$$T = D + D^2 = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Competitors are ranked by the **row sums of** T, from largest to smallest:

$$\text{total} = A:5, B:4, C:2, D:3 \Rightarrow A, B, D, C.$$

The one-step ties are now broken. A finishes above B (both had two direct wins, but A's wins carried more two-step weight), and D finishes above C — D's single win, over the strong team A, was worth more than C's win over the weak team D. So **two-step dominance can break ties in one-step dominance**.

Points to note.

- Always list the competitors in the **same order** down the rows and across the columns, and keep that order for D, D^2 and T.
- D^2 is found by matrix multiplication $D \times D$; in practice you enter D into your CAS and compute D^2 and $D + D^2$ directly, then read off the row sums.
- Ranking by dominance is a model, not a law: it can still leave genuine ties. In a **three-competitor** round robin in which X beat Y, Y beat Z and Z beat X, all three scores are equal and the competitors cannot be separated. A cycle inside a larger competition does **not** by itself force a tie — in the example above A beat B, B beat D

and D beat A, yet those three finish first, second and third. Competitors are left level exactly when their row sums of T come out equal. Symmetric results — such as the three-competitor cycle above — guarantee that, but equal totals can also arise by coincidence between competitors with quite different records.

Worked examples

Example 1 — scaffolded

Four netball teams — Aces (A), Bolts (B), Comets (C) and Darts (D) — played a round robin. Aces beat all three other teams; Bolts beat Comets and Darts; and Comets beat Darts. Construct the dominance matrix and rank the teams by one-step dominance.

Working	Comment
Order the teams A, B, C, D down the rows and across the columns.	Same order for rows and columns.
Row A: Aces beat $B, C, D \Rightarrow [0111]$.	Put a 1 in each column Aces beat.
Row B: Bolts beat $C, D \Rightarrow [0011]$.	B did not beat A (Aces beat Bolts).
Row C: Comets beat $D \Rightarrow [0001]$.	One direct win.
Row D: Darts beat nobody $\Rightarrow [0000]$.	A winless team gives a zero row.
$D=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$	Diagonal all zeros; a valid dominance matrix.
One-step dominance = row sums: $A:3, B:2, C:1, D:0$.	Number of direct wins for each team.
Ranking: A, B, C, D.	Largest row sum first.

Here the results are **transitive**, so one-step dominance ranks the teams outright and no tie-breaking is needed.

Example 2 — scaffolded

Four debating teams A, B, C, D played a round robin with results: B beat A; A beat C and D; B beat C; D beat B; and C beat D. Rank the teams by total dominance.

Working	Comment
Row A: beat $C, D \Rightarrow [0011]$.	A lost to B.
Row B: beat $A, C \Rightarrow [1010]$.	B lost to D.
Row C: beat $D \Rightarrow [0001]$.	
Row D: beat $B \Rightarrow [0100]$.	
$D=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}$	One-step: $A:2, B:2, C:1, D:1$ — A and B are tied.
$D^2=\begin{bmatrix}0&1&0&1\\0&0&1&2\\0&1&0&0\\1&0&1&0\end{bmatrix}$	Compute $D \times D$ (on CAS). Two-step: $A:2, B:3, C:1, D:2$.

Working	Comment
$D+D^2=\begin{bmatrix}0&1&1&2\\1&0&2&2\\0&1&0&1\\1&1&1&0\end{bmatrix}$	Add corresponding entries.
Total dominance = row sums: $A:4,B:5,C:2,D:3$.	
Ranking: B,A,D,C.	Two-step dominance breaks the A–B tie.

Although A and B each won two games, B is ranked first: B beat A head-to-head and its wins carried more two-step weight.

Example 3 — unscaffolded

Five teams A, B, C, D, E played a round robin. The results were: A beat B, C and D; C beat B; B beat D and E; C beat D and E; E beat A and D. Rank the teams by total dominance.

