# General Mathematics Units 3 & 4

## Chapter 7 — Reducing-balance loans, annuities and investments



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## Chapter 7 Reducing-balance loans, annuities and investments — 7A Compound interest investments with regular additions

### Theory

An earlier section modelled a **compound-interest** investment with the recurrence $V_{n+1} = R V_n$: the balance is *multiplied* by a fixed growth factor $R = 1 + \frac{r}{100}$ each period, which is geometric growth. A different model, simple interest, *added* a fixed amount each period, $V_{n+1} = V_n + D$, which is linear growth. This section combines the two into a **savings plan**: you begin with an amount, earn compound interest, and also add a fixed deposit every period. Such a plan is a *compound-interest investment with regular additions*, sometimes called an **annuity investment**.

**The recurrence relation.** Let $V_0$ be the initial investment, let $R = 1 + \frac{r}{100}$ be the growth factor for one compounding period (where $r\,\%$ is the interest rate per period), and let $D$ be the fixed amount deposited each period. In each period the balance is first multiplied by $R$ (interest), then increased by $D$ (the deposit), so

$$V_{n+1} = R V_n + D, V_0 = \text{initial investment}.$$

Here $D > 0$, because money is being **added**. (When money is instead withdrawn or repaid each period — a loan or an annuity — the same form is used with $-D$; that is developed later in this chapter.)

**The order of events in each period.** Within one compounding period, two things happen in order:

- interest is calculated on the **current balance** and added — this multiplies the balance by $R$;
- the regular deposit $D$ is then added at the **end** of the period.

So a deposit made at the end of a period does **not** earn interest during that period; it begins to earn interest in the *next* period. As before, the rate per period is $r = \frac{\text{nominal annual rate}}{k}$ when interest is compounded $k$ times per year.

**Generating the balance sequence.** As with every recurrence relation, the terms are found by **iteration** — applying the rule one period at a time. It is convenient to set the work out in a **growth table** with a row for each period and columns for the *interest earned*, the *deposit*, and the *new balance*. Reading down the balance column gives the sequence $V_1, V_2, V_3, \ldots$

**Rounding convention.** A bank balance is always a whole number of cents, so at the end of each period we round the **interest** to the nearest cent before adding it and the deposit. Every balance in the growth table is then exact to the cent, and each row reconciles as

$$\text{new balance} = \text{previous balance} + \text{interest (to the nearest cent)} + \text{deposit}.$$

(A calculator's built-in finance solver works from a single formula and does *not* round period by period, so its answer can differ from a hand-built table by a cent or two.)

**Why the balance grows faster than compound interest alone.** Compare two accounts that start with the same amount and earn the same rate, one with regular deposits and one without. Each deposit not only adds to the balance, it then **earns compound interest** for every remaining period it stays invested. So the plan with additions steadily pulls ahead: it gains the deposits *and* the interest those deposits generate. The account with no additions earns interest only on its original principal.

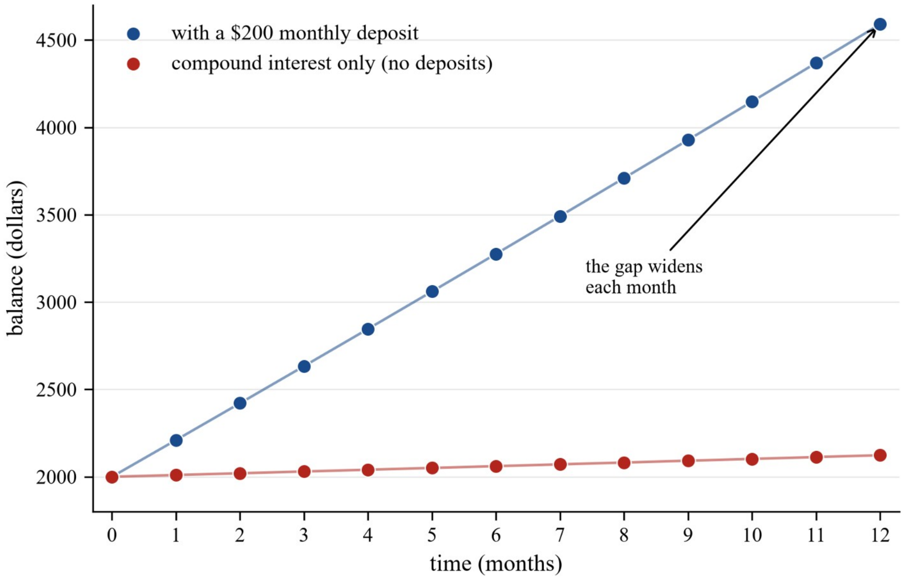
**Total interest earned.** Over $n$ periods the amount you *contribute* is the initial investment plus every deposit, that is $V_0 + nD$. Everything above that in the final balance is interest, so

$$\text{total interest} = V_n - (V_0 + nD).$$

This is also the sum of the interest column in the growth table.

**Using technology.** General Mathematics is a technology-active course. A CAS **finance solver** (TVM) uses the variables $N$ (number of periods), $I\%$ (nominal annual rate), $PV$ (present value), $PMT$ (payment each period), $FV$ (future value) and $P/Y = C/Y$ (periods per year). Money paid **in** to the investment is entered as **negative** (it is an outflow from you), so both $PV$ and $PMT$ are negative and the $FV$ the solver returns is **positive** — the value of the investment. The solver can find the future value directly, or the number of periods $N$ for the balance to reach a target. You should still be able to build the recurrence and iterate it by hand, both to understand the model and to check the technology.

The figure below compares two accounts that each start at the same amount and earn the same interest. The lower points are a compound-interest investment with **no** additions; the upper points add a fixed deposit at the end of every month. The gap widens every period — the additions, and the interest they earn, make the balance climb faster and along a steeper, upward-curving path.




## Worked examples

### *Example 1 — scaffolded*

Priya opens a savings account with $\$4\,000$. It pays interest of 6% p.a. compounded monthly, and at the end of every month she deposits a further $\$150$. (a) Write a recurrence relation for the balance $V_n$ after $n$ months. (b) Find the balance after each of the first three months.

| Working | Comment |
|---|---|
| $r = \frac{6}{12} = 0.5\%$ per month, so $R = 1 + \frac{0.5}{100} = 1.005$. | Monthly rate = annual rate ÷ 12; growth factor $R = 1 + \frac{r}{100}$. |
| (a) $V_{n+1} = 1.005\,V_n + 150,\ V_0 = 4000$. | Multiply by $R$ for the interest, then add the $\$150$ deposit. |
| (b) Month 1: interest $= 0.005 \times 4000 = 20$, so $V_1 = 4000 + 20 + 150 = 4170$. | Interest on the opening balance, then add the deposit: $\$4\,170.00$. |
| Month 2: interest $= 0.005 \times 4170 = 20.85$, so $V_2 = 4170 + 20.85 + 150 = 4340.85$. | Use the new balance to find the next interest: $\$4\,340.85$. |
| Month 3: interest $= 0.005 \times 4340.85 = 21.70425$, so $V_3 = 4340.85 + 21.70 + 150 = 4512.55$. | Round the interest to $\$21.70$; balance $\$4\,512.55$. |

### *Example 2 — scaffolded*

An investor deposits $\$8\,000$ into an account earning 8% p.a. compounded quarterly, and adds $\$500$ at the end of each quarter. Construct a growth table for the first year (four quarters), and hence find the value of the investment after one year and the total interest earned.

| Working | Comment |
| --- | --- |
| $r=\frac{8}{4}=2\%$ per quarter, so $R=1.02$. | Quarterly rate = annual rate $\div 4$. |
| Recurrence: $V_{n+1}=1.02V_n+500, V_0=8000$. | Each quarter add $2\%$ interest, then the $\$500$ deposit. |

Iterating the recurrence one quarter at a time gives the growth table.

| Quarter | Interest earned | Deposit | Balance |
| --- | --- | --- | --- |
| 0 | — | — | $\$8\,000.00$ |
| 1 | $\$160.00$ | $\$500.00$ | $\$8\,660.00$ |
| 2 | $\$173.20$ | $\$500.00$ | $\$9\,333.20$ |
| 3 | $\$186.66$ | $\$500.00$ | $\$10\,019.86$ |
| 4 | $\$200.40$ | $\$500.00$ | $\$10\,720.26$ |

After one year the investment is worth $\$10\,720.26$. The amount contributed is $8000+4\times500=10\,000$, so the total interest earned is $10\,720.26-10\,000=\$720.26$ — the same as the sum of the interest column, $160+173.20+186.66+200.40=720.26$.

### *Example 3 — unscaffolded*

$\$5\,000$ is invested at 4.8% p.a. compounded monthly. Compare the value after six months if (a) no further deposits are made, with (b) a deposit of $\$100$ at the end of each month. How much of the difference is interest?

The monthly rate is $\frac{4.8}{12}=0.4\%$, so $R=1.004$.

With **no deposits** the recurrence is $V_{n+1}=1.004V_n$. Iterating for six months, and rounding the interest to the nearest cent each month, gives

$$5020.00, 5040.08, 5060.24, 5080.48, 5100.80, 5121.20,$$

so the balance is $\$5\,121.20$.

With a $\$100$ **monthly deposit** the recurrence is $V_{n+1}=1.004V_n+100$. Iterating gives

$$5120.00, 5240.48, 5361.44, 5482.89, 5604.82, 5727.24,$$

so the balance is $\$5\,727.24$.

The difference is $5727.24-5121.20=606.04$. Of this, $6\times100=600$ is the deposits themselves; the remaining $\$6.04$ is the extra interest those deposits earned while invested. The account with additions grows faster than compound interest alone, because it gains both the deposits and the interest on them.

$\therefore$ after six months the investment is worth $\$5\,727.24$ with the deposits, compared with $\$5\,121.20$ without — $\$606.04$ more, of which $\$6.04$ is additional interest.

### *Example 4 — unscaffolded*

A saver invests $\$2\,000$ at 3.6% p.a. compounded monthly and deposits $\$250$ at the end of each month. (a) Find the value of the investment after one year. (b) In which month does the balance first exceed $\$6\,000$?

The monthly rate is $\frac{3.6}{12}=0.3\%$, so $R=1.003$ and the recurrence is $V_{n+1}=1.003V_n+250$ with $V_0=2000$.

(a) Iterating for twelve months (interest to the nearest cent each month):

$$2256.00, 2512.77, 2770.31, 3028.62, 3287.71, 3547.57,$$

$$3808.21, 4069.63, 4331.84, 4594.84, 4858.62, 5123.20.$$

After one year the investment is worth $\$5\,123.20$. A CAS finance solver, with $N=12$, $I\%=3.6$, $PV=-2000$, $PMT=-250$ and $P/Y=C/Y=12$, returns $FV=\$5\,123.20$, confirming the by-hand result.

(b) Continuing the iteration past twelve months:

$$V_{13}=5388.57, V_{14}=5654.74, V_{15}=5921.70, V_{16}=6189.47.$$

At the end of the 15th month the balance is $\$5\,921.70$, still below $\$6\,000$; at the end of the 16th month it is $\$6\,189.47$, the first balance above $\$6\,000$.

∴ the investment is worth $\$5\,123.20$ after one year, and its balance first exceeds $\$6\,000$ during the 16th month.

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## Practice questions

### *Scaffolded*

**Q1.** $\$3\,000$ is invested at 12% p.a. compounded monthly, with $\$100$ added at the end of each month. (a) Find the monthly interest rate and the growth factor $R$. (b) Write a recurrence relation for the balance $V_n$. (c) Find $V_1$. (d) Find $V_2$.

**Q2.** A savings plan is modelled by $V_{n+1}=1.02V_n+400$ with $V_0=5000$. (a) State the interest rate per period and the regular deposit. (b) Find $V_1$, $V_2$ and $V_3$ by iteration.

**Q3.** $\$10\,000$ is invested at 6% p.a. compounded quarterly, with $\$250$ added at the end of each quarter. (a) Copy and complete the growth table for the first year.

| Quarter | Interest earned | Deposit | Balance |
|---|---|---|---|
| 0 | — | — | $\$10\,000.00$ |
| 1 | | $\$250.00$ | |
| 2 | | $\$250.00$ | |
| 3 | | $\$250.00$ | |
| 4 | | $\$250.00$ | |

(b) State the value of the investment after one year.

(c) Find the total interest earned.

**Q4.** $\$6\,000$ is invested at 4% p.a. compounded half-yearly for two years (four half-year periods). (a) Find the value after two years if no further deposits are made. (b) Find the value after two years if $\$300$ is deposited at the end of each half-year. (c) State the difference, and how much of it is interest.

**Q5.** $\$1\,000$ is invested at 6% p.a. compounded monthly, with $\$200$ added at the end of each month. (a) Write the recurrence relation. (b) Find the balance at the end of months 1, 2 and 3. (c) Continue the iteration and state the first month in which the balance exceeds $\$3\,000$.

**Q6.** $\$8\,000$ is invested at 4.8% p.a. compounded monthly (0.4% per month), with a fixed deposit $D$ added at the end of each month. After one month the balance is $\$8\,532$. (a) Find the interest earned in the first month. (b) Write an equation for $D$ and solve it. (c) State the recurrence relation for the balance.

### *Unscaffolded*

**Q7.** $\$2500$ is invested at 9% p.a. compounded monthly, with $\$180$ added at the end of each month. Write a recurrence relation for the balance and find the balance after three months.

**Q8.** $\$15000$ is invested at 5% p.a. compounded annually, with $\$2000$ added at the end of each year. Find the value of the investment after five years and the total interest earned.

**Q9.** $\$4000$ is invested at 8% p.a. compounded quarterly for one year. Find the value after one year (a) with no additional deposits and (b) with $\$250$ deposited at the end of each quarter, and (c) state the difference between the two.

**Q10.** A savings plan starts with nothing ($V_0 = 0$) and $\$300$ is deposited at the end of each month into an account paying 6% p.a. compounded monthly. Find the first month in which the balance exceeds $\$2000$.

**Q11.** $\$5000$ is invested at 6% p.a. compounded monthly. Find the balance after six months if the monthly deposit is (a) $\$100$ and (b) $\$200$, and (c) state the difference between the two balances.

**Q12.** $\$10000$ is invested at 4% p.a. compounded quarterly, with a fixed quarterly deposit $D$. After the first quarter the balance is $\$10700$. (a) Find $D$. (b) Find the balance after the second quarter.

**Q13.** $\$20000$ is invested at 3% p.a. compounded monthly, with $\$500$ added at the end of each month. Find the balance after six months and the total interest earned.

**Q14.** $\$8000$ is invested at 7% p.a. compounded annually, with $\$1500$ added at the end of each year. Find the value of the investment after four years.

**Q15.** A compound-interest investment with regular additions grows faster than the same investment earning compound interest with no additions. (a) Explain, in terms of the deposits and the interest they earn, why this is so. (b) A deposit is made at the end of each compounding period. Explain why that deposit earns no interest during the period in which it is made.

**Q16.** Two plans each pay 9% p.a. compounded monthly. Plan X starts with $\$5000$ and makes no deposits. Plan Y starts with $\$2000$ and deposits $\$250$ at the end of each month. Find the balance of each plan after one year and state which is larger.

**Q17.** An annuity investment earns 1.2% per quarter with a $\$400$ deposit at the end of each quarter. Part of its growth table is shown.

| Quarter | Interest earned | Deposit | Balance |
|---|---|---|---|
| 0 | — | — | $\$9000.00$ |
| 1 | $\$108.00$ | $\$400.00$ | $\$9508.00$ |
| 2 | $\$114.10$ | $\$400.00$ | $A$ |
| 3 | $B$ | $\$400.00$ | $C$ |

Find the values $A$, $B$ and $C$.

**Q18.** $\$3000$ is invested at 2% per period, with a fixed deposit added at the end of each period. After the first period the balance is $\$3260$. (a) Find the deposit $D$. (b) Assuming the same deposit, find the balance after the third period.

**Q19.** $\$1200$ is invested at 3.6% p.a. compounded monthly, with $\$150$ added at the end of each month. Find the balance after eight months and the total interest earned over that time.

**Q20.** $\$10000$ is invested at 6% p.a. compounded monthly, with $\$500$ added at the end of each month. (a) Write a recurrence relation for the balance. (b) Find the balance after six months. (c) Find the total interest earned in the first six months. (d) A finance solver returns a future value of $\$13341.53$ for this investment after six months. Explain why this differs by one cent from your answer to (b).

## Answers

**Q1.** (a) 1% per month, $R=1.01$; (b) $V_{n+1}=1.01V_n+100$, $V_0=3000$; (c) $V_1=\$3130.00$; (d) $V_2=\$3261.30$. **Q2.** (a) 2% per period, deposit $\$400$; (b) $V_1=\$5500.00$, $V_2=\$6010.00$, $V_3=\$6530.20$. **Q3.** (a) interest $\$150.00, \$156.00, \$162.09, \$168.27$; balances $\$10400.00, \$10806.00, \$11218.09, \$11636.36$; (b) $\$11636.36$; (c) $\$636.36$. **Q4.** (a) $\$6494.60$; (b) $\$7731.08$; (c) difference $\$1236.48$, of which $\$1200$ is deposits and $\$36.48$ is interest. **Q5.** (a) $V_{n+1}=1.005V_n+200$, $V_0=1000$; (b) $\$1205.00, \$1411.03, \$1618.09$; (c) month 10 ($\$3096.76$). **Q6.** (a) $\$32.00$; (b) $8000+32+D=8532$, so $D=\$500$; (c) $V_{n+1}=1.004V_n+500$, $V_0=8000$. **Q7.** $V_{n+1}=1.0075V_n+180$, $V_0=2500$; $\$3100.73$. **Q8.** $\$30195.49$; total interest $\$5195.49$. **Q9.** (a) $\$4329.73$; (b) $\$5360.13$; (c) $\$1030.40$. **Q10.** Month 7 ($\$2131.77$). **Q11.** (a) $\$5759.44$; (b) $\$6366.99$; (c) $\$607.55$. **Q12.** (a) $D=\$600$; (b) $\$11407.00$. **Q13.** $\$23320.70$; total interest $\$320.70$. **Q14.** $\$17146.28$. **Q15.** (a) Each deposit adds to the balance and then earns compound interest for the rest of the term, so the account gains both the deposits and the interest on them. (b) It is added at the end of the period, after that period's interest has been calculated, so it only earns interest from the next period on. **Q16.** Plan X $\$5469.03$; Plan Y $\$5314.51$; Plan X is larger. **Q17.** $A=\$10022.10$, $B=\$120.27$, $C=\$10542.37$. **Q18.** (a) $D=\$200$; (b) $\$3795.70$. **Q19.** $\$2441.78$; total interest $\$41.78$. **Q20.** (a) $V_{n+1}=1.005V_n+500$, $V_0=10000$; (b) $\$13341.52$; (c) $\$341.52$; (d) the solver uses an exact formula without rounding to the nearest cent each month, so it finishes one cent above the period-by-period total.

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## Fully-worked solutions

**Q1.** $r=\frac{12}{12}=1\%$ per month, so $R=1.01$ and the recurrence is $V_{n+1}=1.01V_n+100$ with $V_0=3000$. Month 1: interest $=0.01\times3000=30$, so $V_1=3000+30+100=3130$, that is $\$3130.00$. Month 2: interest $=0.01\times3130=31.30$, so $V_2=3130+31.30+100=3261.30$, that is $\$3261.30$.

**Q2.** The factor 1.02 gives an interest rate of 2% per period, and the added constant is the deposit $\$400$. Iterating from $V_0=5000$: $V_1=1.02\times5000+400=5100+400=5500$; $V_2=1.02\times5500+400=5610+400=6010$; $V_3=1.02\times6010+400=6130.20+400=6530.20$. So $V_1=\$5500.00$, $V_2=\$6010.00$ and $V_3=\$6530.20$.

**Q3.** $r=\frac{6}{4}=1.5\%$ per quarter, $R=1.015$; each quarter the interest is 1.5% of the previous balance (to the nearest cent) and $\$250$ is added. Quarter 1: interest $=0.015\times10000=150$, balance $10000+150+250=10400$. Quarter 2: interest $=0.015\times10400=156$, balance $10400+156+250=10806$. Quarter 3: interest $=0.015\times10806=162.09$, balance $10806+162.09+250=11218.09$. Quarter 4: interest $=0.015\times11218.09=168.27135\rightarrow168.27$, balance $11218.09+168.27+250=11636.36$. The value after one year is $\$11636.36$. Contributions are $10000+4\times250=11000$, so the total interest is $11636.36-11000=\$636.36$.

**Q4.** $r=\frac{4}{2}=2\%$ per half-year, $R=1.02$. (a) No deposits, $V_{n+1}=1.02V_n$: the balances are $6120.00, 6242.40, 6367.25, 6494.60$, so the value is $\$6494.60$. (b) With $\$300$ each half-year, $V_{n+1}=1.02V_n+300$: the balances are $6420.00, 6848.40, 7285.37, 7731.08$, so the value is $\$7731.08$. (c) The difference is $7731.08-6494.60=1236.48$. Four deposits total $4\times300=1200$, so of the $\$1236.48$ difference, $\$1200$ is deposits and the remaining $\$36.48$ is the extra interest earned on them.

**Q5.** $r=\frac{6}{12}=0.5\%$ per month, $R=1.005$; the recurrence is $V_{n+1}=1.005V_n+200$ with $V_0=1000$. $V_1=1000+5+200=1205.00$; $V_2=1205+6.03+200=1411.03$ (interest $0.005\times1205=6.025\rightarrow6.03$); $V_3=1411.03+7.06+200=1618.09$. Continuing the iteration:

$$1826.18, 2035.31, 2245.49, 2456.72, 2669.00, 2882.35, 3096.76.$$

At month 9 the balance is $\$2882.35$ (below $\$3000$) and at month 10 it is $\$3096.76$, so the balance first exceeds $\$3000$ in **month 10**.

**Q6.** (a) The interest in the first month is $0.4\,\%$ of $\$8\,000$, that is $0.004 \times 8000 = \$32.00$. (b) The balance after one month is the opening balance plus interest plus the deposit: $8000 + 32 + D = 8532$, so $D = 8532 - 8032 = \$500$. (c) With $R = 1.004$ and $D = 500$, the recurrence is $V_{n+1} = 1.004\,V_n + 500$, $V_0 = 8000$.

**Q7.** $r = \frac{9}{12} = 0.75\,\%$ per month, $R = 1.0075$; the recurrence is $V_{n+1} = 1.0075\,V_n + 180$ with $V_0 = 2500$. Month 1: interest $= 0.0075 \times 2500 = 18.75$, balance $2500 + 18.75 + 180 = 2698.75$. Month 2: interest $= 0.0075 \times 2698.75 = 20.24$ (from $20.2406\ldots$), balance $2698.75 + 20.24 + 180 = 2898.99$. Month 3: interest $= 0.0075 \times 2898.99 = 21.74$, balance $2898.99 + 21.74 + 180 = 3100.73$. So the balance after three months is $\$3\,100.73$.

**Q8.** $R = 1.05$; the recurrence is $V_{n+1} = 1.05\,V_n + 2000$ with $V_0 = 15\,000$. Year 1: interest $750$, balance $17\,750.00$. Year 2: interest $887.50$, balance $20\,637.50$. Year 3: interest $1031.88$ (from $1031.875$), balance $23\,669.38$. Year 4: interest $1183.47$, balance $26\,852.85$. Year 5: interest $1342.64$, balance $30\,195.49$. So the value after five years is $\$30\,195.49$. Contributions are $15\,000 + 5 \times 2000 = 25\,000$, so the total interest is $30\,195.49 - 25\,000 = \$5\,195.49$.

**Q9.** $r = \frac{8}{4} = 2\,\%$ per quarter, $R = 1.02$. (a) No deposits: $4080.00, 4161.60, 4244.83, 4329.73$, so the value is $\$4\,329.73$. (b) With $\$250$ per quarter: $4330.00, 4666.60, 5009.93, 5360.13$, so the value is $\$5\,360.13$. (c) The difference is $5360.13 - 4329.73 = \$1\,030.40$.

**Q10.** $r = 0.5\,\%$ per month, $R = 1.005$; the recurrence is $V_{n+1} = 1.005\,V_n + 300$ with $V_0 = 0$. The first deposit gives $V_1 = 0 + 0 + 300 = 300.00$. Continuing:

$$601.50, 904.51, 1209.03, 1515.08, 1822.66, 2131.77.$$

At month 6 the balance is $\$1\,822.66$ (below $\$2\,000$) and at month 7 it is $\$2\,131.77$, so the balance first exceeds $\$2\,000$ in **month 7**.