Listing the teams in the order A,B,C,D,E, each row records that team's direct wins:

$$D=\begin{bmatrix}0&1&1&1&0\\0&0&0&1&1\\0&1&0&1&1\\0&0&0&0&0\\1&0&0&1&0\end{bmatrix}.$$

The one-step dominances (row sums) are $A:3$, $B:2$, $C:3$, $D:0$, $E:2$, so A and C are tied at the top. Squaring,

$$D^2=\begin{bmatrix}0&1&0&2&2\\1&0&0&1&0\\1&0&0&2&1\\0&0&0&0&0\\0&1&1&1&0\end{bmatrix},$$

with two-step dominances $A:5$, $B:2$, $C:4$, $D:0$, $E:3$. The total dominance matrix is

$$D+D^2=\begin{bmatrix}0&2&1&3&2\\1&0&0&2&1\\1&1&0&3&2\\0&0&0&0&0\\1&1&1&2&0\end{bmatrix},$$

with row sums $A:8$, $B:4$, $C:7$, $D:0$, $E:5$. The two-step scores break both one-step ties: A edges out C at the top, and E climbs above B because E beat the top team A, a high-value win.

$\therefore$ the ranking is A,C,E,B,D.

Example 4 — unscaffolded

Four players P, Q, R, S played a round robin: P beat Q and S; Q beat R; R beat P and S; and S beat Q. Rank the players by total dominance, and explain the value of the entry $\left(D^2\right)_{RQ}$.

In the order P,Q,R,S,

$$D=\begin{bmatrix}0&1&0&1\\0&0&1&0\\1&0&0&1\\0&1&0&0\end{bmatrix}, D^2=\begin{bmatrix}0&1&1&0\\1&0&0&1\\0&2&0&1\\0&0&1&0\end{bmatrix}.$$

One-step dominance: $P:2$, $Q:1$, $R:2$, $S:1$ (P and R tied). Two-step dominance (row sums of D^2): $P:2$, $Q:2$, $R:3$, $S:1$. Total dominance (row sums of $D+D^2$): $P:4$, $Q:3$, $R:5$, $S:2$.

The entry $(D^2)_{RQ}=2$ means R dominates Q in two steps in two different ways: R beat P and P beat Q ($R\to P\to Q$), and R beat S and S beat Q ($R\to S\to Q$). This two-step strength lifts R above P, whom it beat head-to-head.

$\therefore$ the ranking is R, P, Q, S.

Practice questions

Scaffolded

Q1. Three teams — Red, Blue and Green — played a round robin. Red beat Blue and Green, and Blue beat Green. (a) Construct the 3×3 dominance matrix D (order Red, Blue, Green). (b) Find the one-step dominance of each team. (c) Rank the teams.

Q2. A round robin between four players A, B, C, D has dominance matrix

$$D=\begin{bmatrix}0&1&0&1\\0&0&0&1\\1&1&0&1\\0&0&0&0\end{bmatrix}.$$

(a) Find the one-step dominance (row sum) of each player. (b) State which player was undefeated and which player was winless. (c) Rank the players by one-step dominance.

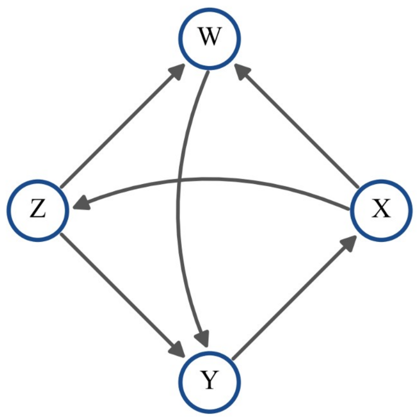
Q3. For the four-team dominance matrix

$$D=\begin{bmatrix}0&1&1&0\\0&0&1&0\\0&0&0&1\\1&1&0&0\end{bmatrix},$$

(a) find the one-step dominance of each team (A, B, C, D); (b) compute D^2; (c) hence find the two-step dominance of each team.

Q4. Using the matrix D from **Q3**: (a) find the total dominance matrix $D+D^2$; (b) find the total dominance (row sum) of each team; (c) rank the four teams, and explain how two-step dominance separates the two pairs of teams that were tied on one-step dominance.