**Q11.** $r = 0.5\,\%$ per month, $R = 1.005$. (a) $D = 100$: $5125.00, 5250.63, 5376.88, 5503.76, 5631.28, 5759.44$, so the balance is $\$5\,759.44$. (b) $D = 200$: $5225.00, 5451.13, 5678.39, 5906.78, 6136.31, 6366.99$, so the balance is $\$6\,366.99$. (c) The difference is $6366.99 - 5759.44 = \$607.55$. The extra $\$100$ each month for six months is $\$600$, and the remaining $\$7.55$ is the extra interest earned on the larger deposits.

**Q12.** $r = \frac{4}{4} = 1\,\%$ per quarter, $R = 1.01$. (a) The interest in the first quarter is $0.01 \times 10\,000 = 100$, so $10\,000 + 100 + D = 10\,700$, giving $D = \$600$. (b) Second quarter: interest $= 0.01 \times 10\,700 = 107$, balance $10\,700 + 107 + 600 = 11\,407$, that is $\$11\,407.00$.

**Q13.** $r = \frac{3}{12} = 0.25\,\%$ per month, $R = 1.0025$; the recurrence is $V_{n+1} = 1.0025\,V_n + 500$ with $V_0 = 20\,000$. Iterating for six months:

$$20\,550.00, 21\,101.38, 21\,654.13, 22\,208.27, 22\,763.79, 23\,320.70,$$

so the balance is $\$23\,320.70$. Contributions are $20\,000 + 6 \times 500 = 23\,000$, so the total interest is $23\,320.70 - 23\,000 = \$320.70$.

**Q14.** $R = 1.07$; the recurrence is $V_{n+1} = 1.07\,V_n + 1500$ with $V_0 = 8000$. Year 1: interest $560$, balance $10\,060.00$. Year 2: interest $704.20$, balance $12\,264.20$. Year 3: interest $858.49$ (from $858.494$), balance $14\,622.69$. Year 4: interest $1023.59$, balance $17\,146.28$. So the value after four years is $\$17\,146.28$.

**Q15.** (a) Every deposit is added to the balance and then earns compound interest for each remaining period it stays invested. So an account with regular additions gains not only the deposits themselves but also the interest those deposits generate, whereas an account with no additions earns interest only on its original principal. The extra principal *and* the extra interest make the balance grow faster. (b) The deposit is made at the **end** of a period, after that period's interest has already been calculated on the opening balance. It was therefore not part of the balance on which that period's interest was worked out, so it earns nothing until the following period.

**Q16.** Both plans use $r=\frac{9}{12}=0.75\%$ per month, $R=1.0075$. Plan X ($V_0=5000$, no deposits): iterating $V_{n+1}=1.0075V_n$ for twelve months gives a final balance of $\$5\,469.03$. Plan Y ($V_0=2000$, $D=250$): iterating $V_{n+1}=1.0075V_n+250$ for twelve months gives $\$5\,314.51$. Since $5469.03>5314.51$, **Plan X** is larger after one year — over this short term its larger starting principal outweighs Plan Y's deposits.

**Q17.** Each quarter the interest is 1.2% of the previous balance (to the nearest cent) and $\$400$ is added. $A=9508.00+114.10+400=\$10\,022.10$. $B=0.012\times10\,022.10=120.2652\rightarrow\$120.27$. $C=10\,022.10+120.27+400=\$10\,542.37$.

**Q18.** (a) The interest in the first period is $0.02\times3000=60$, so $3000+60+D=3260$, giving $D=\$200$. (b) With $R=1.02$ and $D=200$: period 2 interest $=0.02\times3260=65.20$, balance $3260+65.20+200=3525.20$; period 3 interest $=0.02\times3525.20=70.504\rightarrow70.50$, balance $3525.20+70.50+200=3795.70$. So the balance after the third period is $\$3\,795.70$.

**Q19.** $r=\frac{3.6}{12}=0.3\%$ per month, $R=1.003$; the recurrence is $V_{n+1}=1.003V_n+150$ with $V_0=1200$. Iterating for eight months:

$$1353.60, 1507.66, 1662.18, 1817.17, 1972.62, 2128.54, 2284.93, 2441.78,$$

so the balance is $\$2\,441.78$. Contributions are $1200+8\times150=2400$, so the total interest is $2441.78-2400=\$41.78$.

**Q20.** (a) $r=\frac{6}{12}=0.5\%$ per month, $R=1.005$, so the recurrence is $V_{n+1}=1.005V_n+500$ with $V_0=10\,000$. (b) Iterating for six months:

$$10\,550.00, 11\,102.75, 11\,658.26, 12\,216.55, 12\,777.63, 13\,341.52,$$

so the balance is $\$13\,341.52$. (c) Contributions are $10\,000+6\times500=13\,000$, so the total interest is $13\,341.52-13\,000=\$341.52$. (d) The finance solver evaluates a single exponential formula and rounds only at the end, whereas the growth table rounds the interest to the nearest cent every month. Those small monthly roundings accumulate to one cent, so the solver's $\$13\,341.53$ is one cent above the table's $\$13\,341.52$.

### Theory

An earlier section used the recurrence $V_{n+1}=RV_n$ to model **compound interest**: the balance is multiplied by a fixed growth factor $R=1+\frac{r}{100}$ each period, so it grows geometrically. A later addition, a regular deposit, gave $V_{n+1}=RV_n+D$: the balance grows even faster because money is *added* every period. This section changes the sign of $D$: instead of adding money each period we **take money out**, so the balance **falls**.

**The recurrence relation.** Let $V_0$ be the starting amount, let $R=1+\frac{r}{100}$ be the growth factor for one period (where $r\,\%$ is the interest rate per period), and let $D$ be the fixed amount removed each period. In each period the balance first earns interest (multiply by $R$) and then has $D$ taken out, so

$$V_{n+1}=RV_n-D,\,V_0=\text{starting amount}.$$

Because $D$ is subtracted, the balance decreases period by period until it reaches zero.

**Two contexts, one recurrence.** The same rule describes two situations that look very different:

- a **reducing-balance loan** — you *borrow* $V_0$, interest is *charged* on what you still owe, and a regular **repayment** $D$ reduces the debt. Each period the balance owing falls, until the loan is repaid (balance $=0$). It is called a *reducing-balance* loan because interest is charged on the **reduced** balance each period, so as the debt shrinks the interest charged shrinks with it;
- an **annuity** — you *invest* a lump sum $V_0$, it *earns* interest, and you draw a regular **withdrawal** $D$ as income. Each period the balance falls, until the fund is exhausted (balance $=0$).

In both cases the balance is multiplied by $R$ and then $D$ is subtracted, so **the recurrence is identical**; only the story attached to $V_0$, $R$ and $D$ differs. Whether you say “amount owed” or “amount invested”, the arithmetic is the same.

**The order of events in each period.** Within one period two things happen, in order:

- interest is calculated on the **current balance** and applied — this multiplies the balance by $R$;
- the repayment or withdrawal $D$ is then made at the **end** of the period.

So a repayment reduces the balance *after* that period’s interest has been charged. As before, the rate per period is $r=\frac{\text{nominal annual rate}}{k}$ when interest is compounded $k$ times per year.

**Generating the balance sequence.** As with every recurrence relation, the terms are found by **iteration** — applying the rule one period at a time. It is convenient to set the work out in a table with a row for each period and columns for the *interest*, the *payment* (repayment or withdrawal), and the *new balance*. Reading down the balance column gives the sequence $V_1, V_2, V_3, \ldots$, which falls towards zero. (A fuller **amortisation table**, which also splits each payment into the interest part and the principal-reduction part, is developed in the next section; here we simply generate the balance.)

**Rounding convention.** Following the standard used throughout this chapter, at the end of each period we round the **interest** to the nearest cent before applying it, and carry the rounded balance forward. Every balance is then exact to the cent and each row reconciles as

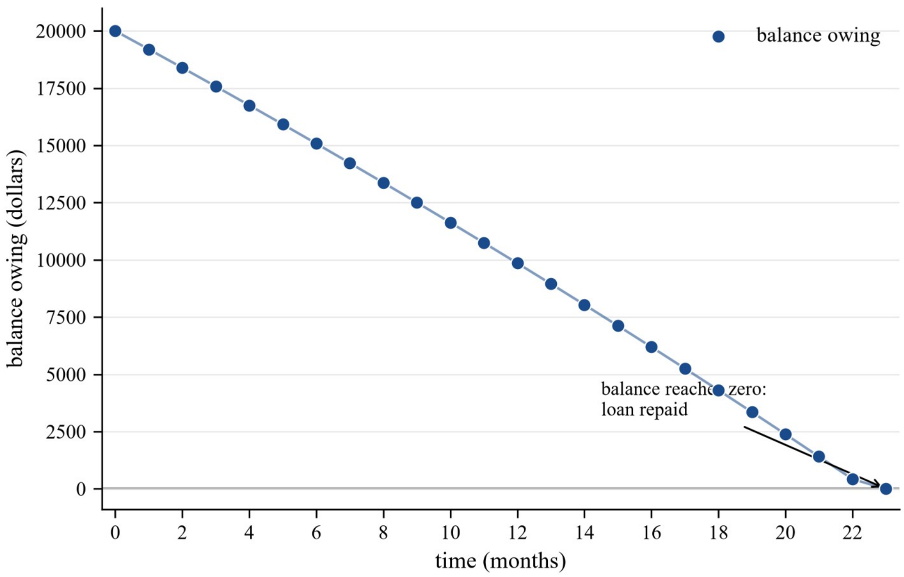
$$\text{new balance}=\text{previous balance}+\text{interest (to the nearest cent)}-\text{payment}.$$

**Interpreting when the balance reaches zero.** The loan is repaid — or the annuity is exhausted — when the balance first reaches $0$. Usually the balance does **not** land exactly on zero: after a number of full payments a small balance is left that is *less* than $D$. That period’s interest is added, and the **final payment** needed is just that remaining balance plus its interest, which is smaller than the usual $D$. To find it, iterate until the balance (after adding the next interest) is at or below $D$; the final payment equals that reduced amount, and it brings the balance to exactly $0$.

**A boundary case.** If $D$ is chosen to equal exactly the interest earned each period, the subtraction cancels the growth and the balance never changes — an investment that pays a fixed income forever. That special case (a *perpetuity*) is looked at later in this chapter. If $D$ is smaller than the interest, the balance actually **grows**; only when $D$ exceeds the interest does the balance fall to zero.

**Using technology.** General Mathematics is a technology-active course. A CAS **finance solver** (TVM) uses the variables $N$ (number of periods), $I\%$ (nominal annual rate), $PV$ (present value), $PMT$ (payment each period), $FV$ (future value) and $P/Y = C/Y$ (periods per year). The sign convention records the direction of each cash flow: money that comes **to** you is positive and money that goes **from** you is negative. For a loan you receive $PV$ (positive) and make repayments $PMT$ (negative); for an annuity you invest $PV$ (negative) and receive withdrawals $PMT$ (positive); in each case $FV = 0$ at the moment the balance reaches zero. The solver evaluates a single formula and does **not** round period by period, so its answer can differ from a hand-built table by a cent or two — where this happens it is worth noting, not hiding. A non-whole value of $N$ tells you the balance reaches zero **partway** through a period, i.e. the final payment is smaller than the rest. You should still be able to build the recurrence and iterate it by hand, both to understand the model and to check the technology.

The figure below shows the balance owing on a reducing-balance loan falling to zero. Because the interest charged shrinks as the balance shrinks, each repayment clears a little more of the principal than the one before, so the balance falls faster and faster and the final payment is smaller than the rest.




## Worked examples

### *Example 1 — scaffolded*

Sam borrows $\$20\,000$ for a car. The loan is charged interest of 12% p.a. compounded monthly, and Sam repays $\$1\,000$ at the end of every month. (a) Write a recurrence relation for the balance $V_n$ owing after $n$ months. (b) Find the balance owing after each of the first three months.

| Working | Comment |
|---|---|
| $r = \frac{12}{12} = 1\%$ per month, so $R = 1 + \frac{1}{100} = 1.01$. | Monthly rate = annual rate ÷ 12; growth factor $R = 1 + \frac{r}{100}$. |
| (a) $V_{n+1} = 1.01\,V_n - 1000,\ V_0 = 20\,000$. | Multiply by $R$ for the interest charged, then subtract the $\$1\,000$ repayment. |
| (b) Month 1: interest $= 0.01 \times 20\,000 = 200$, so | Interest on the opening balance, then subtract the |

| Working | Comment |
|---|---|
| $V_1 = 20\,000 + 200 - 1000 = 19\,200$. | repayment: $\$19\,200.00$. |
| Month 2: interest $= 0.01 \times 19\,200 = 192$, so $V_2 = 19\,200 + 192 - 1000 = 18\,392$. | Use the reduced balance to find the next interest: $\$18\,392.00$. |
| Month 3: interest $= 0.01 \times 18\,392 = 183.92$, so $V_3 = 18\,392 + 183.92 - 1000 = 17\,575.92$. | Balance $\$17\,575.92$. The interest is smaller each month as the balance falls. |

### *Example 2 — scaffolded*

Mai invests $\$10\,000$ in an annuity paying 6% p.a. compounded quarterly, and withdraws $\$2\,000$ at the end of each quarter as income. Construct a balance table, and hence find how long the annuity lasts and the size of the final withdrawal.

| Working | Comment |
|---|---|
| $r = \frac{6}{4} = 1.5\,\%$ per quarter, so $R = 1.015$. | Quarterly rate = annual rate $\div 4$. |
| Recurrence: $V_{n+1} = 1.015\,V_n - 2000\,, V_0 = 10\,000$. | Each quarter add $1.5\,\%$ interest, then subtract the $\$2\,000$ withdrawal. |

Iterating the recurrence one quarter at a time gives the balance table.

| Quarter | Interest earned | Withdrawal | Balance |
|---|---|---|---|
| 0 | — | — | $\$10\,000.00$ |
| 1 | $\$150.00$ | $\$2\,000.00$ | $\$8\,150.00$ |
| 2 | $\$122.25$ | $\$2\,000.00$ | $\$6\,272.25$ |
| 3 | $\$94.08$ | $\$2\,000.00$ | $\$4\,366.33$ |
| 4 | $\$65.49$ | $\$2\,000.00$ | $\$2\,431.82$ |
| 5 | $\$36.48$ | $\$2\,000.00$ | $\$468.30$ |

After five full withdrawals the balance is only $\$468.30$, which is less than $\$2\,000$, so a sixth full withdrawal is impossible. In the sixth quarter the interest is $0.015 \times 468.30 = 7.02$ (to the nearest cent), giving a balance of $468.30 + 7.02 = \$475.32$ before any withdrawal. Taking this $\$475.32$ as the final withdrawal empties the account exactly. So the annuity lasts **six quarters** — five withdrawals of $\$2\,000$ and a final withdrawal of $\$475.32$.

### *Example 3 — unscaffolded*

A short-term loan of $\$3\,000$ is charged 2% interest per month, and is repaid by monthly instalments of $\$800$. How many repayments are needed to clear the loan, what is the size of the final repayment, and how much interest is paid in total?

The recurrence is $V_{n+1} = 1.02\,V_n - 800$ with $V_0 = 3000$. Iterating, and rounding the interest to the nearest cent each month:

$$V_1 = 3000 + 60 - 800 = 2260.00\,,$$

$$V_2 = 2260 + 45.20 - 800 = 1505.20\,,$$

$$V_3 = 1505.20 + 30.10 - 800 = 735.30\,.$$

After three repayments $\$735.30$ is still owing. In month 4 the interest is $0.02 \times 735.30 = 14.71$ (from 14.706), giving $735.30 + 14.71 = \$750.01$ owing before the repayment. This is less than $\$800$, so the final repayment is just $\$750.01$, which clears the loan.

The loan therefore takes **four months**: three repayments of $\$800$ and a final repayment of $\$750.01$. The total repaid is $3 \times 800 + 750.01 = \$3\,150.01$, so the interest paid is $3150.01 - 3000 = \$150.01$.

$\therefore$ four repayments clear the loan (a final repayment of $\$750.01$), and the total interest paid is $\$150.01$.

### *Example 4 — unscaffolded*

The recurrence $V_{n+1}=1.0075V_n-1500$ with $V_0=8000$ can model two situations: a loan of $\$8\,000$ at 9% p.a. compounded monthly repaid at $\$1\,500$ per month, or an annuity of $\$8\,000$ at the same rate paying $\$1\,500$ per month. Show that both reach zero together, and interpret the result in each context.

The monthly rate is $\frac{9}{12}=0.75\%$, so $R=1.0075$. Iterating once per month (interest to the nearest cent):

$$V_1=6560.00\,,V_2=5109.20\,,V_3=3647.52\,,V_4=2174.88\,,V_5=691.19\,.$$

After five payments the balance is $\$691.19$, less than $\$1\,500$. In month 6 the interest is $0.0075\times691.19=5.18$, giving $691.19+5.18=\$696.37$ before the payment, so a final payment of $\$696.37$ brings the balance to exactly zero. The sequence is generated by the **one** recurrence, so it is the same whether the $\$8\,000$ is borrowed or invested.

Read as a **loan**, the balance is the amount still owed; it reaches zero when the debt is repaid, after five repayments of $\$1\,500$ and a final repayment of $\$696.37$ — six months in all. Read as an **annuity**, the balance is the fund remaining; it reaches zero when the fund is exhausted, after five withdrawals of $\$1\,500$ and a final withdrawal of $\$696.37$ — the annuity provides income for six months. A CAS finance solver, asked for $N$ with $FV=0$, returns a value between $5$ and $6$, confirming that the balance clears partway through the sixth month.

$\therefore$ the same recurrence models both; the balance reaches zero in the sixth month, meaning the loan is repaid (as a loan) or the fund is exhausted (as an annuity), with a final payment of $\$696.37$.

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## Practice questions

### *Scaffolded*

**Q1.** A loan of $\$16\,000$ is charged 12% p.a. compounded monthly, and is repaid by $\$1\,800$ at the end of each month. (a) Find the monthly interest rate and the growth factor $R$. (b) Write a recurrence relation for the balance $V_n$ owing. (c) Find $V_1$. (d) Find $V_2$.

**Q2.** An annuity is modelled by $V_{n+1}=1.02V_n-1500$ with $V_0=12\,000$. (a) State the interest rate per period and the regular withdrawal. (b) Find $V_1$, $V_2$ and $V_3$ by iteration.

**Q3.** A loan of $\$8\,000$ is charged 6% p.a. compounded quarterly, and is repaid by $\$1\,000$ at the end of each quarter. (a) Copy and complete the balance table for the first year.

| Quarter | Interest charged | Repayment | Balance |
|---|---|---|---|
| 0 | — | — | $\$8\,000.00$ |
| 1 | | $\$1\,000.00$ | |
| 2 | | $\$1\,000.00$ | |
| 3 | | $\$1\,000.00$ | |
| 4 | | $\$1\,000.00$ | |

(b) State the balance owing after one year.
(c) Find the total interest charged in the first year.

**Q4.** An annuity of $\$5\,000$ pays 8% p.a. compounded quarterly, and $\$1\,100$ is withdrawn at the end of each quarter. (a) Write the recurrence relation. (b) Find the balance after each of the first four quarters. (c) Show that a fifth full withdrawal of $\$1\,100$ is not possible, and find the final withdrawal that empties the account.

**Q5.** A loan of $\$14\,000$ is charged 2% interest per month and repaid by equal monthly instalments $D$. After the first repayment the balance owing is $\$12\,680$. (a) Find the interest charged in the first month. (b) Write an equation for $D$ and solve it. (c) State the recurrence relation for the balance owing. (d) Find the balance owing after two months.

**Q6.** A balance is modelled by $V_{n+1}=1.005\,V_n-400$ with $V_0=6000$. (a) If this models a reducing-balance loan, state the amount borrowed, the monthly interest rate and the monthly repayment. (b) If instead it models an annuity, state what $\$6\,000$, the $0.5\,\%$ and the $\$400$ each represent. (c) Find $V_1$ and $V_2$ (the same for both interpretations).

### *Unscaffolded*

**Q7.** A loan of $\$25\,000$ is charged 9% p.a. compounded monthly and repaid by $\$2\,500$ at the end of each month. Write a recurrence relation for the balance owing and find the balance after three months.

**Q8.** An annuity of $\$50\,000$ pays 6% p.a. compounded monthly, with $\$3\,500$ withdrawn at the end of each month. Find the balance remaining after four months.

**Q9.** A loan of $\$7\,200$ is charged 1.5% interest per month and repaid by monthly instalments of $\$1\,600$. Find how many repayments clear the loan, the size of the final repayment, and the total interest paid.

**Q10.** An annuity of $\$30\,000$ pays 4.8% p.a. compounded monthly, with $\$2\,000$ withdrawn at the end of each month. Find the balance after six months and the total interest earned over that time.

**Q11.** A loan of $\$14\,500$ is charged 1% interest per month. Find the balance owing after four months if the monthly repayment is (a) $\$2\,600$ and (b) $\$3\,200$, and (c) state the difference between the two balances.

**Q12.** A reducing-balance loan of $\$18\,000$ is charged 1.5% interest per quarter and repaid by equal quarterly instalments $D$. After the first repayment the balance owing is $\$16\,770$. (a) Find $D$. (b) Find the balance owing after the second quarter.

**Q13.** An annuity of $\$20\,000$ earns 0.6% interest per month, with $\$2\,500$ withdrawn at the end of each month. Part of its balance table is shown.

| Month | Interest earned | Withdrawal | Balance |
|---|---|---|---|
| 0 | — | — | $\$20\,000.00$ |
| 1 | $\$120.00$ | $\$2\,500.00$ | $\$17\,620.00$ |
| 2 | $A$ | $\$2\,500.00$ | $B$ |
| 3 | $C$ | $\$2\,500.00$ | $D$ |

Find the values $A$, $B$, $C$ and $D$.

**Q14.** A loan of $\$5\,000$ is charged 1% interest per month and repaid by monthly instalments of $\$870$. Find how many repayments are needed to clear the loan, the size of the final repayment, and the total interest paid.

**Q15.** (a) Explain why a reducing-balance loan and an annuity are modelled by the *same* recurrence relation $V_{n+1}=R\,V_n-D$, even though one is money owed and the other is money invested. (b) State what it means, in each context, when the balance first reaches zero.

**Q16.** Two annuities each start with $\$15\,000$ and earn 0.5% interest per month. Annuity P withdraws $\$1\,000$ at the end of each month; annuity Q withdraws $\$1\,600$. Find the balance of each after five months, and state which annuity will be exhausted sooner.

**Q17.** A loan is modelled by $V_{n+1}=R\,V_n-1200$ with $V_0=9000$. After the first repayment the balance owing is $\$8\,070$. (a) Find the interest charged in the first month. (b) Find $R$ and the monthly interest rate. (c) Find the balance owing after the second month.

**Q18.** A retiree invests $\$80\,000$ in an account earning 0.5% interest per month and plans to withdraw a fixed amount $D$ at the end of each month. (a) Find the withdrawal $D$ for which the balance stays at $\$80\,000$ every month, and write the resulting recurrence relation. (b) State what happens to the balance over time if $D$ is instead larger than this amount, and if $D$ is smaller.

**Q19.** An annuity of $\$6\,500$ earns 1% interest per month, with $\$1\,400$ withdrawn at the end of each month. Find how many months the annuity lasts, the size of the final withdrawal, and the total interest earned.

**Q20.** A loan of $\$15\,000$ is charged 12% p.a. compounded monthly and repaid by $\$1\,300$ at the end of each month. (a) Write a recurrence relation for the balance owing. (b) Find the balance owing after six months. (c) A finance solver returns a balance of $\$7\,925.18$ after six months. Explain why this differs by one cent from your answer to (b).