Q5. The network below shows the results of a round robin between four teams W, X, Y, Z (an arrow points from the winner to the loser).



an arrow points from the winner to the loser

(a) Construct the dominance matrix D (order W, X, Y, Z).
(b) Find the one-step dominance of each team.
(c) The total dominance matrix has row sums $W:2$, $X:5$, $Y:3$, $Z:4$. Use these to rank the four teams.

Q6. Four badminton players J, K, L, M played a round robin: J beat L and M; K beat J; L beat K and M; and M beat K. (a) Construct D and find the one-step dominance of each player, showing that two players are tied for first. (b) Compute D^2 and the two-step dominance of each player. (c) Rank the players by total dominance, and identify which player is placed first by the tie-break.

Unscaffolded

Q7. In a round robin between Ava, Ben, Cody and Dan: Ava beat Ben; Ben beat Cody and Dan; Cody beat Ava and Dan; and Dan beat Ava. Construct the dominance matrix and rank the four players by total dominance.

Q8. Four fencers, numbered 1, 2, 3, 4, have dominance matrix

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Compute D^2 and hence find the two-step dominance of each fencer.

Q9. Using the matrix D from **Q8**, find the total dominance matrix $D + D^2$, and rank the four fencers.

Q10. Four netball teams — Aces, Blitz, Chargers and Dukes — played a round robin: Aces beat Blitz and Dukes; Blitz beat Chargers and Dukes; Chargers beat Aces; and Dukes beat Chargers. Show that Aces and Blitz are tied on one-step dominance, and use total dominance to rank all four teams.

Q11. Four squash players R, S, T, U played a round robin, giving

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Explain, in terms of the games played, exactly what the entry $(D^2)_{UR} = 2$ represents. (b) Explain what the entry $(D^2)_{ST} = 0$ tells you about S's results.

Q12. Three chess players X, Y, Z played a round robin: X beat Y, Y beat Z, and Z beat X. Find the one-step and two-step dominance of each player, and explain why a dominance ranking cannot separate the three players.

Q13. Five teams A, B, C, D, E played a round robin: A beat B, D and E; B beat C and D; C beat A, D and E; D beat E; and E beat B. Construct the dominance matrix and rank the five teams by total dominance.

Q14. The grid below records a round robin between five teams (a W in row i, column j means team i beat team j).

	North	South	East	West	Central
North	–	W	W	W	W
South	L	–	W	W	L
East	L	L	–	W	W
West	L	L	L	–	W
Central	L	W	L	L	–

Construct the dominance matrix and rank the five teams by total dominance.

Q15. Five cyclists finished a round-robin series as follows: cyclist 1 beat cyclists 2, 3 and 5; cyclist 2 beat cyclists 3, 4 and 5; cyclist 3 beat cyclists 4 and 5; cyclist 4 beat cyclist 1; and cyclist 5 beat cyclist 4. Rank the cyclists by total dominance as far as possible, and state which two cyclists the ranking fails to separate.

Q16. Let D be the dominance matrix of a round-robin competition with no draws. (a) Explain why every entry on the leading diagonal of D is 0. (b) Explain why, for two different competitors i and j, exactly one of d_{ij} and d_{ji} equals 1. (c) Is D necessarily symmetric? Justify your answer.

Q17. In a round-robin competition, explain why ranking the competitors by total dominance ($D+D^2$) can give a different order from ranking them by one-step dominance (the row sums of D) alone. Refer to the idea of the "quality" of a win.

Q18. In a chess round robin, Nadia beat Omar, Priya, Quinn and Rafi; Omar beat Priya, Quinn and Rafi; Priya beat Quinn and Rafi; and Quinn beat Rafi. Rank the five players by total dominance, showing all matrices used, and explain why total dominance gives exactly the same order as one-step dominance here.

Q19. A four-team round robin (A, B, C, D) has

$$D=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&1\\1&1&0&0\end{bmatrix}, D^2=\begin{bmatrix}0&0&1&0\\1&0&0&1\\1&2&0&0\\0&1&1&0\end{bmatrix}.$$

(a) Find the total dominance of each team and rank the four teams. (b) State which pairs of teams are tied on one-step dominance, and hence whose relative ranking is decided only by two-step dominance.