## Answers

**Q1.** (a) 1% per month, $R=1.01$; (b) $V_{n+1}=1.01V_n-1800$, $V_0=16\,000$; (c) $V_1=\$14\,360.00$; (d) $V_2=\$12\,703.60$. **Q2.** (a) 2% per period, withdrawal $\$1\,500$; (b) $V_1=\$10\,740.00$, $V_2=\$9\,454.80$, $V_3=\$8\,143.90$. **Q3.** (a) interest $\$120.00, \$106.80, \$93.40, \$79.80$; balances $\$7\,120.00, \$6\,226.80, \$5\,320.20, \$4\,400.00$; (b) $\$4\,400.00$; (c) $\$400.00$. **Q4.** (a) $V_{n+1}=1.02V_n-1100$, $V_0=5000$; (b) $\$4\,000.00, \$2\,980.00, \$1\,939.60, \$878.39$; (c) balance before the fifth withdrawal is $878.39+17.57=\$895.96<\$1\,100$, so the final withdrawal is $\$895.96$. **Q5.** (a) $\$280.00$; (b) $14\,000+280-D=12\,680$, so $D=\$1\,600$; (c) $V_{n+1}=1.02V_n-1600$, $V_0=14\,000$; (d) $\$11\,333.60$. **Q6.** (a) borrowed $\$6\,000$, 0.5% per month, repayment $\$400$ per month; (b) $\$6\,000$ invested, 0.5% interest earned per month, $\$400$ withdrawn each month; (c) $V_1=\$5\,630.00$, $V_2=\$5\,258.15$. **Q7.** $V_{n+1}=1.0075V_n-2500$, $V_0=25\,000$; $\$18\,010.34$. **Q8.** $\$36\,902.18$. **Q9.** Five repayments (four of $\$1\,600$ and a final of $\$1\,112.82$); total interest $\$312.82$. **Q10.** $\$18\,606.60$; total interest earned $\$606.60$. **Q11.** (a) $\$4\,531.71$; (b) $\$2\,095.47$; (c) $\$2\,436.24$. **Q12.** (a) $D=\$1\,500$; (b) $\$15\,521.55$. **Q13.** $A=\$105.72$, $B=\$15\,225.72$, $C=\$91.35$, $D=\$12\,817.07$. **Q14.** Six repayments (five of $\$870$ and a final of $\$825.34$); total interest $\$175.34$. **Q15.** (a) Both multiply the balance by $R$ (interest) and then subtract a fixed $D$ (a repayment or a withdrawal), so the arithmetic — and hence the recurrence — is identical; only what $V_0$, $R$ and $D$ describe changes. (b) For a loan, the balance reaching zero means the debt is fully repaid; for an annuity, it means the invested fund is exhausted. **Q16.** Annuity P $\$10\,328.53$; annuity Q $\$7\,298.37$; annuity Q (the larger withdrawals) will be exhausted sooner. **Q17.** (a) $\$270.00$; (b) $R=1.03$, 3% per month; (c) $\$7\,112.10$. **Q18.** (a) $D=\$400$ (the monthly interest), giving $V_{n+1}=1.005V_n-400$, which leaves the balance unchanged at $\$80\,000$; (b) if $D>\$400$ the balance falls and is eventually exhausted; if $D<\$400$ the balance grows. **Q19.** Five months (four withdrawals of $\$1\,400$ and a final withdrawal of $\$1\,090.16$); total interest earned $\$190.16$. **Q20.** (a) $V_{n+1}=1.01V_n-1300$, $V_0=15\,000$; (b) $\$7\,925.19$; (c) the solver evaluates one formula without rounding to the cent each month, whereas the table rounds every month; those roundings accumulate to one cent, so the table finishes one cent above the solver.

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## Fully-worked solutions

**Q1.** $r=\frac{12}{12}=1\%$ per month, so $R=1.01$ and the recurrence is $V_{n+1}=1.01V_n-1800$ with $V_0=16\,000$. Month 1: interest $=0.01\times16\,000=160$, so $V_1=16\,000+160-1800=14\,360$, that is $\$14\,360.00$. Month 2: interest $=0.01\times14\,360=143.60$, so $V_2=14\,360+143.60-1800=12\,703.60$, that is $\$12\,703.60$.

**Q2.** The factor 1.02 is an interest rate of 2% per period, and the subtracted constant is the withdrawal $\$1\,500$. Iterating from $V_0=12\,000$: $V_1=12\,000+240-1500=10\,740$; $V_2=10\,740+214.80-1500=9\,454.80$; $V_3=9\,454.80+189.10-1500=8\,143.90$ (interest $0.02\times9\,454.80=189.096\to189.10$). So $V_1=\$10\,740.00$, $V_2=\$9\,454.80$ and $V_3=\$8\,143.90$.

**Q3.** $r=\frac{6}{4}=1.5\%$ per quarter, $R=1.015$; each quarter the interest is 1.5% of the previous balance (to the nearest cent) and $\$1\,000$ is repaid. Quarter 1: interest $=0.015\times8000=120$, balance $8000+120-1000=7120$. Quarter 2: interest $=0.015\times7120=106.80$, balance $7120+106.80-1000=6226.80$. Quarter 3: interest $=0.015\times6226.80=93.402\to93.40$, balance $6226.80+93.40-1000=5320.20$. Quarter 4: interest $=0.015\times5320.20=79.803\to79.80$, balance $5320.20+79.80-1000=4400.00$. The balance after one year is $\$4\,400.00$. The interest column sums to $120+106.80+93.40+79.80=\$400.00$, the total interest charged in the year.

**Q4.** $r=\frac{8}{4}=2\%$ per quarter, $R=1.02$; the recurrence is $V_{n+1}=1.02V_n-1100$ with $V_0=5000$. Quarter 1: interest 100, balance $5000+100-1100=4000.00$. Quarter 2: interest 80, balance $4000+80-1100=2980.00$. Quarter 3: interest 59.60, balance $2980+59.60-1100=1939.60$. Quarter 4: interest $=0.02\times1939.60=38.792\to38.79$, balance $1939.60+38.79-1100=878.39$. After four withdrawals $\$878.39$ remains, which is less than $\$1100$. In quarter 5 the interest is $0.02 \times 878.39 = 17.5678 \to 17.57$, giving $878.39 + 17.57 = \$895.96$ before any withdrawal. Since this is below $\$1\,100$, the final withdrawal is $\$895.96$, which empties the account.

**Q5.** (a) The interest in the first month is $0.02 \times 14\,000 = \$280.00$. (b) The balance after one month is opening balance plus interest minus repayment: $14\,000 + 280 - D = 12\,680$, so $D = 14\,280 - 12\,680 = \$1\,600$. (c) With $R = 1.02$ and $D = 1600$, the recurrence is $V_{n+1} = 1.02V_n - 1600$, $V_0 = 14\,000$. (d) Month 2: interest $= 0.02 \times 12\,680 = 253.60$, balance $12\,680 + 253.60 - 1600 = 11\,333.60$, that is $\$11\,333.60$.

**Q6.** The factor $1.005$ is 0.5% interest per month and the subtracted $400$ is the fixed payment. (a) As a loan: $\$6\,000$ is borrowed, interest is charged at 0.5% per month, and $\$400$ is repaid each month. (b) As an annuity: $\$6\,000$ is the amount invested, 0.5% is the interest earned each month, and $\$400$ is the amount withdrawn each month. (c) Iterating: $V_1 = 6000 + 30 - 400 = 5630.00$; interest next month $= 0.005 \times 5630 = 28.15$, so $V_2 = 5630 + 28.15 - 400 = 5258.15$. So $V_1 = \$5\,630.00$ and $V_2 = \$5\,258.15$ for both interpretations.

**Q7.** $r = \frac{9}{12} = 0.75\,\%$ per month, $R = 1.0075$; the recurrence is $V_{n+1} = 1.0075V_n - 2500$ with $V_0 = 25\,000$. Month 1: interest $= 0.0075 \times 25\,000 = 187.50$, balance $25\,000 + 187.50 - 2500 = 22\,687.50$. Month 2: interest $= 0.0075 \times 22\,687.50 = 170.15625 \to 170.16$, balance $22\,687.50 + 170.16 - 2500 = 20\,357.66$. Month 3: interest $= 0.0075 \times 20\,357.66 = 152.68245 \to 152.68$, balance $20\,357.66 + 152.68 - 2500 = 18\,010.34$. So the balance after three months is $\$18\,010.34$.

**Q8.** $r = \frac{6}{12} = 0.5\,\%$ per month, $R = 1.005$; the recurrence is $V_{n+1} = 1.005V_n - 3500$ with $V_0 = 50\,000$. Iterating for four months:

$$46\,750.00, 43\,483.75, 40\,201.17, 36\,902.18,$$

so the balance remaining after four months is $\$36\,902.18$.

**Q9.** The recurrence is $V_{n+1} = 1.015V_n - 1600$ with $V_0 = 7200$. $V_1 = 7200 + 108 - 1600 = 5708.00$; $V_2 = 5708 + 85.62 - 1600 = 4193.62$; $V_3 = 4193.62 + 62.90 - 1600 = 2656.52$ (interest $62.9043 \to 62.90$); $V_4 = 2656.52 + 39.85 - 1600 = 1096.37$ (interest $39.8478 \to 39.85$). After four repayments $\$1\,096.37$ remains. In month 5 the interest is $0.015 \times 1096.37 = 16.44555 \to 16.45$, giving $1096.37 + 16.45 = \$1\,112.82$ owing before the repayment, which is below $\$1\,600$. So the loan takes **five months**: four repayments of $\$1\,600$ and a final repayment of $\$1\,112.82$. The total repaid is $4 \times 1600 + 1112.82 = \$7\,512.82$, so the interest paid is $7512.82 - 7200 = \$312.82$.

**Q10.** $r = \frac{4.8}{12} = 0.4\,\%$ per month, $R = 1.004$; the recurrence is $V_{n+1} = 1.004V_n - 2000$ with $V_0 = 30\,000$. Iterating for six months (interest to the cent):

$$28\,120.00, 26\,232.48, 24\,337.41, 22\,434.76, 20\,524.50, 18\,606.60,$$

so the balance after six months is $\$18\,606.60$. The interest earned is the sum of the interest column, $120 + 112.48 + 104.93 + 97.35 + 89.74 + 82.10 = \$606.60$. (Check: six withdrawals total $\$12\,000$ and the balance fell by $30\,000 - 18\,606.60 = 11\,393.40$, and $12\,000 - 11\,393.40 = \$606.60$.)

**Q11.** $r = 1\,\%$ per month, $R = 1.01$, $V_0 = 14\,500$. (a) $D = 2600$: $14\,645 - 2600 = 12\,045.00$; then $12\,165.45 - 2600 = 9565.45$; $9661.10 - 2600 = 7061.10$; $7131.71 - 2600 = 4531.71$. So $V_4 = \$4\,531.71$. (b) $D = 3200$: $14\,645 - 3200 = 11\,445.00$; then $11\,559.45 - 3200 = 8359.45$; $8443.04 - 3200 = 5243.04$; $5295.47 - 3200 = 2095.47$. So $V_4 = \$2\,095.47$. (c) The difference is $4531.71 - 2095.47 = \$2\,436.24$; the larger repayment reduces the balance faster.

**Q12.** $r = 1.5\,\%$ per quarter, $R = 1.015$. (a) The interest in the first quarter is $0.015 \times 18\,000 = 270$, so $18\,000 + 270 - D = 16\,770$, giving $D = 18\,270 - 16\,770 = \$1\,500$. (b) Quarter 2: interest $= 0.015 \times 16\,770 = 251.55$, balance $16\,770 + 251.55 - 1500 = 15\,521.55$, that is $\$15\,521.55$.

**Q13.** Each month the interest is 0.6% of the previous balance (to the nearest cent) and $\$2500$ is withdrawn. $A=0.006\times 17620=\$105.72$, so $B=17620+105.72-2500=\$15225.72$. $C=0.006\times 15225.72=91.35432\rightarrow \$91.35$, so $D=15225.72+91.35-2500=\$12817.07$.

**Q14.** The recurrence is $V_{n+1}=1.01V_n-870$ with $V_0=5000$. $V_1=5000+50-870=4180.00$; $V_2=4180+41.80-870=3351.80$; $V_3=3351.80+33.52-870=2515.32$ (interest $33.518\rightarrow 33.52$); $V_4=2515.32+25.15-870=1670.47$ (interest $25.1532\rightarrow 25.15$); $V_5=1670.47+16.70-870=817.17$ (interest $16.7047\rightarrow 16.70$). After five repayments $\$817.17$ remains. In month 6 the interest is $0.01\times 817.17=8.1717\rightarrow 8.17$, giving $817.17+8.17=\$825.34$ owing, which is below $\$870$. So the loan takes **six months**: five repayments of $\$870$ and a final repayment of $\$825.34$. The total repaid is $5\times 870+825.34=\$5175.34$, so the interest paid is $5175.34-5000=\$175.34$.

**Q15.** (a) In each period a reducing-balance loan and an annuity do exactly the same two things: the balance earns (or is charged) interest, which multiplies it by $R=1+\frac{r}{100}$, and then a fixed amount $D$ is taken out — a repayment for the loan, a withdrawal for the annuity. Because both operations are the same, the recurrence $V_{n+1}=RV_n-D$ is the same; only the meaning of $V_0$ (amount owed vs amount invested) and of $D$ (repayment vs withdrawal) differs. (b) When the balance first reaches zero, a loan has been fully repaid — nothing more is owed — while an annuity has been fully drawn down — the invested fund is exhausted and no further income can be taken.

**Q16.** Both annuities use $r=0.5\%$ per month, $R=1.005$, $V_0=15000$. Annuity P ($D=1000$): iterating $V_{n+1}=1.005V_n-1000$ for five months gives $14075.00, 13145.38, 12211.11, 11272.17, 10328.53$, so $\$10328.53$. Annuity Q ($D=1600$): iterating $V_{n+1}=1.005V_n-1600$ gives $13475.00, 11942.38, 10402.09, 8854.10, 7298.37$, so $\$7298.37$. Annuity Q's balance is falling much faster (its withdrawals are larger while both earn the same interest), so **annuity Q** will be exhausted sooner.

**Q17.** (a) The balance after one month is $9000+\text{interest}-1200=8070$, so the interest is $8070-9000+1200=\$270.00$. (b) The interest is $r\times 9000=270$, so $r=\frac{270}{9000}=0.03=3\%$ per month and $R=1.03$. (c) Month 2: interest $=0.03\times 8070=242.10$, balance $8070+242.10-1200=7112.10$, that is $\$7112.10$.

**Q18.** (a) The interest earned each month on $\$80000$ is $0.005\times 80000=\$400$. If the withdrawal is exactly $\$400$, then each month $V_{n+1}=80000+400-400=80000$, so the balance never changes; the recurrence is $V_{n+1}=1.005V_n-400$ and the balance stays at $\$80000$. (This is a *perpetuity*, a special case examined later in this chapter.) (b) If $D>\$400$, more is taken out than the interest replaces, so the balance falls each month and is eventually exhausted (a finite annuity). If $D<\$400$, less is taken out than the interest adds, so the balance grows.

**Q19.** The recurrence is $V_{n+1}=1.01V_n-1400$ with $V_0=6500$. $V_1=6500+65-1400=5165.00$; $V_2=5165+51.65-1400=3816.65$; $V_3=3816.65+38.17-1400=2454.82$ (interest $38.1665\rightarrow 38.17$); $V_4=2454.82+24.55-1400=1079.37$ (interest $24.5482\rightarrow 24.55$). After four withdrawals $\$1079.37$ remains. In month 5 the interest is $0.01\times 1079.37=10.7937\rightarrow 10.79$, giving $1079.37+10.79=\$1090.16$ before withdrawal, which is below $\$1400$. So the annuity lasts **five months**: four withdrawals of $\$1400$ and a final withdrawal of $\$1090.16$. The total received is $4\times 1400+1090.16=\$6690.16$, so the interest earned is $6690.16-6500=\$190.16$.

**Q20.** (a) $r=\frac{12}{12}=1\%$ per month, $R=1.01$, so the recurrence is $V_{n+1}=1.01V_n-1300$ with $V_0=15000$. (b) Iterating for six months (interest to the nearest cent each month):

$$13850.00, 12688.50, 11515.39, 10330.54, 9133.85, 7925.19,$$

so the balance owing after six months is $\$7925.19$. (c) The finance solver evaluates a single formula and rounds only at the end, giving $\$7925.18$, whereas the table rounds the interest to the nearest cent every month. Those small monthly roundings accumulate to one cent, so the table's $\$7925.19$ is one cent above the solver's $\$7925.18$.

### Theory

An earlier section modelled a **reducing-balance loan** (and, with the same arithmetic, an **annuity**) by the recurrence

$$V_{n+1} = RV_n - D, V_0 = \text{starting amount},$$

where $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r\%$ is the interest rate per period) and $D$ is the fixed payment made at the end of each period — a **repayment** for a loan or a **withdrawal** for an annuity. Iterating the recurrence generates the balance $V_1, V_2, V_3, \ldots$, which falls towards zero. This section looks *inside* each payment.

**What each payment does.** When a payment is made, two things happen. First the period's **interest** is charged on the balance carried into the period; then the payment is applied — part of it covers that interest and the **rest reduces the balance owing**. So every payment splits into two parts:

$$\text{payment} = \text{interest part} + \text{principal-reduction part}.$$

The **interest part** is what the lender charges for the period; the **principal-reduction part** (often just called the *principal reduction*) is the amount by which the debt — the *principal* — actually shrinks. For an annuity, read "interest earned" and "reduction in the capital of the fund".

**The amortisation table.** Setting this out period by period gives an **amortisation table**, with one row per payment and five columns:

- **payment number** — the period, $1, 2, 3, \ldots$ (row $0$ records the opening balance);
- **payment** — the amount paid that period (the fixed $D$, except for a smaller final payment);
- **interest** — the interest for the period, $\text{interest} = \frac{r}{100} \times (\text{previous balance})$, rounded to the nearest cent;
- **principal reduction** — the part of the payment that reduces the balance, $\text{principal reduction} = \text{payment} - \text{interest}$;
- **balance** — the amount still owing after the payment, $\text{balance} = \text{previous balance} - \text{principal reduction}$.

The balance column adds no new information: taking off the principal reduction is the same as adding the interest and subtracting the payment, because $\text{previous balance} - (\text{payment} - \text{interest}) = \text{previous balance} + \text{interest} - \text{payment}$ — exactly the recurrence $V_{n+1} = RV_n - D$. The extra columns simply record *how* each payment was split.

**Rounding convention.** Following the standard used throughout this chapter, the interest is rounded to the nearest cent **each period** before it is used, and the rounded balance is carried forward. Then every entry is exact to the cent and each row reconciles:

$$\text{balance} = \text{previous balance} + \text{interest (to the nearest cent)} - \text{payment}.$$

**Building the table row by row.** Start with the opening balance. For each period: work out the interest on the current balance and round it; subtract that interest from the payment to get the principal reduction; subtract the principal reduction from the balance. Carry the new balance down to the next row and repeat. A short example — borrow $\$2\,000$ at $1\%$ per month and repay $\$700$ at the end of each month:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$2\,000.00$ |
| 1 | $\$700.00$ | $\$20.00$ | $\$680.00$ | $\$1\,320.00$ |
| 2 | $\$700.00$ | $\$13.20$ | $\$686.80$ | $\$633.20$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 3 | $\$639.53$ | $\$6.33$ | $\$633.20$ | $\$0.00$ |

**The pattern down the table.** As the balance falls, the interest charged on it falls too, so a larger slice of each equal payment is left to reduce the principal. Reading down the table the **interest column decreases** and the **principal-reduction column increases**, payment after payment — the debt is cleared faster and faster near the end. (This is why the interest part of an early repayment on a long loan can be much larger than the part that reduces the debt.)

**The final payment.** The balance rarely lands exactly on zero after a whole number of equal payments. Usually a small balance is left that, together with its final period's interest, is *less* than the regular payment. The **last payment** is then just that reduced amount — enough to clear the balance to exactly $\$0.00$ and no more.

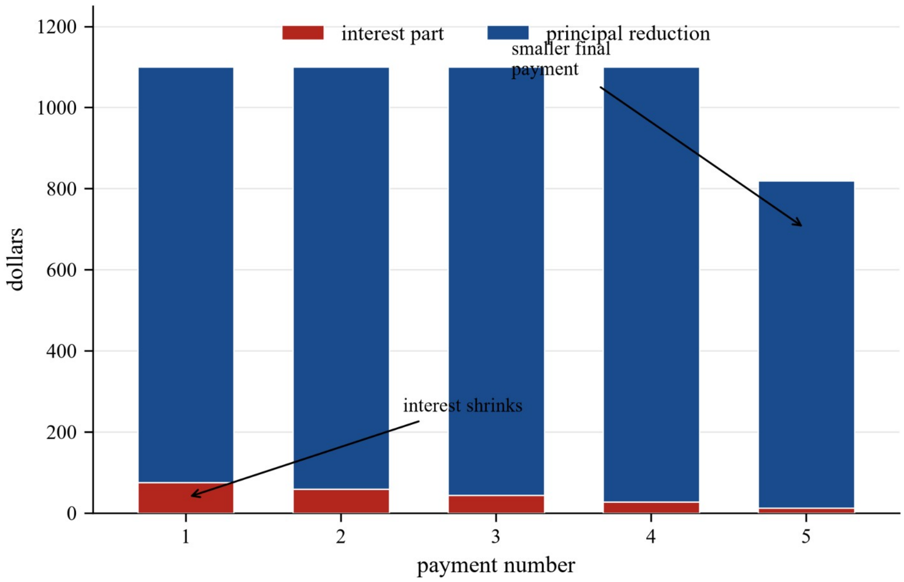
**Totals.** Two totals are read straight off the completed table:

- the **total paid** is the sum of the payment column — for a loan this is the *total cost* of borrowing; for an annuity it is the total income received;
- the **total interest** is the sum of the interest column. Because the principal-reduction column must add up to the original balance $V_0$ (the debt is reduced to zero), the total interest is also

$$\text{total interest} = \text{total paid} - V_0.$$

**Using technology.** In a technology-active course a CAS **finance solver** (TVM) gives the regular payment, the number of periods or a balance directly, and many calculators will generate a full amortisation table. The solver works from a single formula and does **not** round period by period, so a balance or a final payment it reports can differ from a hand-built table by a cent or two; where that happens it is worth noting, not hiding. You should still be able to build and read the table by hand, both to understand where each figure comes from and to check the technology.

The figure below breaks each equal payment of a reducing-balance loan into its two parts. Every bar is the same height (equal payments), but the interest part at the foot of each bar shrinks and the principal-reduction part stacked above it grows from one payment to the next; the final bar is shorter because the last payment is smaller.




## Worked examples

### *Example 1 — scaffolded*

A loan of $\$12\,000$ is charged interest of $12\%$ p.a. compounded monthly and is repaid by $\$1\,500$ at the end of each month. Construct the first three rows of an amortisation table, showing the interest, the principal reduction and the balance owing after each repayment.

| Working | Comment |
|---|---|
| $r=\frac{12}{12}=1\%$ per month, so the interest each period is $0.01\times$ (previous balance). | Monthly rate = annual rate $\div 12$. |
| Month 1: interest $=0.01\times 12\,000=120$; principal reduction $=1500-120=1380$; balance $=12\,000-1380=10\,620$. | Interest on the opening balance; the rest of the $\$1\,500$ reduces the debt. |
| Month 2: interest $=0.01\times 10\,620=106.20$; principal reduction $=1500-106.20=1393.80$; balance $=10\,620-1393.80=9\,226.20$. | Use the *new* balance for the next interest. |
| Month 3: interest $=0.01\times 9\,226.20=92.262\rightarrow 92.26$; principal reduction $=1500-92.26=1407.74$; balance $=9\,226.20-1407.74=7\,818.46$. | Round the interest to the cent first. |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$12\,000.00$ |
| 1 | $\$1\,500.00$ | $\$120.00$ | $\$1\,380.00$ | $\$10\,620.00$ |
| 2 | $\$1\,500.00$ | $\$106.20$ | $\$1\,393.80$ | $\$9\,226.20$ |
| 3 | $\$1\,500.00$ | $\$92.26$ | $\$1\,407.74$ | $\$7\,818.46$ |

Notice the interest falls ($\$120\rightarrow\$106.20\rightarrow\$92.26$) while the principal reduction rises, as expected.

### *Example 2 — scaffolded*

A loan of $\$5\,000$ is charged $6\%$ p.a. compounded quarterly and is repaid by $\$1\,100$ at the end of each quarter. Construct the full amortisation table, and hence find the size of the final repayment, the total amount repaid and the total interest charged.

| Working | Comment |
|---|---|
| $r=\frac{6}{4}=1.5\%$ per quarter, so interest $=0.015\times$ (previous balance). | Quarterly rate = annual rate $\div 4$. |
| Each row: interest $=0.015\times$ balance (to the cent); principal reduction $=1100-\text{interest}$; new balance $=\text{balance}-\text{principal reduction}$. | The recurrence, split into its two parts. |
| Final quarter: only $\$806.83$ is owed. Its interest is $0.015\times 806.83=12.10$, so $\$806.83+\$12.10=\$818.93$ clears the loan — a smaller last repayment. | Iterate until the balance plus its interest is no more than $\$1\,100$. |

Iterating one quarter at a time gives the amortisation table.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,000.00$ |
| 1 | $\$1\,100.00$ | $\$75.00$ | $\$1\,025.00$ | $\$3\,975.00$ |
| 2 | $\$1\,100.00$ | $\$59.63$ | $\$1\,040.37$ | $\$2\,934.63$ |
| 3 | $\$1\,100.00$ | $\$44.02$ | $\$1\,055.98$ | $\$1\,878.65$ |
| 4 | $\$1\,100.00$ | $\$28.18$ | $\$1\,071.82$ | $\$806.83$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 5 | $\$818.93$ | $\$12.10$ | $\$806.83$ | $\$0.00$ |

The final repayment is $\$818.93$. The total repaid is the sum of the payment column, $4 \times 1100 + 818.93 = \$5\,218.93$, so the total interest charged is $5218.93 - 5000 = \$218.93$ — the same as the sum of the interest column, $75 + 59.63 + 44.02 + 28.18 + 12.10 = \$218.93$. (The principal-reduction column adds to $\$5\,000$, the amount borrowed.)

### *Example 3 — unscaffolded*

The amortisation table below records the first four monthly repayments of a reducing-balance loan of $\$9\,000$ charged $1\%$ interest per month and repaid by $\$1\,600$ per month. Find the missing entries $A$, $B$ and $C$, state how much of the third repayment went towards reducing the debt, and find the total interest charged over the four months.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,000.00$ |
| 1 | $\$1\,600.00$ | $\$90.00$ | $\$1\,510.00$ | $\$7\,490.00$ |
| 2 | $\$1\,600.00$ | $\$74.90$ | $\$1\,525.10$ | $\$5\,964.90$ |
| 3 | $\$1\,600.00$ | $A$ | $B$ | $\$4\,424.55$ |
| 4 | $\$1\,600.00$ | $\$44.25$ | $C$ | $\$2\,868.80$ |

Entry $A$ is the interest for month 3, charged on the balance carried in from month 2: $A = 0.01 \times 5964.90 = 59.649 \rightarrow \$59.65$. The principal reduction is the rest of the payment, $B = 1600 - 59.65 = \$1\,540.35$ (and as a check, $5964.90 - 1540.35 = 4424.55$, the balance shown). Entry $C$ is the principal reduction for month 4, $C = 1600 - 44.25 = \$1\,555.75$ (check: $4424.55 - 1555.75 = 2868.80$).

The third repayment reduced the debt by $B = \$1\,540.35$. The total interest over the four months is the sum of the interest column, $90 + 74.90 + 59.65 + 44.25 = \$268.80$.

$\therefore A = \$59.65$, $B = \$1\,540.35$, $C = \$1\,555.75$; the third repayment reduced the debt by $\$1\,540.35$, and $\$268.80$ interest was charged over the four months.

### *Example 4 — unscaffolded*

A community club puts a $\$9\,600$ grant into an annuity earning $6\%$ p.a. compounded monthly and draws $\$2\,500$ from the fund at the end of each month to run a coaching program. Construct the amortisation table and find the size of the final withdrawal and the total interest earned. Compare the income the fund provides with the income the same $\$9\,600$ would provide if it earned no interest at all.

The monthly rate is $\frac{6}{12} = 0.5\%$, so each month the interest earned is $0.005 \times$ (previous balance) and the rest of the $\$2\,500$ withdrawal is drawn from the fund's capital.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,600.00$ |
| 1 | $\$2\,500.00$ | $\$48.00$ | $\$2\,452.00$ | $\$7\,148.00$ |
| 2 | $\$2\,500.00$ | $\$35.74$ | $\$2\,464.26$ | $\$4\,683.74$ |
| 3 | $\$2\,500.00$ | $\$23.42$ | $\$2\,476.58$ | $\$2\,207.16$ |
| 4 | $\$2\,218.20$ | $\$11.04$ | $\$2\,207.16$ | $\$0.00$ |

After three full withdrawals only $\$2207.16$ remains. In the fourth month it earns $0.005 \times 2207.16 = 11.0358 \to 11.04$ (to the nearest cent), giving $2207.16 + 11.04 = \$2218.20$, so a final withdrawal of $\$2218.20$ empties the fund. The total withdrawn is $3 \times 2500 + 2218.20 = \$9718.20$, so the total interest earned is $9718.20 - 9600 = \$118.20$ (the sum of the interest column, $48 + 35.74 + 23.42 + 11.04$).

With no interest at all the $\$9600$ would simply be shared out as $9600 = 3 \times 2500 + 2100$: the same three full withdrawals, but a final one of only $\$2100.00$. Earning interest therefore does not buy an extra withdrawal here; it makes the last withdrawal $2218.20 - 2100 = \$118.20$ bigger — exactly the total interest earned, since every cent the fund earns is a cent that is eventually withdrawn.

$\therefore$ the annuity provides three withdrawals of $\$2500$ and a final withdrawal of $\$2218.20$, earning $\$118.20$ in interest, which is what lifts the final withdrawal from $\$2100.00$ to $\$2218.20$.

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### Practice questions

#### *Scaffolded*

**Q1.** A loan of $\$14\,000$ is charged $18\,\%$ p.a. compounded monthly and is repaid by $\$1300$ at the end of each month. (a) State the monthly interest rate. (b) Find the interest charged in the first month. (c) Find the principal reduction of the first repayment. (d) Find the balance owing after the first repayment.

**Q2.** A loan of $\$6\,000$ is charged $1\,\%$ interest per month and repaid by $\$1\,100$ per month. Copy and complete the first three rows of the amortisation table.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$6\,000.00$ |
| 1 | $\$1\,100.00$ | | | |
| 2 | $\$1\,100.00$ | | | |
| 3 | $\$1\,100.00$ | | | |

**Q3.** A loan of $\$5\,400$ is charged $1.25\,\%$ interest per month and repaid by $\$1\,900$ per month. (a) Copy and complete the amortisation table until the loan is repaid.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,400.00$ |
| 1 | $\$1\,900.00$ | | | |
| 2 | $\$1\,900.00$ | | | |
| 3 | | | | $\$0.00$ |

(b) State the size of the final repayment.

(c) Find the total interest charged.

**Q4.** The completed amortisation table below is for a loan of $\$10\,000$ charged $2\,\%$ interest per quarter and repaid by $\$2\,800$ per quarter.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$10\,000.00$ |
| 1 | $\$2\,800.00$ | $\$200.00$ | $\$2\,600.00$ | $\$7\,400.00$ |
| 2 | $\$2\,800.00$ | $\$148.00$ | $\$2\,652.00$ | $\$4\,748.00$ |
| 3 | $\$2\,800.00$ | $\$94.96$ | $\$2\,705.04$ | $\$2\,042.96$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 4 | $\$2083.82$ | $\$40.86$ | $\$2042.96$ | $\$0.00$ |

Use the table to answer the following. (a) How much interest was charged in the second quarter? (b) How much of the third repayment reduced the amount owed? (c) What was the balance owing after the third repayment? (d) State the size of the final repayment. (e) Find the total interest charged over the life of the loan.

**Q5.** A loan of $\$15000$ is charged $0.6\%$ interest per month and repaid by $\$2600$ per month. Row 3 of its amortisation table has been left blank.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$15000.00$ |
| 1 | $\$2600.00$ | $\$90.00$ | $\$2510.00$ | $\$12490.00$ |
| 2 | $\$2600.00$ | $\$74.94$ | $\$2525.06$ | $\$9964.94$ |
| 3 | $\$2600.00$ | $A$ | $B$ | $C$ |

(a) Charge $0.6\%$ on the balance carried in from month 2, rounding to the nearest cent, to get $A$.
(b) Take $A$ off the $\$2600$ repayment to get the principal reduction $B$.
(c) Take $B$ off the month-2 balance to get $C$, then check your row by confirming that $C$ is also the month-2 balance plus $A$ minus $\$2600$.

**Q6.** A loan of $\$7000$ is charged $6\%$ p.a. compounded quarterly and repaid by $\$1600$ per quarter. (a) Construct the amortisation table until the loan is repaid. (b) State the size of the final repayment. (c) Find the total interest charged. (d) Find the total cost of the loan (the total amount repaid).

### *Unscaffolded*

**Q7.** A loan of $\$8000$ is charged $9\%$ p.a. compounded monthly and repaid by $\$900$ per month. Construct the first three rows of the amortisation table, and state the interest and the principal reduction of the third repayment.

**Q8.** A loan of $\$3000$ is charged $2\%$ interest per month and repaid by $\$650$ per month. By constructing the amortisation table, find how many repayments are needed to clear the loan, the size of the final repayment, and the total interest charged.

**Q9.** An annuity of $\$9000$ earns $1.5\%$ interest per period, and $\$2000$ is withdrawn at the end of each period. By constructing the amortisation table, find the size of the final withdrawal, the total amount withdrawn, and the total interest earned.

**Q10.** The completed amortisation table below is for a loan of $\$20000$ charged $1\%$ interest per month and repaid by $\$3500$ per month.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$20000.00$ |
| 1 | $\$3500.00$ | $\$200.00$ | $\$3300.00$ | $\$16700.00$ |
| 2 | $\$3500.00$ | $\$167.00$ | $\$3333.00$ | $\$13367.00$ |
| 3 | $\$3500.00$ | $\$133.67$ | $\$3366.33$ | $\$10000.67$ |
| 4 | $\$3500.00$ | $\$100.01$ | $\$3399.99$ | $\$6600.68$ |
| 5 | $\$3500.00$ | $\$66.01$ | $\$3433.99$ | $\$3166.69$ |
| 6 | $\$3198.36$ | $\$31.67$ | $\$3166.69$ | $\$0.00$ |

(a) How much interest was charged in the fourth month?
(b) How much of the fifth repayment reduced the amount owed?

(c) What was the balance owing after the third repayment?
(d) State the size of the final repayment.
(e) Find the total interest charged over the life of the loan.

**Q11.** A loan of $\$10\,000$ is charged $1\%$ interest per month and repaid by $\$1\,200$ per month. Construct the first three rows of the amortisation table, and describe how the interest and the principal reduction change from one repayment to the next.

**Q12.** A reducing-balance loan charged $0.9\%$ interest per month is repaid by equal monthly instalments. In one month the repayment takes $\$1\,127.10$ off the debt, leaving $\$7\,872.90$ still owing at the end of that month. (a) Find the balance owing at the start of that month. (b) Find the interest charged that month. (c) Hence find the size of the monthly instalment.

**Q13.** A loan of $\$12\,000$ is charged $1\%$ interest per month and repaid by $\$2\,050$ per month. The first three rows of its amortisation table are below, with row 3 incomplete.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$12\,000.00$ |
| 1 | $\$2\,050.00$ | $\$120.00$ | $\$1\,930.00$ | $\$10\,070.00$ |
| 2 | $\$2\,050.00$ | $\$100.70$ | $\$1\,949.30$ | $\$8\,120.70$ |
| 3 | $\$2\,050.00$ | $A$ | $B$ | $C$ |

Complete row 3 by finding $A$, $B$ and $C$, then add the interest column to get $D$, the total interest charged over the first three repayments.

**Q14.** The amortisation table below is for a loan of $\$8\,000$ charged $1\%$ interest per month and repaid by $\$1\,700$ per month. Exactly one entry in it has been typed incorrectly.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$8\,000.00$ |
| 1 | $\$1\,700.00$ | $\$80.00$ | $\$1\,620.00$ | $\$6\,380.00$ |
| 2 | $\$1\,700.00$ | $\$63.80$ | $\$1\,636.20$ | $\$4\,743.80$ |
| 3 | $\$1\,700.00$ | $\$47.44$ | $\$1\,652.56$ | $\$3\,091.24$ |
| 4 | $\$1\,700.00$ | $\$30.19$ | $\$1\,669.09$ | $\$1\,422.15$ |
| 5 | $\$1\,436.37$ | $\$14.22$ | $\$1\,422.15$ | $\$0.00$ |

Identify the incorrect entry, give two separate checks that show it must be wrong, and state its correct value.

**Q15.** In the amortisation table of a reducing-balance loan repaid by equal instalments: (a) explain why the interest part of each repayment gets smaller while the principal-reduction part gets larger, payment after payment; (b) explain why the final repayment is smaller than the others.

**Q16.** A loan of $\$10\,000$ is charged $1.5\%$ interest per quarter and repaid by $\$1\,800$ per quarter. By constructing the amortisation table, find the total amount repaid (the total cost of the loan) and the total interest charged.

**Q17.** A reducing-balance loan is repaid by monthly instalments of $\$1\,500$. At the end of one month the balance owing is $\$8\,000$, and in the following month the interest charged is $\$120.00$. (a) Find the monthly interest rate. (b) Find the principal reduction of the next repayment. (c) Find the balance owing after that repayment.

**Q18.** An annuity of $\$12\,000$ earns $1\%$ interest per period, and $\$2\,500$ is withdrawn at the end of each period until the fund is empty. Its amortisation table is shown in full below.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$12\,000.00$ |
| 1 | $\$2\,500.00$ | $\$120.00$ | $\$2\,380.00$ | $\$9\,620.00$ |
| 2 | $\$2\,500.00$ | $\$96.20$ | $\$2\,403.80$ | $\$7\,216.20$ |
| 3 | $\$2\,500.00$ | $\$72.16$ | $\$2\,427.84$ | $\$4\,788.36$ |
| 4 | $\$2\,500.00$ | $\$47.88$ | $\$2\,452.12$ | $\$2\,336.24$ |
| 5 | $\$2\,359.60$ | $\$23.36$ | $\$2\,336.24$ | $\$0.00$ |

(a) Split the third withdrawal into the interest earned that period and the amount taken from the capital of the fund.
(b) State how many full withdrawals of $\$2\,500$ the fund provides, and the size of the final withdrawal.
(c) Add the interest column to find the total interest earned, and check that total against the total withdrawn minus the $\$12\,000$ invested.

**Q19.** A loan of $\$26\,000$ is charged $1\%$ interest per month and repaid by $\$500$ at the end of each month. (a) Show that in the first repayment the interest exceeds the principal reduction. (b) Find the first repayment for which the principal reduction exceeds the interest.

**Q20.** A loan of $\$16\,000$ is charged $12\%$ p.a. compounded monthly and repaid by $\$1\,500$ per month. (a) Construct the amortisation table for the first six months. (b) Find the total interest charged in the first six months. (c) State the balance owing after six months. (d) A finance solver returns a balance of $\$7\,756.30$ after six months. Explain why this differs by one cent from your answer to (c).

## Answers

**Q1.** (a) $1.5\%$ per month; (b) $\$210.00$; (c) $\$1090.00$; (d) $\$12910.00$. **Q2.** Interest $\$60.00, \$49.60, \$39.10$; principal reduction $\$1040.00, \$1050.40, \$1060.90$; balances $\$4960.00, \$3909.60, \$2848.70$. **Q3.** (a) interest $\$67.50, \$44.59, \$21.40$; principal reduction $\$1832.50, \$1855.41, \$1712.09$; balances $\$3567.50, \$1712.09, \$0.00$; (b) $\$1733.49$; (c) $\$133.49$. **Q4.** (a) $\$148.00$; (b) $\$2705.04$; (c) $\$2042.96$; (d) $\$2083.82$; (e) $\$483.82$. **Q5.** (a) $A=\$59.79$; (b) $B=\$2540.21$; (c) $C=\$7424.73$ (check: $9964.94+59.79-2600=7424.73$). **Q6.** (a) interest $\$105.00, \$82.58, \$59.81, \$36.71, \$13.26$; balances $\$5505.00, \$3987.58, \$2447.39, \$884.10, \$0.00$; (b) $\$897.36$; (c) $\$297.36$; (d) $\$7297.36$. **Q7.** Interest $\$60.00, \$53.70, \$47.35$; balances $\$7160.00, \$6313.70, \$5461.05$; third repayment: interest $\$47.35$, principal reduction $\$852.65$. **Q8.** Five repayments (four of $\$650$ and a final of $\$579.62$); total interest $\$179.62$. **Q9.** Five withdrawals (four of $\$2000$ and a final of $\$1391.03$); total withdrawn $\$9391.03$; total interest earned $\$391.03$. **Q10.** (a) $\$100.01$; (b) $\$3433.99$; (c) $\$10000.67$; (d) $\$3198.36$; (e) $\$698.36$. **Q11.** Interest $\$100.00, \$89.00, \$77.89$; principal reduction $\$1100.00, \$1111.00, \$1122.11$; balances $\$8900.00, \$7789.00, \$6666.89$. The interest falls and the principal reduction rises each month. **Q12.** (a) $\$9000.00$; (b) $\$81.00$; (c) $\$1208.10$. **Q13.** $A=\$81.21$, $B=\$1968.79$, $C=\$6151.91$; $D=\$301.91$. **Q14.** The interest in row 4 is wrong: it should be $\$30.91$, not $\$30.19$. Checks: $0.01\times 3091.24=\$30.91$, and interest + principal reduction must be the $\$1700$ repayment, but $30.19+1669.09=\$1699.28$. **Q15.** (a) The balance falls each period, so the interest charged on it falls; with equal repayments the part left over to reduce the principal therefore rises. (b) After the last full repayment the balance owing (plus its interest) is less than the usual instalment, so the final repayment is only that reduced amount. **Q16.** Total repaid $\$10521.24$; total interest $\$521.24$. **Q17.** (a) $1.5\%$ per month; (b) $\$1380.00$; (c) $\$6620.00$. **Q18.** (a) interest earned $\$72.16$, drawn from capital $\$2427.84$; (b) four full withdrawals of $\$2500$ and a final withdrawal of $\$2359.60$; (c) $\$359.60$ (check: $12359.60-12000=359.60$). **Q19.** (a) interest $\$260.00 >$ principal reduction $\$240.00$; (b) the sixth repayment (principal reduction $\$252.24 >$ interest $\$247.76$). **Q20.** (a) balances $\$14660.00, \$13306.60, \$11939.67, \$10559.07, \$9164.66, \$7756.31$; (b) $\$756.31$; (c) $\$7756.31$; (d) the solver uses one formula without rounding to the cent each month, whereas the table rounds every month; those roundings accumulate to one cent, so the table finishes one cent above the solver.

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## Fully-worked solutions

**Q1.** $r=\frac{18}{12}=1.5\%$ per month. (a) The monthly rate is $1.5\%$. (b) The first month's interest is $0.015\times 14000=\$210.00$. (c) The principal reduction is $1300-210=\$1090.00$. (d) The balance owing is $14000-1090=\$12910.00$.

**Q2.** Each month the interest is $1\%$ of the previous balance (to the nearest cent) and the principal reduction is $1100-\text{interest}$. Month 1: interest $=0.01\times 6000=60$, principal reduction $1100-60=1040$, balance $6000-1040=4960$. Month 2: interest $=0.01\times 4960=49.60$, principal reduction $1050.40$, balance $3909.60$. Month 3: interest $=0.01\times 3909.60=39.096\rightarrow 39.10$, principal reduction $1060.90$, balance $2848.70$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$6000.00$ |
| 1 | $\$1100.00$ | $\$60.00$ | $\$1040.00$ | $\$4960.00$ |
| 2 | $\$1100.00$ | $\$49.60$ | $\$1050.40$ | $\$3909.60$ |
| 3 | $\$1100.00$ | $\$39.10$ | $\$1060.90$ | $\$2848.70$ |

**Q3.** Interest is $1.25\%$ of the previous balance each month. Month 1: interest $=0.0125\times 5400=67.50$, principal reduction $1832.50$, balance $3567.50$. Month 2: interest $=0.0125\times 3567.50=44.59375\rightarrow 44.59$, principal reduction $1855.41$, balance $1712.09$. Month 3: only $\$1712.09$ is owed; its interest is $0.0125\times 1712.09=21.401125\rightarrow 21.40$, so $1712.09+21.40=\$1733.49$ clears the loan (principal reduction $1712.09$, balance $0$).

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,400.00$ |
| 1 | $\$1\,900.00$ | $\$67.50$ | $\$1\,832.50$ | $\$3\,567.50$ |
| 2 | $\$1\,900.00$ | $\$44.59$ | $\$1\,855.41$ | $\$1\,712.09$ |
| 3 | $\$1\,733.49$ | $\$21.40$ | $\$1\,712.09$ | $\$0.00$ |

(b) The final repayment is $\$1\,733.49$. (c) The total interest is $67.50+44.59+21.40=\$133.49$ (equivalently $2\times1900+1733.49-5400=5533.49-5400=\$133.49$).

**Q4.** Reading straight off the table: (a) the interest in quarter 2 is $\$148.00$; (b) the principal reduction in quarter 3 is $\$2\,705.04$; (c) the balance after quarter 3 is $\$2\,042.96$; (d) the final repayment (quarter 4) is $\$2\,083.82$. (e) The total interest is the sum of the interest column, $200+148+94.96+40.86=\$483.82$ (check: $3\times2800+2083.82-10\,000=10\,483.82-10\,000=\$483.82$).

**Q5.** The interest in the third month is charged on the balance carried in from month 2, $\$9\,964.94$. (a) $A=0.006\times9964.94=59.78964\rightarrow\$59.79$. (b) The principal reduction is the rest of the $\$2\,600$ repayment, $B=2600-59.79=\$2\,540.21$. (c) $C=9964.94-2540.21=\$7\,424.73$. As a check, the month-2 balance plus the interest minus the repayment gives the same figure: $9964.94+59.79-2600=\$7\,424.73$.

**Q6.** $r=\dfrac{6}{4}=1.5\,\%$ per quarter, so interest $=0.015\times$ (previous balance). Quarter 1: interest $105.00$, principal reduction $1495.00$, balance $5505.00$. Quarter 2: interest $=0.015\times5505=82.575\rightarrow82.58$, principal reduction $1517.42$, balance $3987.58$. Quarter 3: interest $=0.015\times3987.58=59.8137\rightarrow59.81$, principal reduction $1540.19$, balance $2447.39$. Quarter 4: interest $=0.015\times2447.39=36.71085\rightarrow36.71$, principal reduction $1563.29$, balance $884.10$. Quarter 5: only $\$884.10$ is owed; its interest is $0.015\times884.10=13.2615\rightarrow13.26$, so $884.10+13.26=\$897.36$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$7\,000.00$ |
| 1 | $\$1\,600.00$ | $\$105.00$ | $\$1\,495.00$ | $\$5\,505.00$ |
| 2 | $\$1\,600.00$ | $\$82.58$ | $\$1\,517.42$ | $\$3\,987.58$ |
| 3 | $\$1\,600.00$ | $\$59.81$ | $\$1\,540.19$ | $\$2\,447.39$ |
| 4 | $\$1\,600.00$ | $\$36.71$ | $\$1\,563.29$ | $\$884.10$ |
| 5 | $\$897.36$ | $\$13.26$ | $\$884.10$ | $\$0.00$ |

(b) The final repayment is $\$897.36$. (c) The total interest is $105+82.58+59.81+36.71+13.26=\$297.36$. (d) The total cost of the loan is $4\times1600+897.36=\$7\,297.36$ (and $7297.36-7000=\$297.36$, the interest).

**Q7.** $r=\dfrac{9}{12}=0.75\,\%$ per month, so interest $=0.0075\times$ (previous balance). Month 1: interest $=0.0075\times8000=60$, principal reduction $840$, balance $7160$. Month 2: interest $=0.0075\times7160=53.70$, principal reduction $846.30$, balance $6313.70$. Month 3: interest $=0.0075\times6313.70=47.35275\rightarrow47.35$, principal reduction $852.65$, balance $5461.05$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$8\,000.00$ |
| 1 | $\$900.00$ | $\$60.00$ | $\$840.00$ | $\$7\,160.00$ |
| 2 | $\$900.00$ | $\$53.70$ | $\$846.30$ | $\$6\,313.70$ |
| 3 | $\$900.00$ | $\$47.35$ | $\$852.65$ | $\$5\,461.05$ |

The third repayment has interest $\$47.35$ and principal reduction $\$852.65$.

**Q8.** Interest is $2\%$ of the previous balance each month; the principal reduction is $650-\text{interest}$. Month 1: interest 60, principal reduction 590, balance 2410. Month 2: interest 48.20, principal reduction 601.80, balance 1808.20. Month 3: interest $=0.02\times1808.20=36.164\to36.16$, principal reduction 613.84, balance 1194.36. Month 4: interest $=0.02\times1194.36=23.8872\to23.89$, principal reduction 626.11, balance 568.25. Month 5: only $\$568.25$ is owed; its interest is $0.02\times568.25=11.365\to11.37$, so $568.25+11.37=\$579.62$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$3\,000.00$ |
| 1 | $\$650.00$ | $\$60.00$ | $\$590.00$ | $\$2\,410.00$ |
| 2 | $\$650.00$ | $\$48.20$ | $\$601.80$ | $\$1\,808.20$ |
| 3 | $\$650.00$ | $\$36.16$ | $\$613.84$ | $\$1\,194.36$ |
| 4 | $\$650.00$ | $\$23.89$ | $\$626.11$ | $\$568.25$ |
| 5 | $\$579.62$ | $\$11.37$ | $\$568.25$ | $\$0.00$ |

So the loan takes **five repayments** — four of $\$650$ and a final repayment of $\$579.62$. The total interest is $60+48.20+36.16+23.89+11.37=\$179.62$ (check: $4\times650+579.62-3000=3179.62-3000=\$179.62$).