Q20. Five teams A, B, C, D, E played a round robin: A beat B, C and D; B beat C and E; C beat D and E; D beat B and E; and E beat A. (a) Find the total dominance of each team. (b) Rank the teams as far as possible, and explain why three of them cannot be separated even after using two-step dominance.

Answers

Q1. (a) $\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$; (b) Red 2, Blue 1, Green 0; (c) Red, Blue, Green. **Q2.** (a) $A:2, B:1, C:3, D:0$; (b) C undefeated, D winless; (c) C, A, B, D. **Q3.** (a) $A:2, B:1, C:1, D:2$; (b) $D^2 = \begin{bmatrix}0&0&1&1\\0&0&0&1\\1&1&0&0\\0&1&2&0\end{bmatrix}$; (c) $A:2, B:1, C:2, D:3$. **Q4.** (a) $\begin{bmatrix}0&1&2&1\\0&0&1&1\\1&1&0&1\\1&2&2&0\end{bmatrix}$; (b) $A:4, B:2, C:3, D:5$; (c) D, A, C, B — A and D were tied on 2 direct wins and B and C on 1; the two-step scores separate both pairs ($D:3$ ahead of $A:2$, and $C:2$ ahead of $B:1$). **Q5.** (a) $\begin{bmatrix}0&0&1&0\\1&0&0&1\\0&1&0&0\\1&0&1&0\end{bmatrix}$; (b) $W:1, X:2, Y:1, Z:2$; (c) X, Z, Y, W. **Q6.** (a) $J:2, K:1, L:2, M:1$ (J, L tied); (b) $D^2 = \begin{bmatrix}0&2&0&1\\0&0&1&1\\1&1&0&0\\1&0&0&0\end{bmatrix}$, two-step $J:3, K:2, L:2, M:1$; (c) J, L, K, M — J first. **Q7.** One-step Ava 1, Ben 2, Cody 2, Dan 1; total dominance Ava 3, Ben 5, Cody 4, Dan 2; ranking Ben, Cody, Ava, Dan. **Q8.** $D^2 = \begin{bmatrix}0&1&0&2\\1&0&0&0\\1&0&0&1\\0&1&1&0\end{bmatrix}$; two-step $1:3, 2:1, 3:2, 4:2$. **Q9.** $D + D^2 = \begin{bmatrix}0&2&1&2\\1&0&0&1\\1&1&0&2\\1&1&1&0\end{bmatrix}$; totals $1:5, 2:2, 3:4, 4:3$; ranking $1, 3, 4, 2$. **Q10.** One-step Aces 2, Blitz 2 (tied), Chargers 1, Dukes 1; totals Aces 5, Blitz 4, Chargers 3, Dukes 2; ranking Aces, Blitz, Chargers, Dukes. **Q11.** (a) U dominated R in two steps in 2 ways: $U \to S \to R$ and $U \to T \to R$. (b) S beat only R, and R did not beat T, so there is no chain $S \to k \to T$: S has no two-step dominance over T. **Q12.** One-step all 1, two-step all 1, total all 2; each player won one game and lost one, so with only three players the three records are identical and cannot be separated. **Q13.** One-step $A:3, B:2, C:3, D:1, E:1$; totals $A:7, B:6, C:8, D:2, E:3$; ranking C, A, B, E, D. **Q14.** $D = \begin{bmatrix}0&1&1&1&1\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\\0&1&0&0&0\end{bmatrix}$; totals North 10, South 5, East 4, West 2, Central 3; ranking North, South, East, Central, West. **Q15.** Total dominance $1:9, 2:7, 3:4, 4:4, 5:2$; ranking 1, 2, then 3 and 4 level, then 5 — cyclists 3 and 4 cannot be separated. **Q16.** (a) No competitor plays itself, so $d_{ii} = 0$. (b) Each game has exactly one winner. (c) No: if i beat j then $d_{ij} = 1$ but $d_{ji} = 0$, so $D \neq D^T$ in general. **Q17.** One-step counts direct wins equally; two-step rewards beating strong opponents, so total dominance can reorder teams with equal (or fewer) direct wins. **Q18.** One-step Nadia 4, Omar 3, Priya 2, Quinn 1, Rafi 0; totals Nadia 10, Omar 6, Priya 3, Quinn 1, Rafi 0; ranking Nadia, Omar, Priya, Quinn, Rafi — the results are transitive, so the two-step scores fall in the same order and no tie-break is needed. **Q19.** (a) $A:2, B:3, C:5, D:4$; ranking C, D, B, A. (b) C, D tied one-step and A, B tied one-step, so both orders are decided by two-step dominance. **Q20.** (a) $A:9, B:5, C:5, D:5, E:4$. (b) A first, E last; B, C and D all score 5, so they cannot be separated — each lost to A, each beat E, and each won one and lost one of the games among themselves, so their records are interchangeable.