**Q9.** Interest is $1.5\%$ of the previous balance each period; the amount drawn from the capital is $2000-\text{interest}$. Period 1: interest 135, capital drawn 1865, balance 7135. Period 2: interest $=0.015\times7135=107.025\to107.03$, capital drawn 1892.97, balance 5242.03. Period 3: interest $=0.015\times5242.03=78.63045\to78.63$, capital drawn 1921.37, balance 3320.66. Period 4: interest $=0.015\times3320.66=49.8099\to49.81$, capital drawn 1950.19, balance 1370.47. Period 5: only $\$1\,370.47$ remains; its interest is $0.015\times1370.47=20.557\to20.56$, so the final withdrawal is $1370.47+20.56=\$1\,391.03$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,000.00$ |
| 1 | $\$2\,000.00$ | $\$135.00$ | $\$1\,865.00$ | $\$7\,135.00$ |
| 2 | $\$2\,000.00$ | $\$107.03$ | $\$1\,892.97$ | $\$5\,242.03$ |
| 3 | $\$2\,000.00$ | $\$78.63$ | $\$1\,921.37$ | $\$3\,320.66$ |
| 4 | $\$2\,000.00$ | $\$49.81$ | $\$1\,950.19$ | $\$1\,370.47$ |
| 5 | $\$1\,391.03$ | $\$20.56$ | $\$1\,370.47$ | $\$0.00$ |

The final withdrawal is $\$1\,391.03$. The total withdrawn is $4\times2000+1391.03=\$9\,391.03$, so the total interest earned is $9391.03-9000=\$391.03$.

**Q10.** Reading straight off the table: (a) the interest in month 4 is $\$100.01$; (b) the principal reduction in month 5 is $\$3\,433.99$; (c) the balance after month 3 is $\$10\,000.67$; (d) the final repayment (month 6) is $\$3\,198.36$. (e) The total interest is $200+167+133.67+100.01+66.01+31.67=\$698.36$ (check: $5\times3500+3198.36-20\,000=20\,698.36-20\,000=\$698.36$).

**Q11.** Interest is $1\%$ of the previous balance each month. Month 1: interest 100, principal reduction 1100, balance 8900. Month 2: interest 89, principal reduction 1111, balance 7789. Month 3: interest $=0.01\times7789=77.89$, principal reduction 1122.11, balance 6666.89.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$10\,000.00$ |
| 1 | $\$1\,200.00$ | $\$100.00$ | $\$1\,100.00$ | $\$8\,900.00$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 2 | $\$1200.00$ | $\$89.00$ | $\$1111.00$ | $\$7789.00$ |
| 3 | $\$1200.00$ | $\$77.89$ | $\$1122.11$ | $\$6666.89$ |

The interest falls each month ($\$100 \rightarrow \$89 \rightarrow \$77.89$) because it is charged on a smaller balance, so with the same $\$1200$ repayment the principal reduction rises ($\$1100 \rightarrow \$1111 \rightarrow \$1122.11$).

**Q12.** (a) The principal reduction is the amount by which the balance fell during the month, so the balance at the start was the closing balance plus that reduction, $7872.90 + 1127.10 = \$9000.00$. (b) The interest is $0.9\%$ of the balance carried into the month, $0.009 \times 9000 = \$81.00$. (c) The instalment is the interest plus the principal reduction, $81 + 1127.10 = \$1208.10$.

**Q13.** The interest in month 3 is charged on the month-2 balance $\$8120.70$. $A = 0.01 \times 8120.70 = 81.207 \rightarrow \$81.21$ ; $B = 2050 - 81.21 = \$1968.79$; $C = 8120.70 - 1968.79 = \$6151.91$. The total interest over the first three repayments is $D = 120 + 100.70 + 81.21 = \$301.91$.

**Q14.** Every row of an amortisation table has to satisfy two independent relationships, and the faulty entry is the one that breaks both. *Check 1 — the interest is* $1\%$ *of the balance carried in.* Rows 1, 2, 3 and 5 pass: $0.01 \times 8000 = 80.00$, $0.01 \times 6380 = 63.80$, $0.01 \times 4743.80 = 47.438 \rightarrow 47.44$ and $0.01 \times 1422.15 = 14.2215 \rightarrow 14.22$. Row 4 fails: $0.01 \times 3091.24 = 30.9124 \rightarrow \$30.91$, not the $\$30.19$ printed. *Check 2 — the interest and the principal reduction must add to the payment.* In row 4, $30.19 + 1669.09 = \$1699.28$, which is 72 cents short of the $\$1700$ repayment; with $\$30.91$ the row adds up exactly, $30.91 + 1669.09 = \$1700.00$.

The rest of row 4 is sound: $3091.24 - 1669.09 = \$1422.15$ is the balance shown, and that balance carries correctly into row 5. So the single incorrect entry is the interest in row 4 (the digits of $\$30.91$ have been transposed), and its correct value is $\$30.91$.

**Q15.** (a) With each repayment the balance owing falls, and the interest for a period is a fixed percentage of that balance, so the interest charged gets smaller period after period. The repayments are equal, so the part of each that is *not* interest — the principal reduction — must get correspondingly larger. (b) The instalments are all the same size and are fixed in advance, while the balance is driven down by the recurrence — interest added, then the instalment taken off — so there is no reason for it to land on exactly zero after a whole number of instalments. After the last full repayment a small balance is left which, once its final period's interest is added, comes to less than a full instalment; paying a whole instalment would overpay the debt, so the final repayment is just that reduced amount, which brings the balance to exactly zero.

**Q16.** $r = 1.5\%$ per quarter, so interest $= 0.015 \times$ (previous balance). Quarter 1: interest $150.00$, principal reduction $1650.00$, balance $8350.00$. Quarter 2: interest $= 0.015 \times 8350 = 125.25$, principal reduction $1674.75$, balance $6675.25$. Quarter 3: interest $= 0.015 \times 6675.25 = 100.12875 \rightarrow 100.13$, principal reduction $1699.87$, balance $4975.38$. Quarter 4: interest $= 0.015 \times 4975.38 = 74.6307 \rightarrow 74.63$, principal reduction $1725.37$, balance $3250.01$. Quarter 5: interest $= 0.015 \times 3250.01 = 48.75015 \rightarrow 48.75$, principal reduction $1751.25$, balance $1498.76$. Quarter 6: only $\$1498.76$ is owed; its interest is $0.015 \times 1498.76 = 22.4814 \rightarrow 22.48$, so a final repayment of $1498.76 + 22.48 = \$1521.24$ clears the loan. The total amount repaid is $5 \times 1800 + 1521.24 = \$10521.24$, so the total interest is $10521.24 - 10000 = \$521.24$.

**Q17.** (a) The interest is charged on the $\$8000$ balance, so the monthly rate is $\frac{120}{8000} = 0.015 = 1.5\%$ per month. (b) The principal reduction is the rest of the repayment, $1500 - 120 = \$1380.00$. (c) The balance after that repayment is $8000 - 1380 = \$6620.00$.

**Q18.** (a) Reading row 3: the interest earned is $\$72.16$ and the amount drawn from the capital (the principal reduction) is $\$2427.84$; together they make up the $\$2500$ withdrawal. (b) Rows 1 to 4 each pay a full $\$2500$, so the fund provides **four** full withdrawals; row 5 is the reduced final withdrawal of $\$2359.60$ that empties it. (c) The total interest earned is the sum of the interest column, $120+96.20+72.16+47.88+23.36=\$359.60$. The check agrees: the total withdrawn is $4\times2500+2359.60=\$12\,359.60$, and $12\,359.60-12\,000=\$359.60$.

**Q19.** Interest is $1\%$ of the previous balance each month; the principal reduction is $500-\text{interest}$. (a) In month 1 the interest is $0.01\times26\,000=\$260.00$, so the principal reduction is $500-260=\$240.00$. Since $260>240$, the interest exceeds the principal reduction. (b) Continuing the table (interest, then principal reduction, then balance):

| Payment number | Interest | Principal reduction | Balance |
|---|---|---|---|
| 1 | $\$260.00$ | $\$240.00$ | $\$25\,760.00$ |
| 2 | $\$257.60$ | $\$242.40$ | $\$25\,517.60$ |
| 3 | $\$255.18$ | $\$244.82$ | $\$25\,272.78$ |
| 4 | $\$252.73$ | $\$247.27$ | $\$25\,025.51$ |
| 5 | $\$250.26$ | $\$249.74$ | $\$24\,775.77$ |
| 6 | $\$247.76$ | $\$252.24$ | $\$24\,523.53$ |

For payments 1–5 the interest is still larger than the principal reduction; at payment 6 the interest ($\$247.76$) first drops below the principal reduction ($\$252.24$). So the **sixth repayment** is the first for which the principal reduction exceeds the interest.

**Q20.** $r=\frac{12}{12}=1\%$ per month, so interest $=0.01\times$ (previous balance). Month 1: interest $160.00$, principal reduction $1340.00$, balance $14\,660.00$. Month 2: interest $146.60$, principal reduction $1353.40$, balance $13\,306.60$. Month 3: interest $=0.01\times13\,306.60=133.066\rightarrow133.07$, principal reduction $1366.93$, balance $11\,939.67$. Month 4: interest $=0.01\times11\,939.67=119.3967\rightarrow119.40$, principal reduction $1380.60$, balance $10\,559.07$. Month 5: interest $=0.01\times10\,559.07=105.5907\rightarrow105.59$, principal reduction $1394.41$, balance $9\,164.66$. Month 6: interest $=0.01\times9\,164.66=91.6466\rightarrow91.65$, principal reduction $1408.35$, balance $7\,756.31$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$16\,000.00$ |
| 1 | $\$1\,500.00$ | $\$160.00$ | $\$1\,340.00$ | $\$14\,660.00$ |
| 2 | $\$1\,500.00$ | $\$146.60$ | $\$1\,353.40$ | $\$13\,306.60$ |
| 3 | $\$1\,500.00$ | $\$133.07$ | $\$1\,366.93$ | $\$11\,939.67$ |
| 4 | $\$1\,500.00$ | $\$119.40$ | $\$1\,380.60$ | $\$10\,559.07$ |
| 5 | $\$1\,500.00$ | $\$105.59$ | $\$1\,394.41$ | $\$9\,164.66$ |
| 6 | $\$1\,500.00$ | $\$91.65$ | $\$1\,408.35$ | $\$7\,756.31$ |

(b) The total interest charged in the first six months is $160+146.60+133.07+119.40+105.59+91.65=\$756.31$.

(c) The balance owing after six months is $\$7\,756.31$.

(d) The finance solver evaluates a single formula and rounds only at the end, giving $\$7\,756.30$, whereas the amortisation table rounds the interest to the nearest cent every month. Those small monthly roundings accumulate to one cent, so the table's $\$7\,756.31$ is one cent above the solver's $\$7\,756.30$.

## Theory

An earlier section set out the **amortisation table** of a reducing-balance loan (and, with the same arithmetic, an annuity), the recurrence

$$V_{n+1} = RV_n - D, V_0 = \text{starting amount},$$

written out one payment per row. Recall that $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r\%$ is the interest rate per period) and $D$ is the fixed payment — a **repayment** for a loan or a **withdrawal** for an annuity. Each row of the table splits a payment into two parts, in five columns:

- **payment number** — the period $1, 2, 3, \ldots$ (row $0$ records the opening balance);
- **payment** — the amount paid that period (the fixed $D$, except for a smaller final payment);
- **interest** — the interest charged that period, $\frac{r}{100} \times (\text{previous balance})$, rounded to the nearest cent;
- **principal reduction** — the part of the payment that reduces the balance, $\text{payment} - \text{interest}$;
- **balance** — the amount still owing after the payment, $\text{previous balance} - \text{principal reduction}$.

This section is about **using** such a table — completed, or partly completed — to *analyse a financial situation* and answer questions: how much is owed at a certain stage, how much of a particular payment is interest, what the whole arrangement costs, and how the answers change if the payment or the interest rate is changed.

**Rounding convention.** As throughout this chapter, the interest is rounded to the nearest cent **each period** before it is used, and the rounded balance is carried forward. Every entry is then exact to the cent and each row reconciles as

$$\text{balance} = \text{previous balance} + \text{interest (to the nearest cent)} - \text{payment}.$$

**Reading a value straight from the table.** Once a table is built, many questions are answered by simply looking up the right cell:

- the **balance owing after the $k$th payment** is the balance-column entry in row $k$;
- the **interest** contained in the $k$th payment is the interest-column entry in row $k$;
- the **principal reduction** (how much the $k$th payment actually took off the debt, or drew from an annuity's capital) is the principal-reduction entry in row $k$.

The small table below is for a loan of $\$5\,000$ charged $1\%$ interest per month and repaid by $\$1\,300$ per month. It is read as follows: after the second repayment $\$2\,487.50$ is still owed; the third repayment contained $\$24.88$ of interest and reduced the debt by $\$1\,275.12$; and the loan is cleared by a smaller fourth repayment of $\$1\,224.50$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,000.00$ |
| 1 | $\$1\,300.00$ | $\$50.00$ | $\$1\,250.00$ | $\$3\,750.00$ |
| 2 | $\$1\,300.00$ | $\$37.50$ | $\$1\,262.50$ | $\$2\,487.50$ |
| 3 | $\$1\,300.00$ | $\$24.88$ | $\$1\,275.12$ | $\$1\,212.38$ |
| 4 | $\$1\,224.50$ | $\$12.12$ | $\$1\,212.38$ | $\$0.00$ |

**Totals over the whole arrangement.** Two totals summarise the situation:

- the **total paid** is the sum of the payment column. For a loan this is the **total cost** of borrowing; for an annuity it is the total income drawn.

- the **total interest** is the sum of the interest column. Because the principal-reduction column must add up to the original balance $V_0$, the total interest is also

$$\text{total interest} = \text{total paid} - V_0.$$

For the loan above, the total cost is $4$ payments summing to $\$5\,124.50$, so the total interest is $5124.50 - 5000 = \$124.50$ — the same as the sum of the interest column, $50 + 37.50 + 24.88 + 12.12$.

**Filling in missing cells.** A partly completed table can be finished by using the four relationships in either direction. Reading *across* a row: interest = $\frac{r}{100}$ × (previous balance); principal reduction = payment − interest; balance = previous balance − principal reduction. Working *backwards* from a known interest figure, the balance it was charged on is

$$\text{previous balance} = \frac{\text{interest}}{r/100},$$

and a known interest and principal reduction add back to the payment. A quick way to check any completed row is that its balance equals the previous balance plus the interest minus the payment.

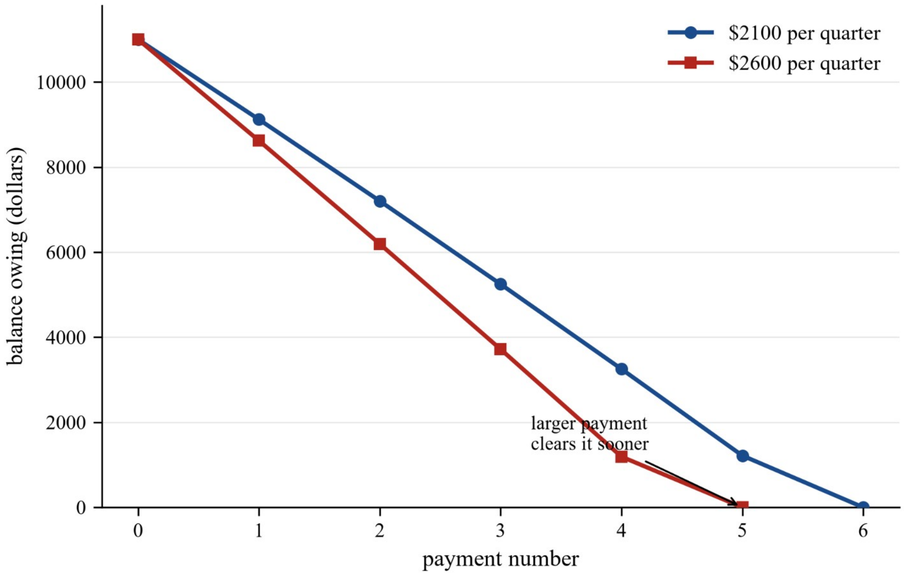
**Effect of changing the regular payment.** Suppose the same loan is repaid by a **larger** regular payment. Each period's interest is unchanged to start with, so more of every payment is left to reduce the principal; the balance falls faster, the loan is cleared in **fewer periods**, and less interest is charged overall — interest is being paid on the debt for a shorter time. A **smaller** payment does the opposite: the debt lingers, so more interest accrues and the total cost rises. If a payment is ever *less than or equal to* the first period's interest, none of it reduces the principal and the balance never falls — the loan is never repaid.

**Effect of changing the interest rate.** A **higher** interest rate charges more interest on the same balance every period, so a larger slice of each payment is swallowed by interest and a smaller slice reduces the principal. The balance falls more slowly: the loan takes **longer** to repay and costs **more** in total interest. Compared over the *same* number of equal payments, a higher rate therefore leaves a **higher** balance still owing. A lower rate has the opposite effect. (This is exactly why a small change in a mortgage rate can change the total interest, the time to repay and the size of the repayment noticeably.)

**Annuities read as income.** The identical table models an **annuity** — a lump sum invested and drawn down by regular withdrawals. Here the "interest" column is the interest **earned** each period and the "principal reduction" column is the amount **drawn from the capital** of the fund; the balance column is the capital remaining. The fund empties with a smaller final withdrawal, and the **total interest earned** is the total withdrawn minus the amount originally invested. A related question is *how long the income lasts* — the number of full withdrawals before the fund runs out.

**Using technology.** A CAS **finance solver** (TVM) will return a regular payment, a balance or the number of periods $N$ directly, and many will generate a full table. Because the solver works from a single formula and does **not** round period by period, a balance or final payment it reports can differ from a hand-built table by a cent or two — worth noting, not hiding. A solver's answer for $N$ is often **not a whole number**: a value between, say, $6$ and $7$ means six full payments are made and the arrangement finishes *partway* through the seventh period, with a smaller final payment. Building and reading the table by hand is what lets you interpret and check the technology.

The graph below compares the loan of $\$11\,000$ at $2\%$ per quarter of a later example, repaid at two different rates: $\$2\,100$ per quarter and $\$2\,600$ per quarter. The larger payment drives the balance down more steeply, clearing the debt one quarter sooner.




## Worked examples

### *Example 1 — scaffolded*

The amortisation table below is for a loan of $\$6\,000$ charged $1\%$ interest per month and repaid by $\$1\,050$ at the end of each month.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$6\,000.00$ |
| 1 | $\$1\,050.00$ | $\$60.00$ | $\$990.00$ | $\$5\,010.00$ |
| 2 | $\$1\,050.00$ | $\$50.10$ | $\$999.90$ | $\$4\,010.10$ |
| 3 | $\$1\,050.00$ | $\$40.10$ | $\$1\,009.90$ | $\$3\,000.20$ |
| 4 | $\$1\,050.00$ | $\$30.00$ | $\$1\,020.00$ | $\$1\,980.20$ |
| 5 | $\$1\,050.00$ | $\$19.80$ | $\$1\,030.20$ | $\$950.00$ |
| 6 | $\$959.50$ | $\$9.50$ | $\$950.00$ | $\$0.00$ |

Use the table to find the balance owing after the second repayment, the interest and the principal reduction contained in the third repayment, the total interest charged, and the total cost of the loan.

| Working | Comment |
|---|---|
| Balance after repayment 2 $= \$4\,010.10$. | Read the balance column in row 2. |
| Interest in repayment 3 $= \$40.10$. | Read the interest column in row 3. |
| Principal reduction in repayment 3 $= \$1\,009.90$. | Read the principal-reduction column in row 3. |
| Total interest $= 60 + 50.10 + 40.10 + 30 + 19.80 + 9.50 = \$209.50$. | Sum of the interest column. |
| Total cost $= 5 \times 1050 + 959.50 = \$6\,209.50$. | Sum of the payment column (five full repayments and the smaller last one). |

As a check, total paid $- V_0 = 6209.50 - 6000 = \$209.50$, the total interest.

### *Example 2 — scaffolded*

A loan of $\$11\,000$ is charged $8\%$ p.a. compounded quarterly. Compare repaying it by $\$2\,100$ per quarter with repaying it by $\$2\,600$ per quarter: in each case find the number of repayments, the size of the final repayment and the total interest charged, and hence state how much interest the larger repayment saves.

| Working | Comment |
| --- | --- |
| $r=\frac{8}{4}=2\%$ per quarter, so interest $=0.02\times$ (previous balance). | Quarterly rate = annual rate $\div 4$. |
| Build each table to payoff, taking a smaller final repayment. | The recurrence, split row by row. |

Repaying by $\$2\,100$ per quarter:

| Payment number | Payment | Interest | Principal reduction | Balance |
| --- | --- | --- | --- | --- |
| 0 | — | — | — | $\$11\,000.00$ |
| 1 | $\$2\,100.00$ | $\$220.00$ | $\$1\,880.00$ | $\$9\,120.00$ |
| 2 | $\$2\,100.00$ | $\$182.40$ | $\$1\,917.60$ | $\$7\,202.40$ |
| 3 | $\$2\,100.00$ | $\$144.05$ | $\$1\,955.95$ | $\$5\,246.45$ |
| 4 | $\$2\,100.00$ | $\$104.93$ | $\$1\,995.07$ | $\$3\,251.38$ |
| 5 | $\$2\,100.00$ | $\$65.03$ | $\$2\,034.97$ | $\$1\,216.41$ |
| 6 | $\$1\,240.74$ | $\$24.33$ | $\$1\,216.41$ | $\$0.00$ |

Repaying by $\$2\,600$ per quarter:

| Payment number | Payment | Interest | Principal reduction | Balance |
| --- | --- | --- | --- | --- |
| 0 | — | — | — | $\$11\,000.00$ |
| 1 | $\$2\,600.00$ | $\$220.00$ | $\$2\,380.00$ | $\$8\,620.00$ |
| 2 | $\$2\,600.00$ | $\$172.40$ | $\$2\,427.60$ | $\$6\,192.40$ |
| 3 | $\$2\,600.00$ | $\$123.85$ | $\$2\,476.15$ | $\$3\,716.25$ |
| 4 | $\$2\,600.00$ | $\$74.33$ | $\$2\,525.67$ | $\$1\,190.58$ |
| 5 | $\$1\,214.39$ | $\$23.81$ | $\$1\,190.58$ | $\$0.00$ |

At $\$2\,100$ per quarter there are **six** repayments (five of $\$2\,100$ and a final $\$1\,240.74$), with total interest $220+182.40+144.05+104.93+65.03+24.33=\$740.74$. At $\$2\,600$ per quarter there are **five** repayments (four of $\$2\,600$ and a final $\$1\,214.39$), with total interest $220+172.40+123.85+74.33+23.81=\$614.39$. The larger repayment therefore saves $740.74-614.39=\$126.35$ in interest and clears the loan one quarter sooner, because the debt is paid off in less time.

### *Example 3 — unscaffolded*

A loan of $\$14\,000$ is repaid by $\$1\,200$ at the end of each month. Compare the first six months under a rate of $12\%$ p.a. compounded monthly with the first six months under $18\%$ p.a. compounded monthly: find the balance still owing after six months and the total interest charged over the six months in each case.

At $12\%$ p.a. the monthly rate is $\frac{12}{12}=1\%$; at $18\%$ p.a. it is $\frac{18}{12}=1.5\%$. Each month the interest is that percentage of the previous balance (to the nearest cent) and the rest of the $\$1\,200$ reduces the principal.

At $1\%$ per month:

| Payment number | Payment | Interest | Principal reduction | Balance |
| --- | --- | --- | --- | --- |
| 0 | — | — | — | $\$14\,000.00$ |
| 1 | $\$1\,200.00$ | $\$140.00$ | $\$1\,060.00$ | $\$12\,940.00$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 2 | $\$1200.00$ | $\$129.40$ | $\$1070.60$ | $\$11869.40$ |
| 3 | $\$1200.00$ | $\$118.69$ | $\$1081.31$ | $\$10788.09$ |
| 4 | $\$1200.00$ | $\$107.88$ | $\$1092.12$ | $\$9695.97$ |
| 5 | $\$1200.00$ | $\$96.96$ | $\$1103.04$ | $\$8592.93$ |
| 6 | $\$1200.00$ | $\$85.93$ | $\$1114.07$ | $\$7478.86$ |

At $1.5\%$ per month:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$14000.00$ |
| 1 | $\$1200.00$ | $\$210.00$ | $\$990.00$ | $\$13010.00$ |
| 2 | $\$1200.00$ | $\$195.15$ | $\$1004.85$ | $\$12005.15$ |
| 3 | $\$1200.00$ | $\$180.08$ | $\$1019.92$ | $\$10985.23$ |
| 4 | $\$1200.00$ | $\$164.78$ | $\$1035.22$ | $\$9950.01$ |
| 5 | $\$1200.00$ | $\$149.25$ | $\$1050.75$ | $\$8899.26$ |
| 6 | $\$1200.00$ | $\$133.49$ | $\$1066.51$ | $\$7832.75$ |

At $1\%$ per month the balance after six months is $\$7478.86$ and the interest charged is $140+129.40+118.69+107.88+96.96+85.93=\$678.86$. At $1.5\%$ per month the balance after six months is $\$7832.75$ and the interest charged is $210+195.15+180.08+164.78+149.25+133.49=\$1032.75$. The higher rate leaves $7832.75-7478.86=\$353.89$ more still owing and costs $1032.75-678.86=\$353.89$ more in interest — the same figure, since the six payments total $\$7200$ either way, so every extra dollar of interest is a dollar less taken off the debt.

$\therefore$ after six months $\$7478.86$ is owed and $\$678.86$ interest has been charged at $1\%$ per month, compared with $\$7832.75$ owed and $\$1032.75$ interest at $1.5\%$ per month.

### *Example 4 — unscaffolded*

A retiree invests $\$15000$ in an annuity paying $8\%$ p.a. compounded quarterly and withdraws $\$2600$ at the end of each quarter. Part of its amortisation table is shown, with the interest in quarter 4 marked $A$. Find $A$; state how much of the third withdrawal is interest and how much is drawn from the fund's capital; find the size of the final withdrawal and the total interest earned; and explain what a finance solver means when it returns a non-whole number of periods.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$15000.00$ |
| 1 | $\$2600.00$ | $\$300.00$ | $\$2300.00$ | $\$12700.00$ |
| 2 | $\$2600.00$ | $\$254.00$ | $\$2346.00$ | $\$10354.00$ |
| 3 | $\$2600.00$ | $\$207.08$ | $\$2392.92$ | $\$7961.08$ |
| 4 | $\$2600.00$ | $A$ | $\$2440.78$ | $\$5520.30$ |
| 5 | $\$2600.00$ | $\$110.41$ | $\$2489.59$ | $\$3030.71$ |
| 6 | $\$2600.00$ | $\$60.61$ | $\$2539.39$ | $\$491.32$ |
| 7 | $\$501.15$ | $\$9.83$ | $\$491.32$ | $\$0.00$ |

The quarterly rate is $\frac{8}{4} = 2\%$, so the interest in quarter $4$ is charged on the balance carried in from quarter $3$: $A = 0.02 \times 7961.08 = 159.2216 \rightarrow \$159.22$ (and as a check, $2600 - 159.22 = 2440.78$, the principal reduction shown).

Reading row $3$, the third withdrawal earns $\$207.08$ in interest and draws $\$2\,392.92$ from the capital of the fund; together these make up the $\$2\,600$ withdrawal. The fund is exhausted by a smaller seventh withdrawal of $\$501.15$. The total withdrawn is $6 \times 2600 + 501.15 = \$16\,101.15$, so the total interest earned is $16\,101.15 - 15\,000 = \$1\,101.15$.

A finance solver asked for $N$ with $PV = -15\,000$, $PMT = 2600$, $I\% = 8$, $P/Y = C/Y = 4$ and $FV = 0$ returns a value between $6$ and $7$. A non-whole $N$ means the fund runs out **partway** through the seventh quarter: six full withdrawals of $\$2\,600$ are made and the seventh is the reduced amount $\$501.15$ that empties the fund.

$\therefore A = \$159.22$; the third withdrawal is $\$207.08$ interest and $\$2\,392.92$ from capital; the final withdrawal is $\$501.15$; the annuity earns $\$1\,101.15$ interest; and the solver's non-whole $N$ signals that the last withdrawal is smaller than the rest.

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## Practice questions

### *Scaffolded*

**Q1.** The amortisation table below is for a loan of $\$5\,000$ charged $1\%$ interest per month and repaid by $\$1\,100$ per month.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,000.00$ |
| 1 | $\$1\,100.00$ | $\$50.00$ | $\$1\,050.00$ | $\$3\,950.00$ |
| 2 | $\$1\,100.00$ | $\$39.50$ | $\$1\,060.50$ | $\$2\,889.50$ |
| 3 | $\$1\,100.00$ | $\$28.90$ | $\$1\,071.10$ | $\$1\,818.40$ |
| 4 | $\$1\,100.00$ | $\$18.18$ | $\$1\,081.82$ | $\$736.58$ |
| 5 | $\$743.95$ | $\$7.37$ | $\$736.58$ | $\$0.00$ |

(a) State the balance owing after the second repayment.
(b) State the interest included in the third repayment.
(c) State the principal reduction of the third repayment.
(d) Find the total interest charged over the life of the loan.
(e) Find the total cost of the loan.

**Q2.** A loan of $\$4\,500$ is charged $2\%$ interest per month. (a) If it is repaid by $\$1\,000$ per month, construct the amortisation table and state the number of repayments and the total interest charged. (b) If it is instead repaid by $\$1\,600$ per month, construct the amortisation table and state the number of repayments and the total interest charged. (c) State how much interest is saved by making the larger repayment, and why.

**Q3.** A loan of $\$9\,000$ is charged $1\%$ interest per month and repaid by $\$1\,500$ per month. Part of its amortisation table is shown.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,000.00$ |
| 1 | $\$1\,500.00$ | $\$90.00$ | $\$1\,410.00$ | $\$7\,590.00$ |
| 2 | $\$1\,500.00$ | $\$75.90$ | $\$1\,424.10$ | $\$6\,165.90$ |
| 3 | $\$1\,500.00$ | $A$ | $B$ | $C$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|

(a) Find the interest $A$ charged in the third month.
(b) Find the principal reduction $B$ of the third repayment.
(c) Find the balance $C$ owing after the third repayment.

**Q4.** A loan of $\$13\,000$ is repaid by $\$2\,200$ at the end of each quarter. (a) If the rate is $1\%$ per quarter, construct the first three rows of the amortisation table and state the balance owing after three repayments and the total interest charged so far. (b) If the rate is instead $1.25\%$ per quarter, do the same. (c) State the effect of the higher rate on the balance owing and on the interest charged.

**Q5.** The amortisation table below is for an annuity of $\$10\,000$ earning $1.5\%$ interest per period, from which $\$2\,200$ is withdrawn at the end of each period.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$10\,000.00$ |
| 1 | $\$2\,200.00$ | $\$150.00$ | $\$2\,050.00$ | $\$7\,950.00$ |
| 2 | $\$2\,200.00$ | $\$119.25$ | $\$2\,080.75$ | $\$5\,869.25$ |
| 3 | $\$2\,200.00$ | $\$88.04$ | $\$2\,111.96$ | $\$3\,757.29$ |
| 4 | $\$2\,200.00$ | $\$56.36$ | $\$2\,143.64$ | $\$1\,613.65$ |
| 5 | $\$1\,637.85$ | $\$24.20$ | $\$1\,613.65$ | $\$0.00$ |

(a) In the third withdrawal, how much is interest earned and how much is drawn from the capital of the fund?
(b) State the size of the final withdrawal.
(c) Find the total interest earned over the life of the annuity.

**Q6.** A loan of $\$7\,000$ is charged $1\%$ interest per month and repaid by $\$1\,500$ per month. (a) Construct the amortisation table until the loan is repaid. (b) Find the total cost of the loan. (c) Find the total interest charged, and verify it two ways.

### *Unscaffolded*

**Q7.** A loan of $\$20\,000$ is charged $6\%$ p.a. compounded monthly ($0.5\%$ per month) and repaid by $\$1\,800$ per month. By constructing the amortisation table, find the balance owing after the fifth repayment and the interest included in the fifth repayment.

**Q8.** A loan of $\$17\,000$ is charged $1\%$ interest per month. Compare repaying it by $\$2\,400$ per month with repaying it by $\$2\,800$ per month: for each, state the number of repayments and the total interest charged, and hence find how much interest the larger repayment saves.

**Q9.** A loan of $\$6\,000$ is repaid by $\$1\,200$ per month. Compare the first four months at $1\%$ per month with the first four months at $2\%$ per month: find the balance owing after four months and the total interest charged over those months in each case, and comment on the difference.

**Q10.** An annuity of $\$8\,000$ earns $2\%$ interest per quarter, and $\$1\,500$ is withdrawn at the end of each quarter. By constructing the amortisation table, find how many withdrawals the fund provides, the size of the final withdrawal, and the total interest earned.

**Q11.** During one month of a reducing-balance loan charged $1.2\%$ interest per month, the interest included in that month's repayment is $\$48.00$; the repayment is $\$700$. (a) Find the balance owing at the start of that month. (b) Find the principal reduction of that repayment. (c) Find the balance owing at the end of that month.

**Q12.** A loan of $\$5\,000$ is charged $1.5\%$ interest per quarter and repaid by $\$1\,300$ per quarter. By constructing the amortisation table, find the total cost of the loan and the total interest charged, and show that the total interest agrees whether it is found as the sum of the interest column or as the total paid minus the amount borrowed.

**Q13.** A loan of $\$18\,000$ is charged $1\%$ interest per month and repaid by $\$2\,500$ per month. Part of its amortisation table is shown.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$18\,000.00$ |
| 1 | $\$2\,500.00$ | $\$180.00$ | $\$2\,320.00$ | $\$15\,680.00$ |
| 2 | $\$2\,500.00$ | $\$156.80$ | $\$2\,343.20$ | $\$13\,336.80$ |
| 3 | $\$2\,500.00$ | $\$133.37$ | $\$2\,366.63$ | $\$10\,970.17$ |
| 4 | $\$2\,500.00$ | $A$ | $B$ | $C$ |

Find $A$, $B$ and $C$, and hence find the total interest charged over the first four months.

**Q14.** A loan of $\$30\,000$ is charged $1\%$ interest per month. (a) Find the interest charged in the first month. (b) Explain what happens to the balance owing if the monthly repayment is exactly $\$300$. (c) If the repayment is $\$600$, find the principal reduction of the first repayment and the balance owing after the first month.

**Q15.** An annuity of $\$19\,000$ earns $1\%$ interest per period, and $\$3\,200$ is withdrawn at the end of each period. (a) By constructing the amortisation table, find the number of withdrawals, the size of the final withdrawal, and the total interest earned. (b) A finance solver, asked for the number of periods $N$ with $PV=-19\,000$, $PMT=3200$ and $FV=0$, returns a value between $6$ and $7$. Explain what this non-whole number means.

**Q16.** The amortisation table below is for a loan of $\$22\,000$ charged $1\%$ interest per month and repaid by $\$3\,900$ per month.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$22\,000.00$ |
| 1 | $\$3\,900.00$ | $\$220.00$ | $\$3\,680.00$ | $\$18\,320.00$ |
| 2 | $\$3\,900.00$ | $\$183.20$ | $\$3\,716.80$ | $\$14\,603.20$ |
| 3 | $\$3\,900.00$ | $\$146.03$ | $\$3\,753.97$ | $\$10\,849.23$ |
| 4 | $\$3\,900.00$ | $\$108.49$ | $\$3\,791.51$ | $\$7\,057.72$ |
| 5 | $\$3\,900.00$ | $\$70.58$ | $\$3\,829.42$ | $\$3\,228.30$ |
| 6 | $\$3\,260.58$ | $\$32.28$ | $\$3\,228.30$ | $\$0.00$ |

(a) State the balance owing after the third repayment.
(b) State how much of the fourth repayment was interest.
(c) State the size of the final repayment.
(d) Find the total interest charged over the life of the loan.
(e) Describe how the interest and the principal reduction change down the table, and why.

**Q17.** A loan of $\$9\,000$ is charged $1.5\%$ interest per quarter. (a) If it is repaid by $\$1\,600$ per quarter, find the number of repayments and the size of the final repayment. (b) If it is instead repaid by $\$1\,900$ per quarter, find the number of repayments and the size of the final repayment. (c) State the effect of the larger repayment on the number of repayments and on the total interest charged.

**Q18.** A home loan of $\$40\,000$ is charged $6\%$ p.a. compounded monthly ($0.5\%$ per month) and repaid by $\$3\,000$ per month. (a) Construct the first four rows of the amortisation table. (b) Find the total interest charged in the first four months. (c) State the balance owing after four months. (d) Explain, using the table, why the interest part of each repayment is falling.

**Q19.** For a reducing-balance loan repaid by equal instalments: (a) explain why raising the interest rate, with the same regular repayment, increases both the time taken to repay the loan and the total interest paid; (b) explain how both the total cost of the loan and the total interest charged can be read from a completed amortisation table, and how the two are related.

**Q20.** A loan of $\$25\,000$ is charged $1\%$ interest per month and repaid by $\$2\,000$ per month. (a) Construct the first six rows of the amortisation table. (b) Find the total interest charged in the first six months. (c) State the balance owing after six months. (d) Without recomputing the whole table, state whether the balance owing after six months would be higher or lower if the rate were $1.25\%$ per month instead, and explain.

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## Answers

**Q1.** (a) $\$2889.50$; (b) $\$28.90$; (c) $\$1071.10$; (d) $\$143.95$; (e) $\$5143.95$. **Q2.** (a) five repayments, total interest $\$264.33$; (b) three repayments, total interest $\$178.80$; (c) $\$85.53$ saved, because the larger repayment clears the loan in three months instead of five, so interest is charged for less time. **Q3.** (a) $A=\$61.66$; (b) $B=\$1438.34$; (c) $C=\$4727.56$. **Q4.** (a) balance $\$6727.69$, interest so far $\$327.69$; (b) balance $\$6810.77$, interest so far $\$410.77$; (c) the higher rate leaves $\$83.08$ more owing and charges $\$83.08$ more interest. **Q5.** (a) interest earned $\$88.04$, drawn from capital $\$2111.96$; (b) $\$1637.85$; (c) $\$437.85$. **Q6.** (a) see solutions; (b) $\$7205.57$; (c) $\$205.57$. **Q7.** balance $\$11414.57$; interest in the fifth repayment $\$65.74$. **Q8.** $\$2400$/mo: eight repayments, total interest $\$722.96$; $\$2800$/mo: seven repayments, total interest $\$628.40$; the larger repayment saves $\$94.56$. **Q9.** at $1\%$: balance $\$1371.15$, interest $\$171.15$; at $2\%$: balance $\$1548.67$, interest $\$348.67$; the higher rate leaves $\$177.52$ more owing and costs $\$177.52$ more interest over the four months. **Q10.** six withdrawals (five of $\$1500$ and a final $\$1047.11$); total interest earned $\$547.11$. **Q11.** (a) $\$4000.00$; (b) $\$652.00$; (c) $\$3348.00$. **Q12.** total cost $\$5188.64$; total interest $\$188.64$ (sum of the interest column $=5188.64-5000$). **Q13.** $A=\$109.70$, $B=\$2390.30$, $C=\$8579.87$; total interest over four months $\$579.87$. **Q14.** (a) $\$300.00$; (b) the repayment exactly equals the interest, so no principal is repaid and the balance stays at $\$30000$ every month — the loan is never repaid; (c) principal reduction $\$300.00$, balance $\$29700.00$. **Q15.** (a) seven withdrawals (six of $\$3200$ and a final $\$487.25$), total interest earned $\$687.25$; (b) the fund runs out partway through the seventh period — six full withdrawals and a smaller seventh of $\$487.25$. **Q16.** (a) $\$10849.23$; (b) $\$108.49$; (c) $\$3260.58$; (d) $\$760.58$; (e) the interest falls and the principal reduction rises each month, because the interest is a fixed percentage of a shrinking balance. **Q17.** (a) six repayments, final $\$1473.71$; (b) five repayments, final $\$1806.25$; (c) the larger repayment gives one fewer repayment and saves $473.71-406.25=\$67.46$ in interest. **Q18.** (a) see solutions; (b) $\$715.72$; (c) $\$28715.72$; (d) the interest is $0.5\%$ of a balance that falls each month, so it decreases. **Q19.** (a) a higher rate charges more interest each period, so less of each repayment reduces the principal — the balance falls more slowly, taking longer to clear and accruing more interest; (b) the total cost is the sum of the payment column and the total interest is the sum of the interest column, and total interest = total cost − amount borrowed. **Q20.** (a) see solutions; (b) $\$1233.97$; (c) $\$14233.97$; (d) higher — a higher rate charges more interest each month, so less of each $\$2000$ reduces the principal, leaving a larger balance after six months.

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## Fully-worked solutions

**Q1.** Every figure is read straight from the completed table. (a) The balance column in row 2 is $\$2889.50$. (b) The interest column in row 3 is $\$28.90$. (c) The principal-reduction column in row 3 is $\$1071.10$. (d) The total interest is the sum of the interest column, $50+39.50+28.90+18.18+7.37=\$143.95$. (e) The total cost is the sum of the payment column, $4\times1100+743.95=\$5143.95$ (and $5143.95-5000=\$143.95$, the interest).

**Q2.** Interest is $2\%$ of the previous balance each month; the principal reduction is the rest of the repayment. (a) $D=1000$: Month 1: interest 90, principal reduction 910, balance 3590. Month 2: interest 71.80, principal reduction 928.20, balance 2661.80. Month 3: interest $=0.02\times2661.80=53.236\rightarrow53.24$, principal reduction 946.76, balance 1715.04. Month 4: interest $=0.02\times1715.04=34.3008\rightarrow34.30$, principal reduction 965.70, balance 749.34. Month 5: only $\$749.34$ is owed; its interest is $0.02\times749.34=14.9868\rightarrow14.99$, so $749.34+14.99=\$764.33$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$4500.00$ |
| 1 | $\$1000.00$ | $\$90.00$ | $\$910.00$ | $\$3590.00$ |
| 2 | $\$1000.00$ | $\$71.80$ | $\$928.20$ | $\$2661.80$ |
| 3 | $\$1000.00$ | $\$53.24$ | $\$946.76$ | $\$1715.04$ |
| 4 | $\$1000.00$ | $\$34.30$ | $\$965.70$ | $\$749.34$ |
| 5 | $\$764.33$ | $\$14.99$ | $\$749.34$ | $\$0.00$ |

So there are **five** repayments and the total interest is $90 + 71.80 + 53.24 + 34.30 + 14.99 = \$264.33$. (b) $D = 1600$: Month 1: interest 90, principal reduction 1510, balance 2990. Month 2: interest 59.80, principal reduction 1540.20, balance 1449.80. Month 3: only $\$1\,449.80$ is owed; its interest is $0.02 \times 1449.80 = 28.996 \to 29.00$, so $1449.80 + 29 = \$1\,478.80$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$4\,500.00$ |
| 1 | $\$1\,600.00$ | $\$90.00$ | $\$1\,510.00$ | $\$2\,990.00$ |
| 2 | $\$1\,600.00$ | $\$59.80$ | $\$1\,540.20$ | $\$1\,449.80$ |
| 3 | $\$1\,478.80$ | $\$29.00$ | $\$1\,449.80$ | $\$0.00$ |

So there are **three** repayments and the total interest is $90 + 59.80 + 29 = \$178.80$. (c) The larger repayment saves $264.33 - 178.80 = \$85.53$ in interest, because it clears the loan in three months instead of five, so interest is charged on the balance for less time.

**Q3.** The interest in month 3 is charged on the month-2 balance $\$6\,165.90$. $A = 0.01 \times 6165.90 = 61.659 \to \$61.66$; $B = 1500 - 61.66 = \$1\,438.34$; $C = 6165.90 - 1438.34 = \$4\,727.56$.

**Q4.** Interest is the stated percentage of the previous balance each quarter, and the principal reduction is $2200 - \text{interest}$. (a) At $1\%$: Quarter 1: interest 130, principal reduction 2070, balance $10\,930$. Quarter 2: interest 109.30, principal reduction 2090.70, balance 8839.30. Quarter 3: interest $= 0.01 \times 8839.30 = 88.393 \to 88.39$, principal reduction 2111.61, balance 6727.69.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$13\,000.00$ |
| 1 | $\$2\,200.00$ | $\$130.00$ | $\$2\,070.00$ | $\$10\,930.00$ |
| 2 | $\$2\,200.00$ | $\$109.30$ | $\$2\,090.70$ | $\$8\,839.30$ |
| 3 | $\$2\,200.00$ | $\$88.39$ | $\$2\,111.61$ | $\$6\,727.69$ |

The balance after three repayments is $\$6\,727.69$ and the interest so far is $130 + 109.30 + 88.39 = \$327.69$. (b) At $1.25\%$: Quarter 1: interest 162.50, principal reduction 2037.50, balance $10\,962.50$. Quarter 2: interest $= 0.0125 \times 10\,962.50 = 137.03125 \to 137.03$, principal reduction 2062.97, balance 8899.53. Quarter 3: interest $= 0.0125 \times 8899.53 = 111.244125 \to 111.24$, principal reduction 2088.76, balance 6810.77.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$13\,000.00$ |
| 1 | $\$2\,200.00$ | $\$162.50$ | $\$2\,037.50$ | $\$10\,962.50$ |
| 2 | $\$2\,200.00$ | $\$137.03$ | $\$2\,062.97$ | $\$8\,899.53$ |
| 3 | $\$2\,200.00$ | $\$111.24$ | $\$2\,088.76$ | $\$6\,810.77$ |

The balance after three repayments is $\$6\,810.77$ and the interest so far is $162.50 + 137.03 + 111.24 = \$410.77$. (c) The higher rate leaves $6810.77 - 6727.69 = \$83.08$ more owing after three repayments and has charged $410.77 - 327.69 = \$83.08$ more interest — the same figure, since the three repayments total $\$6\,600$ either way.

**Q5.** (a) Reading row 3, the interest earned is $\$88.04$ and the amount drawn from the capital (the principal reduction) is $\$2\,111.96$; together they make up the $\$2\,200$ withdrawal. (b) The final withdrawal (row 5) is $\$1\,637.85$. (c) The total interest earned is the sum of the interest column, $150 + 119.25 + 88.04 + 56.36 + 24.20 = \$437.85$ (check: $4 \times 2200 + 1637.85 - 10\,000 = 10\,437.85 - 10\,000 = \$437.85$).

**Q6.** Interest is $1\%$ of the previous balance each month; the principal reduction is $1500-\text{interest}$. Month 1: interest $70$, principal reduction $1430$, balance $5570$. Month 2: interest $55.70$, principal reduction $1444.30$, balance $4125.70$. Month 3: interest $=0.01\times 4125.70=41.257\to 41.26$, principal reduction $1458.74$, balance $2666.96$. Month 4: interest $=0.01\times 2666.96=26.6696\to 26.67$, principal reduction $1473.33$, balance $1193.63$. Month 5: only $\$1\,193.63$ is owed; its interest is $0.01\times 1193.63=11.9363\to 11.94$, so $1193.63+11.94=\$1\,205.57$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$7\,000.00$ |
| 1 | $\$1\,500.00$ | $\$70.00$ | $\$1\,430.00$ | $\$5\,570.00$ |
| 2 | $\$1\,500.00$ | $\$55.70$ | $\$1\,444.30$ | $\$4\,125.70$ |
| 3 | $\$1\,500.00$ | $\$41.26$ | $\$1\,458.74$ | $\$2\,666.96$ |
| 4 | $\$1\,500.00$ | $\$26.67$ | $\$1\,473.33$ | $\$1\,193.63$ |
| 5 | $\$1\,205.57$ | $\$11.94$ | $\$1\,193.63$ | $\$0.00$ |

(b) The total cost is $4\times 1500+1205.57=\$7\,205.57$. (c) The total interest is $70+55.70+41.26+26.67+11.94=\$205.57$, which agrees with total cost minus the amount borrowed, $7205.57-7000=\$205.57$.

**Q7.** $r=\dfrac{6}{12}=0.5\%$ per month, so interest $=0.005\times$ (previous balance). Month 1: interest $100$, principal reduction $1700$, balance $18\,300$. Month 2: interest $91.50$, principal reduction $1708.50$, balance $16\,591.50$. Month 3: interest $=0.005\times 16\,591.50=82.9575\to 82.96$, principal reduction $1717.04$, balance $14\,874.46$. Month 4: interest $=0.005\times 14\,874.46=74.3723\to 74.37$, principal reduction $1725.63$, balance $13\,148.83$. Month 5: interest $=0.005\times 13\,148.83=65.744\to 65.74$, principal reduction $1734.26$, balance $11\,414.57$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$20\,000.00$ |
| 1 | $\$1\,800.00$ | $\$100.00$ | $\$1\,700.00$ | $\$18\,300.00$ |
| 2 | $\$1\,800.00$ | $\$91.50$ | $\$1\,708.50$ | $\$16\,591.50$ |
| 3 | $\$1\,800.00$ | $\$82.96$ | $\$1\,717.04$ | $\$14\,874.46$ |
| 4 | $\$1\,800.00$ | $\$74.37$ | $\$1\,725.63$ | $\$13\,148.83$ |
| 5 | $\$1\,800.00$ | $\$65.74$ | $\$1\,734.26$ | $\$11\,414.57$ |

The balance owing after the fifth repayment is $\$11\,414.57$ and the interest in that repayment is $\$65.74$.