Fully-worked solutions

Q1. Listing Red, Blue, Green, Red beat both others (row $[0\,1\,1]$), Blue beat Green (row $[0\,0\,1]$) and Green beat nobody (row $[0\,0\,0]$):

$$D = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Row sums give one-step dominance Red 2, Blue 1, Green 0, so the ranking is Red, Blue, Green. (The results are transitive, so no tie-break is needed.)

Q2. The row sums of D are $A: 0+1+0+1=2$, $B: 0+0+0+1=1$, $C: 1+1+0+1=3$, $D: 0$. Player C won all three games (undefeated, row sum 3) and player D won none (winless, zero row). Ordering the row sums from largest gives C, A, B, D.

Q3. (a) Row sums: $A: 2$, $B: 1$, $C: 1$, $D: 2$. (b) Multiplying $D \times D$,

$$D^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

(c) The row sums of D^2 give two-step dominance $A: 2$, $B: 1$, $C: 2$, $D: 3$.

Q4. (a) Adding the matrices from Q3,

$$D + D^2 = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix}.$$

(b) Row sums: $A: 4$, $B: 2$, $C: 3$, $D: 5$. (c) Ranking D, A, C, B. One-step dominance left two ties — A and D on 2 direct wins, B and C on 1 — and the two-step scores ($A: 2$, $B: 1$, $C: 2$, $D: 3$) separate both. D's wins carried more two-step weight than A's, so D is placed first; likewise C is placed above B.

Q5. Reading the arrows out of each node, W beat Y, X beat W and Z, Y beat X, and Z beat W and Y:

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

One-step dominance (row sums) is $W: 1$, $X: 2$, $Y: 1$, $Z: 2$, so X and Z are tied for first and W and Y are tied for third. Using the given total dominance row sums $X: 5$, $Z: 4$, $Y: 3$, $W: 2$, the ranking is X, Z, Y, W — both one-step ties are broken by the two-step scores.

Q6. (a) In the order J, K, L, M,

$$D = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

with one-step dominance $J: 2$, $K: 1$, $L: 2$, $M: 1$ — J and L are tied for first. (b) Squaring,

$$D^2=\begin{bmatrix}0&2&0&1\\0&0&1&1\\1&1&0&0\\1&0&0&0\end{bmatrix},$$

so two-step dominance is $J:3$, $K:2$, $L:2$, $M:1$. (c) Total dominance (row sums of $D+D^2$) is $J:5$, $K:3$, $L:4$, $M:2$, giving the ranking J, L, K, M. The tie-break places J first: J beat L head-to-head, and J's wins carried more two-step weight.

Q7. In the order Ava, Ben, Cody, Dan,

$$D=\begin{bmatrix}0&1&0&0\\0&0&1&1\\1&0&0&1\\1&0&0&0\end{bmatrix}, D^2=\begin{bmatrix}0&0&1&1\\2&0&0&1\\1&1&0&0\\0&1&0&0\end{bmatrix}.$$

One-step dominance is Ava 1, Ben 2, Cody 2, Dan 1 (Ben and Cody tied). Two-step dominance is Ava 2, Ben 3, Cody 2, Dan 1, so total dominance is Ava 3, Ben 5, Cody 4, Dan 2. Ranking: **Ben, Cody, Ava, Dan.**

Q8. Multiplying $D\times D$,

$$D^2=\begin{bmatrix}0&1&0&2\\1&0&0&0\\1&0&0&1\\0&1&1&0\end{bmatrix},$$

so the two-step dominance (row sums) is fencer $1:3$, $2:1$, $3:2$, $4:2$.