**Q8.** Interest is $1\%$ of the previous balance each month. At $\$2\,400$ per month the table runs for eight repayments (seven of $\$2\,400$ and a final $\$922.96$): the interest column is $170+147.70+125.18+102.43+79.45+56.25+32.81+9.14=\$722.96$. At $\$2\,800$ per month it runs for seven repayments (six of $\$2\,800$ and a final $\$828.40$): the interest column is $170+143.70+117.14+90.31+63.21+35.84+8.20=\$628.40$. So the $\$2\,400$ plan charges $\$722.96$ interest over eight months, the $\$2\,800$ plan charges $\$628.40$ over seven months, and the larger repayment saves $722.96-628.40=\$94.56$ — it clears the debt sooner, so interest accrues for less time.

**Q9.** The repayment is $\$1\,200$ each month; the interest is the stated percentage of the previous balance. At $1\%$: balances $4860.00, 3708.60, 2545.69, 1371.15$, with interest column $60+48.60+37.09+25.46=\$171.15$. At $2\%$: balances $4920.00, 3818.40, 2694.77, 1548.67$, with interest column $120+98.40+76.37+53.90=\$348.67$. So after four months $\$1\,371.15$ is owed at $1\%$ compared with $\$1\,548.67$ at $2\%$, and the interest charged is $\$171.15$ compared with $\$348.67$. The higher rate leaves $1548.67-1371.15=\$177.52$ more owing and costs $348.67-171.15=\$177.52$ more interest — the same amount, because the four repayments total $\$4\,800$ either way.

**Q10.** Interest is $2\%$ of the previous balance each quarter; the amount drawn from capital is $1500 - \text{interest}$. Quarter 1: interest 160, capital drawn 1340, balance 6660. Quarter 2: interest 133.20, capital drawn 1366.80, balance 5293.20. Quarter 3: interest $= 0.02 \times 5293.20 = 105.864 \to 105.86$, capital drawn 1394.14, balance 3899.06. Quarter 4: interest $= 0.02 \times 3899.06 = 77.9812 \to 77.98$, capital drawn 1422.02, balance 2477.04. Quarter 5: interest $= 0.02 \times 2477.04 = 49.5408 \to 49.54$, capital drawn 1450.46, balance 1026.58. Quarter 6: only $\$1\,026.58$ remains; its interest is $0.02 \times 1026.58 = 20.5316 \to 20.53$, so the final withdrawal is $1026.58 + 20.53 = \$1\,047.11$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$8\,000.00$ |
| 1 | $\$1\,500.00$ | $\$160.00$ | $\$1\,340.00$ | $\$6\,660.00$ |
| 2 | $\$1\,500.00$ | $\$133.20$ | $\$1\,366.80$ | $\$5\,293.20$ |
| 3 | $\$1\,500.00$ | $\$105.86$ | $\$1\,394.14$ | $\$3\,899.06$ |
| 4 | $\$1\,500.00$ | $\$77.98$ | $\$1\,422.02$ | $\$2\,477.04$ |
| 5 | $\$1\,500.00$ | $\$49.54$ | $\$1\,450.46$ | $\$1\,026.58$ |
| 6 | $\$1\,047.11$ | $\$20.53$ | $\$1\,026.58$ | $\$0.00$ |

So the fund provides **six** withdrawals (five of $\$1\,500$ and a final $\$1\,047.11$). The total withdrawn is $5 \times 1500 + 1047.11 = \$8\,547.11$, so the total interest earned is $8547.11 - 8000 = \$547.11$.

**Q11.** (a) The interest is $1.2\%$ of the balance at the start of the month, so $0.012 \times \text{balance} = 48$ gives $\text{balance} = \dfrac{48}{0.012} = \$4\,000.00$. (b) The principal reduction is the rest of the repayment, $700 - 48 = \$652.00$. (c) The balance at the end of the month is $4000 - 652 = \$3\,348.00$.

**Q12.** $r = 1.5\%$ per quarter, so interest $= 0.015 \times$ (previous balance). Quarter 1: interest 75, principal reduction 1225, balance 3775. Quarter 2: interest $= 0.015 \times 3775 = 56.625 \to 56.63$, principal reduction 1243.37, balance 2531.63. Quarter 3: interest $= 0.015 \times 2531.63 = 37.9745 \to 37.97$, principal reduction 1262.03, balance 1269.60. Quarter 4: only $\$1\,269.60$ is owed; its interest is $0.015 \times 1269.60 = 19.044 \to 19.04$, so $1269.60 + 19.04 = \$1\,288.64$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,000.00$ |
| 1 | $\$1\,300.00$ | $\$75.00$ | $\$1\,225.00$ | $\$3\,775.00$ |
| 2 | $\$1\,300.00$ | $\$56.63$ | $\$1\,243.37$ | $\$2\,531.63$ |
| 3 | $\$1\,300.00$ | $\$37.97$ | $\$1\,262.03$ | $\$1\,269.60$ |
| 4 | $\$1\,288.64$ | $\$19.04$ | $\$1\,269.60$ | $\$0.00$ |

The total cost is $3 \times 1300 + 1288.64 = \$5\,188.64$. The total interest, as the sum of the interest column, is $75 + 56.63 + 37.97 + 19.04 = \$188.64$; as total paid minus the amount borrowed, it is $5188.64 - 5000 = \$188.64$ — the two agree.

**Q13.** The interest in month 4 is charged on the month-3 balance $\$10\,970.17$. $A = 0.01 \times 10\,970.17 = 109.7017 \to \$109.70$; $B = 2500 - 109.70 = \$2\,390.30$; $C = 10\,970.17 - 2390.30 = \$8\,579.87$. The total interest over the first four months is $180 + 156.80 + 133.37 + 109.70 = \$579.87$.

**Q14.** (a) The interest in the first month is $0.01 \times 30\,000 = \$300.00$. (b) If the repayment is exactly $\$300$, it equals the interest, so the principal reduction is $300 - 300 = \$0$: none of the debt is repaid and the balance stays at $\$30\,000$ every month. The loan is never repaid (an interest-only arrangement). (c) If the repayment is $\$600$, the principal reduction is $600 - 300 = \$300.00$, so the balance after the first month is $30\,000 - 300 = \$29\,700.00$.

**Q15.** (a) Interest is $1\%$ of the previous balance each period; the amount drawn from capital is $3200 - \text{interest}$. Period 1: interest $190$, capital drawn $3010$, balance $15\,990$. Period 2: interest $159.90$, capital drawn $3040.10$, balance $12\,949.90$. Period 3: interest $= 0.01 \times 12\,949.90 = 129.499 \to 129.50$, capital drawn $3070.50$, balance $9879.40$. Period 4: interest $= 0.01 \times 9879.40 = 98.794 \to 98.79$, capital drawn $3101.21$, balance $6778.19$. Period 5: interest $= 0.01 \times 6778.19 = 67.7819 \to 67.78$, capital drawn $3132.22$, balance $3645.97$. Period 6: interest $= 0.01 \times 3645.97 = 36.4597 \to 36.46$, capital drawn $3163.54$, balance $482.43$. Period 7: only $\$482.43$ remains; its interest is $0.01 \times 482.43 = 4.8243 \to 4.82$, so the final withdrawal is $482.43 + 4.82 = \$487.25$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$19\,000.00$ |
| 1 | $\$3\,200.00$ | $\$190.00$ | $\$3\,010.00$ | $\$15\,990.00$ |
| 2 | $\$3\,200.00$ | $\$159.90$ | $\$3\,040.10$ | $\$12\,949.90$ |
| 3 | $\$3\,200.00$ | $\$129.50$ | $\$3\,070.50$ | $\$9\,879.40$ |
| 4 | $\$3\,200.00$ | $\$98.79$ | $\$3\,101.21$ | $\$6\,778.19$ |
| 5 | $\$3\,200.00$ | $\$67.78$ | $\$3\,132.22$ | $\$3\,645.97$ |
| 6 | $\$3\,200.00$ | $\$36.46$ | $\$3\,163.54$ | $\$482.43$ |
| 7 | $\$487.25$ | $\$4.82$ | $\$482.43$ | $\$0.00$ |

So there are **seven** withdrawals (six of $\$3\,200$ and a final $\$487.25$), and the total interest earned is $6 \times 3200 + 487.25 - 19\,000 = \$687.25$ (the sum of the interest column). (b) A non-whole $N$ between $6$ and $7$ means the fund is exhausted **partway** through the seventh period: six full withdrawals of $\$3\,200$ are made and the seventh is the smaller amount $\$487.25$ that empties the fund.

**Q16.** Reading straight from the table: (a) the balance after repayment 3 is $\$10\,849.23$; (b) the interest in repayment 4 is $\$108.49$; (c) the final repayment (row 6) is $\$3\,260.58$. (d) The total interest is the sum of the interest column, $220 + 183.20 + 146.03 + 108.49 + 70.58 + 32.28 = \$760.58$ (check: $5 \times 3900 + 3260.58 - 22\,000 = \$760.58$). (e) Down the table the interest column decreases and the principal-reduction column increases, because the interest is a fixed $1\%$ of a balance that falls each month, so with equal repayments the part left to reduce the principal grows.

**Q17.** Interest is $1.5\%$ of the previous balance each quarter. (a) At $\$1\,600$ per quarter the balances are $7535.00, 6048.03, 4538.75, 3006.83, 1451.93$, then a final repayment of $1451.93 + 21.78 = \$1\,473.71$ — **six** repayments in all. The interest column is $135 + 113.03 + 90.72 + 68.08 + 45.10 + 21.78 = \$473.71$. (b) At $\$1\,900$ per quarter the balances are $7235.00, 5443.53, 3625.18, 1779.56$, then a final repayment of $1779.56 + 26.69 = \$1\,806.25$ — **five** repayments in all. The interest column is $135 + 108.53 + 81.65 + 54.38 + 26.69 = \$406.25$. (c) The larger repayment clears the loan in one fewer repayment (five instead of six) and saves $473.71 - 406.25 = \$67.46$ in interest, because the debt is repaid sooner.

**Q18.** $r = \frac{6}{12} = 0.5\%$ per month, so interest $= 0.005 \times$ (previous balance). Month 1: interest $200$, principal reduction $2800$, balance $37\,200$. Month 2: interest $186$, principal reduction $2814$, balance $34\,386$. Month 3: interest $= 0.005 \times 34\,386 = 171.93$, principal reduction $2828.07$, balance $31\,557.93$. Month 4: interest $= 0.005 \times 31\,557.93 = 157.78965 \to 157.79$, principal reduction $2842.21$, balance $28\,715.72$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$40\,000.00$ |
| 1 | $\$3\,000.00$ | $\$200.00$ | $\$2\,800.00$ | $\$37\,200.00$ |
| 2 | $\$3\,000.00$ | $\$186.00$ | $\$2\,814.00$ | $\$34\,386.00$ |
| 3 | $\$3\,000.00$ | $\$171.93$ | $\$2\,828.07$ | $\$31\,557.93$ |
| 4 | $\$3\,000.00$ | $\$157.79$ | $\$2\,842.21$ | $\$28\,715.72$ |

(b) The total interest in the first four months is $200+186+171.93+157.79=\$715.72$.

(c) The balance owing after four months is $\$28\,715.72$.

(d) The interest each month is $0.5\,\%$ of the balance carried into that month. Because every repayment reduces the balance, the balance falls month by month, so $0.5\,\%$ of it — the interest part of the repayment — falls too.

**Q19.** (a) The interest charged in a period is a fixed percentage of the balance owing. A higher rate makes that interest larger, so with the same regular repayment a smaller part of each repayment is left to reduce the principal. The balance therefore falls more slowly, which means more repayments are needed to clear it and, because interest is charged on the outstanding balance for longer, the total interest paid is greater. (b) The **total cost** of the loan is the sum of the payment column — every instalment actually paid, including the smaller final one. The **total interest** is the sum of the interest column. Since the principal-reduction column must add up to the amount originally borrowed, the two totals are related by total interest = total cost − amount borrowed.

**Q20.** $r=1\,\%$ per month, so interest $=0.01\times$ (previous balance). Month 1: interest $250$, principal reduction $1750$, balance $23\,250$. Month 2: interest $232.50$, principal reduction $1767.50$, balance $21\,482.50$. Month 3: interest $=0.01\times21\,482.50=214.825\rightarrow214.83$, principal reduction $1785.17$, balance $19\,697.33$. Month 4: interest $=0.01\times19\,697.33=196.9733\rightarrow196.97$, principal reduction $1803.03$, balance $17\,894.30$. Month 5: interest $=0.01\times17\,894.30=178.943\rightarrow178.94$, principal reduction $1821.06$, balance $16\,073.24$. Month 6: interest $=0.01\times16\,073.24=160.7324\rightarrow160.73$, principal reduction $1839.27$, balance $14\,233.97$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$25\,000.00$ |
| 1 | $\$2\,000.00$ | $\$250.00$ | $\$1\,750.00$ | $\$23\,250.00$ |
| 2 | $\$2\,000.00$ | $\$232.50$ | $\$1\,767.50$ | $\$21\,482.50$ |
| 3 | $\$2\,000.00$ | $\$214.83$ | $\$1\,785.17$ | $\$19\,697.33$ |
| 4 | $\$2\,000.00$ | $\$196.97$ | $\$1\,803.03$ | $\$17\,894.30$ |
| 5 | $\$2\,000.00$ | $\$178.94$ | $\$1\,821.06$ | $\$16\,073.24$ |
| 6 | $\$2\,000.00$ | $\$160.73$ | $\$1\,839.27$ | $\$14\,233.97$ |

(b) The total interest in the first six months is $250+232.50+214.83+196.97+178.94+160.73=\$1\,233.97$.

(c) The balance owing after six months is $\$14\,233.97$.

(d) It would be **higher**. A rate of $1.25\,\%$ per month charges more interest on the same balance each month, so less of each $\$2\,000$ repayment reduces the principal; the balance falls more slowly, so after six months more would still be owing.

### Theory

Earlier sections modelled a **reducing-balance loan** — and, with identical arithmetic, an **annuity** — by the recurrence

$$V_{n+1} = R V_n - D, V_0 = \text{starting amount},$$

where $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r\%$ is the interest rate per period) and $D$ is the fixed payment (a **repayment** for a loan, a **withdrawal** for an annuity). Iterating the recurrence one period at a time generates the balance and, eventually, clears it to zero. General Mathematics is a technology-active course, and for these calculations a CAS **finance solver** (also called a *TVM solver*, for *time value of money*) does the iterating for you. This section is about **driving the solver**: reading the **balance** left after a given number of payments, and finding the **final payment** — the smaller last payment that brings the balance to exactly zero. You should still be able to build the recurrence by hand, both to understand the model and to check the technology.

**The finance-solver variables.** A finance solver stores six quantities. Enter the five you know, then solve for the sixth:

- $N$ — the **number of payments** (compounding periods), counted in periods, not years;
- $I\%$ — the **nominal annual interest rate**, entered as a percentage (the solver, not you, divides it down to the rate per period);
- $PV$ — the **present value**, the amount at the start (the sum borrowed, or the lump sum invested);
- $PMT$ — the **payment** made each period (the repayment or withdrawal);
- $FV$ — the **future value**, the balance at the end, after $N$ payments;
- $P/Y$ and $C/Y$ — the number of **payments per year** and **compounding periods per year**. In this course they are always equal, $P/Y = C/Y = k$ (for example $k = 12$ for monthly, $k = 4$ for quarterly). The solver then uses the per-period rate $\frac{I\%}{k}$, so the growth factor it works with is exactly the recurrence's $R = 1 + \frac{I\%/k}{100}$.

**The sign convention — state it clearly.** A finance solver tracks the *direction* of every cash flow with a sign, always from **your** point of view:

- money that comes **in** to you (that you **receive**) is entered as **positive** $(+)$;
- money that goes **out** from you (that you **pay**) is entered as **negative** $(-)$.

This fixes the signs in the two standard set-ups:

| Situation | $PV$ (start) | $PMT$ (each period) | $FV$ (end) |
|---|---|---|---|
| Reducing-balance **loan** | + you **receive** the amount borrowed | − you **pay** each repayment | − balance still **owed** (money still to pay out) |
| **Annuity** (invest and draw down) | − you **pay** the lump sum in | + you **receive** each withdrawal | + balance still **in the fund** (still yours) |

At the instant a loan is repaid or an annuity is exhausted the balance is zero, so there $FV = 0$.

**Finding the balance ($FV$) after some payments.** To find how much is still owing (or still in the fund) after a given number of payments, set $N$ to that number of payments, enter $PV$, $PMT$, $I\%$ and $P/Y = C/Y$, and **solve for** $FV$. Read the answer by its **size**: that is the balance. The **sign** merely records the direction — for a loan the solver returns a **negative** $FV$ (money you would still have to pay), and for an annuity a **positive** $FV$ (money still yours). So a solver display of $FV = -4\,898.99$ on a loan means $\$4\,898.99$ is **still owing**.

**A note on rounding.** The by-hand amortisation method used throughout this chapter rounds the interest to the nearest cent **each period** and carries the rounded balance forward, so every table entry is exact to the cent. A finance solver instead evaluates a **single formula** and does **not** round period by period. The two therefore agree in most cases but can differ by **a cent or two**; this is not an error in either — it is the accumulated per-period rounding of the hand table. Where it happens it is worth noting, not hiding. For a loan of $\$10\,000$ at $12\%$ p.a. compounded monthly ($1\%$ per month) repaid by $\$1\,100$ per month, the balances build as:

| Month | Payment | Interest | Balance |
|---|---|---|---|
| 0 | — | — | $\$10\,000.00$ |
| 1 | $\$1\,100.00$ | $\$100.00$ | $\$9\,000.00$ |
| 2 | $\$1\,100.00$ | $\$90.00$ | $\$7\,990.00$ |
| 3 | $\$1\,100.00$ | $\$79.90$ | $\$6\,969.90$ |
| 4 | $\$1\,100.00$ | $\$69.70$ | $\$5\,939.60$ |
| 5 | $\$1\,100.00$ | $\$59.40$ | $\$4\,899.00$ |

The hand table gives $\$4\,899.00$ owing after five months; the solver ($N=5$, $I\%=12$, $PV=10\,000$, $PMT=-1100$, $P/Y=C/Y=12$) returns $FV=-4\,898.99$ — one cent lower, because it never rounded along the way.

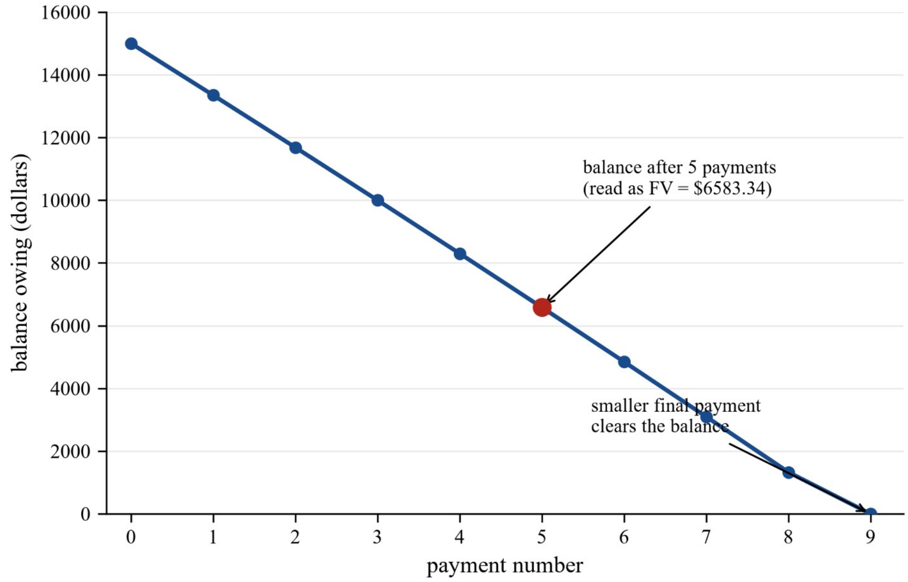
**Finding the final payment.** The regular payment $D$ almost never clears the balance to exactly zero after a whole number of payments; the **last payment differs**. It is smaller than the rest — just enough to clear whatever is left, plus that last period's interest, and no more. With a solver, find it in two steps:

1. **Count the full payments.** Enter $PV$, $PMT$, $I\%$, $P/Y=C/Y$ and $FV=0$, and **solve for** $N$. The answer is usually **not a whole number** — say $N=4.76$. This means the balance reaches zero **partway** through a period: there are **4** full payments (the whole-number part) and then one more, smaller, payment. So the number of full payments is the whole-number part of $N$.
2. **Grow the last balance by one period.** Set $N$ to that whole number of full payments and solve for $FV$: this is the balance still owing after the last full payment. In the next period it earns one more lot of interest, so the **final payment** is that balance multiplied by the growth factor $R$:
$$\text{final payment} = \left(\text{balance after the last full payment}\right) \times R.$$
This reduced amount clears the balance to exactly $\$0.00$.

Because the solver does not round period by period, a final payment found this way can sit a cent or two from the one a full hand-built amortisation table gives; the hand table, rounding each period, is the figure exact to the cent.

**Checking against the recurrence.** Whatever the solver reports, you can confirm it by iterating $V_{n+1}=RV_n-D$ from $V_0$: generate the balances one period at a time (rounding the interest to the cent each period) and read off the balance after the required number of payments, or continue until a small balance is left and add its interest to find the final payment. Reading the solver is fast; iterating the recurrence is the check.

The graph below tracks the balance owing on a reducing-balance loan, payment by payment, down to zero. The height at any payment number is the balance a solver returns as $FV$ for that $N$; the last step is short because the final payment is smaller than the rest.




## Worked examples

### *Example 1 — scaffolded*

A loan of $\$15\,000$ is charged interest of $12\%$ p.a. compounded monthly and is repaid by $\$1\,800$ at the end of each month. Use a finance solver to find the balance still owing after five months, and confirm the answer with the recurrence.

| Working | Comment |
|---|---|
| Set $P/Y = C/Y = 12$, $I\% = 12$, $PV = 15000$, $PMT = -1800$, $N = 5$; solve for $FV$. | Twelve periods a year; enter the **annual** rate as $I\%$. The amount received ($PV$) is +, the repayments paid out ($PMT$) are −. |
| $FV = -6\,583.34$. | The solver returns a **negative** future value. |
| Balance owing after five months $= \$6\,583.34$. | Read the **size**; the minus sign only records that it is money still owed. |
| Check: $r = \frac{12}{12} = 1\%$ per month, so $R = 1.01$ and $V_{n+1} = 1.01V_n - 1800$, $V_0 = 15\,000$. | The per-period rate and growth factor match what the solver uses. |

Iterating the recurrence five times:

| Payment number | Payment | Interest | Balance |
|---|---|---|---|
| 0 | — | — | $\$15\,000.00$ |
| 1 | $\$1\,800.00$ | $\$150.00$ | $\$13\,350.00$ |
| 2 | $\$1\,800.00$ | $\$133.50$ | $\$11\,683.50$ |
| 3 | $\$1\,800.00$ | $\$116.84$ | $\$10\,000.34$ |
| 4 | $\$1\,800.00$ | $\$100.00$ | $\$8\,300.34$ |
| 5 | $\$1\,800.00$ | $\$83.00$ | $\$6\,583.34$ |

The recurrence gives $\$6\,583.34$ after five months, agreeing with the solver to the cent.

### *Example 2 — scaffolded*

A loan of $\$6\,000$ is charged $6\%$ p.a. compounded quarterly and is repaid by $\$1\,400$ at the end of each quarter. Use a finance solver to find how many repayments are needed and the size of the final repayment.

| Working | Comment |
|---|---|
| Set $P/Y=C/Y=4$, $I\%=6$, $PV=6000$, $PMT=-1400$, $FV=0$; solve for $N$. | Four periods a year; $FV=0$ marks the loan being fully repaid. |
| $N=4.46$ (to two decimals). | A non-whole $N$: the balance hits zero **partway** through a period. |
| So there are **4** full repayments, then one smaller repayment — **5 repayments** in all. | The whole-number part of $N$ counts the full repayments. |
| Set $N=4$; solve for $FV$: $FV=-640.92$, so $\$640.92$ is owed after the 4th repayment. | The balance carried into the final quarter. |
| $r=\frac{6}{4}=1.5\%$ per quarter, so $R=1.015$; final repayment $=640.92\times1.015=\$650.53$. | Grow the last balance by one period's interest; that reduced amount clears the loan. |

$\therefore$ five repayments are needed — four of $\$1\,400$ and a final repayment of $\$650.53$.

### *Example 3 — unscaffolded*

A retiree invests $\$22\,000$ in an annuity paying $8\%$ p.a. compounded quarterly and withdraws $\$5\,000$ at the end of each quarter. Use a finance solver to find how much is left in the fund after three withdrawals, how many withdrawals the fund provides, the size of the final withdrawal, and the total interest earned.