Q9. Adding,

$$D+D^2=\begin{bmatrix}0&2&1&2\\1&0&0&1\\1&1&0&2\\1&1&1&0\end{bmatrix},$$

with total dominance $1:5$, $2:2$, $3:4$, $4:3$. Ranking: $1, 3, 4, 2$. (Fencers 1 and 3 were tied on one-step dominance; the two-step scores place 1 ahead.)

Q10. In the order Aces, Blitz, Chargers, Dukes,

$$D=\begin{bmatrix}0&1&0&1\\0&0&1&1\\1&0&0&0\\0&0&1&0\end{bmatrix},$$

with one-step dominance Aces 2, Blitz 2 (tied), Chargers 1, Dukes 1. Then

$$D^2=\begin{bmatrix}0&0&2&1\\1&0&1&0\\0&1&0&1\\1&0&0&0\end{bmatrix},$$

giving two-step dominance Aces 3, Blitz 2, Chargers 2, Dukes 1, and total dominance Aces 5, Blitz 4, Chargers 3, Dukes 2. Ranking: **Aces, Blitz, Chargers, Dukes** — Aces beat Blitz head-to-head and also had the higher two-step score, so the tie-break goes their way.

Q11. (a) The entry $(D^2)_{UR}$ counts the two-step chains $U \to k \to R$, that is, the players k that U beat and who in turn beat R. From D, U beat S and T, and both S and T beat R, giving two chains: $U \to S \to R$ and $U \to T \to R$. So U dominates R in two steps in exactly two different ways. (b) $(D^2)_{ST}$ counts the chains $S \to k \to T$. Reading row S of D, S's only win was over R, and R did not beat T (in fact T beat R), so no such chain exists and the entry is 0: S dominates T neither directly nor in two steps.

Q12. In the order X, Y, Z,

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Each player won exactly one game, so one-step dominance is 1 for all three; each row of D^2 also sums to 1, so two-step dominance is 1 for all three, and total dominance is 2 for all three. With only three players, the cycle (X beat Y, Y beat Z, Z beat X) makes the three records identical — each won one game, and each player's single win was over the player who beat the third — so no player is more dominant than the others and the ranking cannot separate them.

Q13. In the order A, B, C, D, E,

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

with one-step dominance $A:3$, $B:2$, $C:3$, $D:1$, $E:1$ (A and C tied). Then

$$D^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

so two-step dominance is $A:4$, $B:4$, $C:5$, $D:1$, $E:2$. Total dominance is $A:7$, $B:6$, $C:8$, $D:2$, $E:3$. Ranking: C, A, B, E, D — C beat A head-to-head and has the higher two-step score, so it takes first place.

Q14. Reading a W as a 1 and an L or dash as 0, in the order North, South, East, West, Central,

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

One-step dominance is North 4, South 2, East 2, West 1, Central 1; two-step dominance is North 6, South 3, East 2, West 1, Central 2; total dominance is North 10, South 5, East 4, West 2, Central 3. Ranking: **North, South, East, Central, West.** (Both one-step ties are broken by the two-step scores: South above East, and Central above West.)

Q15. In the order $1, 2, 3, 4, 5$,

$$D = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, D^2 = \begin{bmatrix} 0 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

One-step dominance is $1:3$, $2:3$, $3:2$, $4:1$, $5:1$; two-step dominance is $1:6$, $2:4$, $3:2$, $4:3$, $5:1$; total dominance is $1:9$, $2:7$, $3:4$, $4:4$, $5:2$. Ranking: 1**, then** 2**, then** 3 **and** 4 **level, then** 5**.** The ranking fails to separate cyclists 3 and 4: cyclist 3 won two games to cyclist 4's one, but cyclist 4's single win was over cyclist 1 — the strongest rider — which is worth more at the two-step stage, and the two effects cancel exactly. Note that these two are *not* in a cycle with each other (3 beat 4); a tie can arise whenever the totals happen to coincide.

Q16. (a) The entry d_{ii} would record competitor i beating itself, but no competitor plays itself, so $d_{ii}=0$ for every i. (b) Competitors i and j play exactly once and there are no draws, so exactly one of them wins: if i won then $d_{ij}=1$ and $d_{ji}=0$; if j won then $d_{ij}=0$ and $d_{ji}=1$. In either case exactly one of the two entries is 1. (c) No. From part (b), whenever i beat j we have $d_{ij}=1$ but $d_{ji}=0$, so $d_{ij}\neq d_{ji}$; a dominance matrix is therefore not symmetric (except in the trivial case of no games).

Q17. One-step dominance simply counts each competitor's direct wins and treats every win as worth the same amount, so competitors with the same number of wins are tied. Two-step dominance (the row sums of D^2) counts wins over opponents who themselves won games, so it measures the *quality* of a competitor's wins — beating a strong opponent contributes more than beating a weak one. Because total dominance $D+D^2$ adds this quality measure to the count of direct wins, it can promote a competitor who beat strong opponents above one who beat only weak opponents, giving a different order from one-step dominance alone.

Q18. In the order Nadia, Omar, Priya, Quinn, Rafi,

$$D=\begin{bmatrix}0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix},D^2=\begin{bmatrix}0&0&1&2&3\\0&0&0&1&2\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},D+D^2=\begin{bmatrix}0&1&2&3&4\\0&0&1&2&3\\0&0&0&1&2\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}.$$

One-step dominance is Nadia 4, Omar 3, Priya 2, Quinn 1, Rafi 0; two-step dominance is Nadia 6, Omar 3, Priya 1, Quinn 0, Rafi 0; total dominance is Nadia 10, Omar 6, Priya 3, Quinn 1, Rafi 0. Ranking: **Nadia, Omar, Priya, Quinn, Rafi.** The results are **transitive** — every player beat exactly those below them in this order — so the set of players beaten by a stronger player contains the set beaten by a weaker one. Each player's two-step score is the sum of the one-step scores of the players they beat, so those scores fall in the same order as the one-step scores, and adding them cannot reorder anybody. There are no ties to break here, so total dominance simply confirms the win–loss order.

Q19. (a) Adding the given matrices,

$$D+D^2=\begin{bmatrix}0&1&1&0\\1&0&1&1\\2&2&0&1\\1&2&1&0\end{bmatrix},$$

with total dominance $A:2$, $B:3$, $C:5$, $D:4$. Ranking: C, D, B, A. (b) The one-step dominances (row sums of D) are $A:1$, $B:1$, $C:2$, $D:2$, so C and D are tied at the top and A and B are tied at the bottom. Their relative order (C above D, and B above A) is therefore decided only by two-step dominance.

Q20. (a) In the order A, B, C, D, E,

$$D=\begin{bmatrix}0&1&1&1&0\\0&0&1&0&1\\0&0&0&1&1\\0&1&0&0&1\\1&0&0&0&0\end{bmatrix},D^2=\begin{bmatrix}0&1&1&1&3\\1&0&0&1&1\\1&1&0&0&1\\1&0&1&0&1\\0&1&1&1&0\end{bmatrix}.$$

The one-step dominances are $A:3$, $B:2$, $C:2$, $D:2$, $E:1$, and the two-step dominances are $A:6$, $B:3$, $C:3$, $D:3$, $E:3$, giving total dominance $A:9$, $B:5$, $C:5$, $D:5$, $E:4$. (b) The ranking places A first and E last, but B, C and D all have total dominance 5. Their results are interchangeable: each lost to A, each beat E, and each won exactly one and lost exactly one of the three games played among themselves (B beat C, C beat D, D beat B). Because the pattern of results is the same for all three, their one-step scores (2 each) and their two-step scores (3 each) must match as well, so two-step dominance cannot break the tie. Note that the cycle alone is not the reason — in the theory example A, B and D also form a cycle yet finish first, second and third. It is the symmetry of the *whole* set of results that leaves B, C and D level. This shows that a dominance ranking can still produce ties.