For an annuity the lump sum is paid **in**, so $PV=-22\,000$, and the withdrawals are **received**, so $PMT=+5000$; here $I\%=8$ and $P/Y=C/Y=4$, and the per-period rate is $\frac{8}{4}=2\%$, so $R=1.02$.

Setting $N=3$ and solving gives $FV=8\,044.58$; the future value is **positive** because it is money still in the fund, so $\$8\,044.58$ remains after three withdrawals.

To find how long the fund lasts, set $FV=0$ and solve for $N$: $N=4.65$. The whole-number part is $4$, so there are **4** full withdrawals and then one smaller one — **5 withdrawals** in all. Setting $N=4$ and solving gives $FV=3\,205.47$, the amount left after the fourth withdrawal; grown by one quarter's interest, the final withdrawal is $3\,205.47\times1.02=\$3\,269.58$.

The total withdrawn is $4\times5000+3269.58=\$23\,269.58$, so the total interest earned is $23\,269.58-22\,000=\$1\,269.58$.

$\therefore$ $\$8\,044.58$ remains after three withdrawals; the fund provides five withdrawals (four of $\$5\,000$ and a final $\$3\,269.58$), earning $\$1\,269.58$ in interest.

### *Example 4 — unscaffolded*

A loan of $\$25\,000$ is charged $9\%$ p.a. compounded monthly and is repaid by $\$2\,200$ at the end of each month. Use a finance solver to find the number of repayments and the size of the final repayment, and compare the final repayment with the value from a full amortisation table.

Here $PV=25\,000$, $PMT=-2200$, $I\%=9$, $P/Y=C/Y=12$, and the per-period rate is $\frac{9}{12}=0.75\%$, so $R=1.0075$.

Solving for $N$ with $FV=0$ gives $N=11.92$. The whole-number part is 11, so there are 11 full repayments and then a smaller one — **12 repayments** in all. Setting $N=11$ and solving gives $FV=-2\,013.38$, so $\$2\,013.38$ is owed after the 11th repayment; the final repayment is $2\,013.38\times1.0075=\$2\,028.48$.

Building the amortisation table by hand (rounding the interest to the cent each month) leaves $\$2\,013.39$ owing after the 11th repayment, and a final repayment of $2\,013.39+15.10=\$2\,028.49$. The solver's $\$2\,028.48$ and the table's $\$2\,028.49$ differ by one cent: the solver evaluates a single formula without rounding each month, while the table rounds every month, and those roundings accumulate. Taking the table as exact to the cent, the total repaid is $11 \times 2200 + 2028.49 = \$26\,228.49$ and the total interest charged is $26\,228.49 - 25\,000 = \$1\,228.49$.

$\therefore$ the loan takes 12 repayments (eleven of $\$2\,200$ and a final of about $\$2\,028.48$ – $\$2\,028.49$), and the one-cent spread is the per-period rounding of the hand table.

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## Practice questions

### *Scaffolded*

**Q1.** A loan of $\$8\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$1\,500$ at the end of each month. A finance solver is used to find the balance owing after four months. (a) State the values of $P/Y$ and $C/Y$. (b) State whether $PV$ is entered as positive or negative, and why. (c) State whether $PMT$ is entered as positive or negative, and why. (d) With $N=4$, the solver returns $FV = -2\,234.23$. State the balance owing after four months. (e) Explain what the negative sign on $FV$ means.

**Q2.** An annuity of $\$28\,000$ pays $12\%$ p.a. compounded monthly, and $\$4\,000$ is withdrawn at the end of each month. A finance solver is used to find the amount left in the fund after five withdrawals. (a) State the per-period interest rate. (b) State whether $PV$ is entered as positive or negative, and why. (c) State whether $PMT$ is entered as positive or negative, and why. (d) With $N=5$, the solver returns $FV = 9\,024.26$. State the amount left in the fund, and explain why $FV$ is positive here.

**Q3.** A loan of $\$6\,500$ is charged $6\%$ p.a. compounded quarterly and is repaid by $\$1\,500$ at the end of each quarter. (a) With $PV = 6500$, $PMT = -1500$, $I\% = 6$, $P/Y = C/Y = 4$ and $FV = 0$, the solver returns $N = 4.51$. State how many full repayments are made and how many repayments there are in total. (b) Setting $N=4$, the solver returns $FV = -762.51$. State the balance owing after the last full repayment. (c) State the per-period growth factor $R$. (d) Find the final repayment, using final payment $= (\text{last balance}) \times R$.

**Q4.** A loan of $\$12\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$2\,000$ per month. (a) A finance solver ($N=4$) returns $FV = -4\,366.45$. State the balance owing after four months according to the solver. (b) An amortisation table, rounding the interest to the nearest cent each month, gives a balance of $\$4\,366.44$ after four months. State the difference between the two figures. (c) Explain why the solver and the table differ.

**Q5.** An annuity of $\$11\,000$ earns $12\%$ p.a. compounded monthly, and $\$2\,000$ is withdrawn at the end of each month. (a) With $FV=0$, the solver returns $N = 5.69$. Explain what this non-whole number means. (b) Setting $N=5$, the solver returns $FV = 1\,359.10$. State the amount left after the fifth withdrawal. (c) The per-period growth factor is $R = 1.01$. Find the size of the final withdrawal. (d) Find the total interest earned over the life of the annuity.

**Q6.** A loan of $\$16\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$3\,000$ per month. (a) With $FV=0$, the solver returns $N=5.51$. State the number of full repayments and the total number of repayments. (b) Setting $N=5$, the solver returns $FV = -1\,513.15$. State the balance owing after the last full repayment. (c) Find the final repayment. (d) Find the total amount repaid and the total interest charged.

### *Unscaffolded*

**Q7.** A loan of $\$24\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$3\,500$ per month. Use a finance solver to find the balance still owing after four months, and confirm your answer by iterating the recurrence.

**Q8.** An annuity of $\$13\,000$ earns $6\%$ p.a. compounded quarterly, and $\$2\,500$ is withdrawn at the end of each quarter. Use a finance solver to find the amount left in the fund after four withdrawals.

**Q9.** A loan of $\$10\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$2\,600$ per month. Use a finance solver to find the number of repayments and the size of the final repayment.

**Q10.** An annuity of $\$8\,500$ pays $8\%$ p.a. compounded quarterly, and $\$1\,900$ is withdrawn at the end of each quarter. Use a finance solver to find the number of withdrawals, the size of the final withdrawal, and the total interest earned.

**Q11.** A loan of $\$16\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$2\,800$ per month. (a) Use a finance solver to find the final repayment (via the balance after the last full repayment). (b) A full amortisation table gives a final repayment of $\$2\,558.69$. Compare this with your answer to (a) and explain the difference.

**Q12.** A loan of $\$28\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$4\,000$ per month. A finance solver, asked for $N$ with $FV=0$, returns $N=7.29$. Interpret this value, and find the size of the final repayment.

**Q13.** Explain, in terms of the sign convention, why on a finance solver: (a) for a reducing-balance loan $PV$ is entered as positive but $PMT$ as negative; (b) for an annuity $PV$ is entered as negative but $PMT$ as positive; (c) a loan's balance still owing is returned as a negative $FV$.

**Q14.** A loan of $\$21\,000$ is charged $6\%$ p.a. compounded monthly and is repaid by $\$2\,000$ per month. Use a finance solver to find the balance still owing after six months.

**Q15.** A loan of $\$15\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$1\,800$ per month. Use a finance solver to find the number of repayments, the size of the final repayment, and the total interest charged.

**Q16.** An annuity of $\$23\,000$ earns $12\%$ p.a. compounded monthly, and $\$4\,200$ is withdrawn at the end of each month. Use a finance solver to find the number of withdrawals, the size of the final withdrawal, and the total interest earned.

**Q17.** A student sets a finance solver to $N=4$, $I\%=12$, $PV=9\,500$, $PMT=-2\,000$, $P/Y=C/Y=12$ and solves for $FV$, obtaining $FV=-1\,764.94$. (a) Describe the financial situation being modelled. (b) State the balance owing after four months. (c) State what the solver would return for $FV$ if $N$ were set to $0$, and explain.

**Q18.** An annuity of $\$12\,000$ earns $12\%$ p.a. compounded monthly, and $\$2\,000$ is withdrawn at the end of each month. Use a finance solver to find the size of the final withdrawal. A full amortisation table gives a final withdrawal of $\$438.54$; account for any difference.

**Q19.** (a) Explain why a finance solver's balance can differ by a cent or two from a hand-built amortisation table. (b) A finance solver returns a whole-number value of $N$ when $FV=0$. Explain what that tells you about the final payment.

**Q20.** A loan of $\$26\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$3\,500$ per month. (a) Use a finance solver to find the balance still owing after six months. (b) Use a finance solver to find the total number of repayments and the size of the final repayment. (c) Find the total interest charged over the life of the loan.

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## Answers

**Q1.** (a) $P/Y=C/Y=12$; (b) positive — the borrower receives the loan; (c) negative — each repayment is paid out; (d) $\$2234.23$; (e) it is money still owed (an outflow still to come). **Q2.** (a) $1\%$ per month; (b) negative — the lump sum is paid into the fund; (c) positive — each withdrawal is received; (d) $\$9024.26$; $FV$ is positive because it is money still in the fund (still yours). **Q3.** (a) 4 full repayments, 5 in total; (b) $\$762.51$; (c) $R=1.015$; (d) $\$773.95$. **Q4.** (a) $\$4366.45$; (b) one cent; (c) the solver evaluates one formula without rounding each month, while the table rounds the interest to the cent every month, so the two differ by the accumulated rounding. **Q5.** (a) the fund is exhausted partway through the 6th period — 5 full withdrawals and a smaller 6th; (b) $\$1359.10$; (c) $\$1372.69$; (d) $\$372.69$. **Q6.** (a) 5 full repayments, 6 in total; (b) $\$1513.15$; (c) $\$1528.28$; (d) total repaid $\$16528.28$, total interest $\$528.28$. **Q7.** $\$10763.09$. **Q8.** $\$3570.47$. **Q9.** 4 repayments (three of $\$2600$ and a final $\$2449.00$). **Q10.** 5 withdrawals (four of $\$1900$ and a final $\$1397.01$); total interest earned $\$497.01$. **Q11.** (a) $\$2558.68$; (b) the table gives $\$2558.69$, one cent more, because it rounds the interest to the cent each month whereas the solver does not. **Q12.** 7 full repayments and a smaller 8th (8 in all); final repayment $\$1177.31$. **Q13.** (a) the borrower receives $PV$ (in, +) but pays each $PMT$ (out, $-$); (b) the investor pays $PV$ in ($-$) but receives each $PMT$ (+); (c) the balance owing is money still to be paid out, so it carries the negative (outflow) sign. **Q14.** $\$9486.92$. **Q15.** 9 repayments (eight of $\$1800$ and a final $\$1341.93$); total interest $\$741.93$. **Q16.** 6 withdrawals (five of $\$4200$ and a final $\$2776.50$); total interest earned $\$776.50$. **Q17.** (a) a loan (reducing balance) of $\$9500$ at $12\%$ p.a. compounded monthly, repaid by $\$2000$ per month, four months in; (b) $\$1764.94$; (c) $FV=-9500$ — with no payments made, the balance equals the amount borrowed. **Q18.** solver final withdrawal $\$438.55$; the table's $\$438.54$ is one cent lower because the table rounds each month while the solver does not. **Q19.** (a) the table rounds the interest to the nearest cent each period and carries it forward, whereas the solver evaluates a single formula with no per-period rounding, so the two can differ by the accumulated rounding; (b) a whole-number $N$ means the balance reaches exactly zero on a regular payment, so the final payment is a **full** payment, not a reduced one. **Q20.** (a) $\$6067.47$; (b) 8 repayments (seven of $\$3500$ and a final $\$2654.43$); (c) $\$1154.43$.

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## Fully-worked solutions

**Q1.** Monthly compounding means $P/Y=C/Y=12$. (a) $P/Y=C/Y=12$. (b) The borrower **receives** the $\$8000$, an inflow, so $PV=+8000$. (c) Each $\$1500$ repayment is **paid out**, an outflow, so $PMT=-1500$. (d) With $N=4$ the solver returns $FV=-2234.23$, so the balance owing after four months is $\$2234.23$. (e) The negative sign records that this is money still to be paid out — the amount still owed.

**Q2.** The rate is $12\%$ p.a. compounded monthly. (a) $\frac{12}{12}=1\%$ per month. (b) The $\$28000$ is **paid into** the fund, an outflow from the investor, so $PV=-28000$. (c) Each $\$4000$ withdrawal is **received**, an inflow, so $PMT=+4000$. (d) With $N=5$ the solver returns $FV=9024.26$, so $\$9024.26$ is left in the fund. Here $FV$ is **positive** because it is money still in the fund and therefore still the investor's.

**Q3.** $PV=6500$, $PMT=-1500$, $I\%=6$, $P/Y=C/Y=4$. (a) With $FV=0$ the solver returns $N=4.51$; the whole-number part is $4$, so there are **4** full repayments and then one smaller repayment — **5** repayments in total. (b) Setting $N=4$ gives $FV=-762.51$, so $\$762.51$ is owed after the last full repayment. (c) $r=\frac{6}{4}=1.5\%$ per quarter, so $R=1.015$. (d) The final repayment is $762.51\times1.015=\$773.95$, which clears the loan.

**Q4.** (a) With $N=4$ the solver returns $FV=-4366.45$, so it reports $\$4366.45$ owing after four months. (b) The amortisation table gives $\$4366.44$, so the two differ by $4366.45-4366.44=\$0.01$ (one cent). (c) The amortisation table rounds the interest to the nearest cent **each month** and carries the rounded balance forward; the solver evaluates a single formula and rounds only at the end. The small monthly roundings accumulate to one cent, so the two answers differ by that amount.

**Q5.** $PV=-11000$, $PMT=2000$, $1\%$ per month, $R=1.01$. (a) $N=5.69$ is not a whole number, so the fund runs out **partway** through the sixth month: there are **5** full withdrawals and then a smaller sixth withdrawal. (b) Setting $N=5$ gives $FV=1359.10$, so $\$1359.10$ remains after the fifth withdrawal. (c) The final withdrawal is

$1\,359.10 \times 1.01 = \$1\,372.69$, which empties the fund. (d) The total withdrawn is $5 \times 2000 + 1372.69 = \$11\,372.69$, so the total interest earned is $11\,372.69 - 11\,000 = \$372.69$.

**Q6.** $PV = 16\,000$, $PMT = -3000$, $1\%$ per month, $R = 1.01$. (a) $N = 5.51$, so there are **5** full repayments and a smaller sixth — **6** in total. (b) Setting $N = 5$ gives $FV = -1\,513.15$, so $\$1\,513.15$ is owed after the fifth repayment. (c) The final repayment is $1\,513.15 \times 1.01 = \$1\,528.28$. (d) The total repaid is $5 \times 3000 + 1528.28 = \$16\,528.28$, so the total interest is $16\,528.28 - 16\,000 = \$528.28$.

**Q7.** Set $P/Y = C/Y = 12$, $I\% = 12$, $PV = 24\,000$, $PMT = -3500$, $N = 4$; solve for $FV$: $FV = -10\,763.09$, so $\$10\,763.09$ is still owing after four months. Check with the recurrence $V_{n+1} = 1.01V_n - 3500$, $V_0 = 24\,000$:

| Payment number | Payment | Interest | Balance |
|---|---|---|---|
| 0 | — | — | $\$24\,000.00$ |
| 1 | $\$3\,500.00$ | $\$240.00$ | $\$20\,740.00$ |
| 2 | $\$3\,500.00$ | $\$207.40$ | $\$17\,447.40$ |
| 3 | $\$3\,500.00$ | $\$174.47$ | $\$14\,121.87$ |
| 4 | $\$3\,500.00$ | $\$141.22$ | $\$10\,763.09$ |

The recurrence agrees with the solver: $\$10\,763.09$.

**Q8.** For the annuity $PV = -13\,000$, $PMT = 2500$, $I\% = 6$, $P/Y = C/Y = 4$ (so $1.5\%$ per quarter). Setting $N = 4$ and solving gives $FV = 3\,570.47$ (positive — money still in the fund), so $\$3\,570.47$ is left after four withdrawals.

**Q9.** $PV = 10\,000$, $PMT = -2600$, $1\%$ per month, $R = 1.01$. With $FV = 0$ the solver returns $N = 3.94$, so there are **3** full repayments and a smaller fourth — **4** repayments in total. Setting $N = 3$ gives $FV = -2\,424.75$, and the final repayment is $2\,424.75 \times 1.01 = \$2\,449.00$. So there are four repayments: three of $\$2\,600$ and a final $\$2\,449.00$.

**Q10.** $PV = -8\,500$, $PMT = 1900$, $I\% = 8$, $P/Y = C/Y = 4$ (so $2\%$ per quarter, $R = 1.02$). With $FV = 0$ the solver returns $N = 4.73$, so there are **4** full withdrawals and a smaller fifth — **5** withdrawals in all. Setting $N = 4$ gives $FV = 1\,369.62$, and the final withdrawal is $1\,369.62 \times 1.02 = \$1\,397.01$. The total withdrawn is $4 \times 1900 + 1397.01 = \$8\,997.01$, so the total interest earned is $8\,997.01 - 8\,500 = \$497.01$.

**Q11.** $PV = 16\,000$, $PMT = -2800$, $1\%$ per month, $R = 1.01$. (a) With $FV = 0$ the solver returns $N = 5.91$, so there are 5 full repayments; setting $N = 5$ gives $FV = -2\,533.35$, and the final repayment is $2\,533.35 \times 1.01 = \$2\,558.68$. (b) The amortisation table gives $\$2\,558.69$, one cent more than the solver's $\$2\,558.68$. The table rounds the interest to the nearest cent every month and carries the rounded balance forward, whereas the solver evaluates one formula without rounding along the way; the accumulated monthly rounding is the one-cent difference.

**Q12.** $PV = 28\,000$, $PMT = -4000$, $1\%$ per month, $R = 1.01$. $N = 7.29$ is not a whole number, so the loan is cleared **partway** through the eighth month: there are **7** full repayments and then a smaller eighth repayment (8 in all). Setting $N = 7$ gives $FV = -1\,165.65$, so the final repayment is $1\,165.65 \times 1.01 = \$1\,177.31$.

**Q13.** The solver signs every cash flow from the client's point of view: money **received** is positive, money **paid** is negative. (a) For a loan the borrower **receives** the sum borrowed (an inflow), so $PV > 0$, but **pays** each repayment (an outflow), so $PMT < 0$. (b) For an annuity the investor **pays** the lump sum into the fund (an outflow), so $PV < 0$, but **receives** each withdrawal (an inflow), so $PMT > 0$. (c) A loan's outstanding balance is money the borrower must still pay out, so it takes the outflow sign — the solver returns it as a **negative** $FV$.

**Q14.** Set $P/Y = C/Y = 12$, $I\% = 6$ (so $0.5\%$ per month), $PV = 21\,000$, $PMT = -2000$, $N = 6$; solve for $FV$: $FV = -9\,486.92$, so $\$9\,486.92$ is still owing after six months.

**Q15.** $PV = 15\,000$, $PMT = -1800$, $1\%$ per month, $R = 1.01$. With $FV = 0$ the solver returns $N = 8.74$, so there are **8** full repayments and a smaller ninth — **9** repayments in all. Setting $N = 8$ gives $FV = -1\,328.64$, and the final repayment is $1\,328.64 \times 1.01 = \$1\,341.93$. The total repaid is $8 \times 1800 + 1341.93 = \$15\,741.93$, so the total interest is $15\,741.93 - 15\,000 = \$741.93$.

**Q16.** $PV=-23\,000$, $PMT=4200$, $1\%$ per month, $R=1.01$. With $FV=0$ the solver returns $N=5.66$, so there are **5** full withdrawals and a smaller sixth — **6** withdrawals in all. Setting $N=5$ gives $FV=2\,749.01$, and the final withdrawal is $2\,749.01\times1.01=\$2\,776.50$. The total withdrawn is $5\times4200+2776.50=\$23\,776.50$, so the total interest earned is $23\,776.50-23\,000=\$776.50$.

**Q17.** (a) $PV=+9\,500$ (received) with $PMT=-2\,000$ (paid) is a **reducing-balance loan**: $\$9\,500$ borrowed at $12\%$ p.a. compounded monthly, repaid by $\$2\,000$ per month, and $N=4$ looks four months in. (b) The solver returns $FV=-1\,764.94$, so $\$1\,764.94$ is still owing after four months. (c) With $N=0$ no payments have been made, so the balance is just the amount borrowed: $FV=-9\,500$ (the size is $\$9\,500$, the negative sign marking money owed).

**Q18.** $PV=-12\,000$, $PMT=2000$, $1\%$ per month, $R=1.01$. With $FV=0$ the solver returns $N=6.22$, so there are 6 full withdrawals; setting $N=6$ gives $FV=434.21$, and the final withdrawal is $434.21\times1.01=\$438.55$. The amortisation table gives $\$438.54$, one cent lower: the table rounds the interest to the cent each month while the solver does not, and the accumulated rounding is that one cent.

**Q19.** (a) The by-hand amortisation table rounds the interest to the nearest cent **each period** and carries the rounded balance forward, so its balances are exact to the cent step by step. A finance solver evaluates a single formula for the balance and rounds only at the end, so it never does the per-period rounding. Over many periods the small roundings in the table accumulate, so the two can differ by a cent or two — neither is wrong; they simply round at different stages. (b) If $N$ comes out a **whole number** when $FV=0$, the balance reaches **exactly** zero on a regular payment, so no reduced payment is needed — the **final payment is a full payment**, the same size as the rest.

**Q20.** $PV=26\,000$, $PMT=-3500$, $1\%$ per month, $R=1.01$. (a) Setting $N=6$ and solving gives $FV=-6\,067.47$, so $\$6\,067.47$ is still owing after six months. (b) With $FV=0$ the solver returns $N=7.76$, so there are **7** full repayments and a smaller eighth — **8** repayments in all. Setting $N=7$ gives $FV=-2\,628.15$, so the final repayment is $2\,628.15\times1.01=\$2\,654.43$. (c) The total repaid is $7\times3500+2654.43=\$27\,154.43$, so the total interest charged is $27\,154.43-26\,000=\$1\,154.43$. (A hand-built amortisation table, rounding the interest to the cent each month, gives a final repayment of $\$2\,654.42$ and hence $\$1\,154.42$ interest — the usual one-cent difference, since the solver does not round period by period.)

### Theory

An earlier section modelled a **reducing-balance loan** and an **annuity** by the one recurrence

$$V_{n+1} = RV_n - D, V_0 = \text{starting amount},$$

where $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r\%$ is the interest rate per period) and $D$ is the fixed payment — a repayment for a loan or a withdrawal for an annuity. It also introduced the CAS **finance solver** (the *TVM solver*), which works with the same situation through six quantities:

- $N$ — the total **number of payments**;
- $I\%$ — the **nominal annual interest rate**;
- $PV$ — the **present value** (the amount borrowed or invested now);
- $PMT$ — the **regular payment** made each period;
- $FV$ — the **future value** (the balance at the end);
- $P/Y = C/Y$ — the number of **payments per year** (equal to the number of compounds per year).

The solver holds a single equation linking these quantities. Enter any five and it returns the sixth. The **sign convention** records the direction of each cash flow: money that comes **to** you is **positive** and money that goes **from** you is **negative**. For a **loan** you *receive* the principal, so $PV$ is positive, and you *make* repayments, so $PMT$ is negative. For an **annuity** you *invest* the principal, so $PV$ is negative, and you *receive* withdrawals, so $PMT$ is positive. In both cases $FV = 0$ at the moment the balance reaches zero — the loan is repaid or the fund is exhausted.

Reading a **balance** or a **present value** off the solver was covered earlier. This section uses it to find the **other three** unknowns of a loan or annuity that finishes at $FV = 0$: the **payment** $PMT$, the **number of payments** $N$ (the *time*), and the **interest rate** $I\%$.

**Finding the regular payment $PMT$.** To answer "what repayment clears this loan in a set time?", enter $N$, $I\%$, $PV$ and $FV = 0$ (with $P/Y = C/Y$) and solve for $PMT$. The solver returns an exact amount that is almost never a whole number of cents. Because a real repayment must be a whole number of cents, and we want the debt cleared **within** the agreed $N$ payments, we round the solver's payment **up** to the next cent. The payments are then all this rounded amount, except for a slightly **smaller final payment** that brings the balance to exactly $\$0.00$. (Rounding *down* would leave a few cents still owing and force a small extra payment.) For an annuity you likewise solve for the withdrawal $PMT$ that draws the fund to zero over $N$ periods; a regular income is usually quoted to the nearest cent.

**Finding the number of payments $N$ (the time).** To answer "how many payments clear this loan?" or "how long does this annuity last?", enter $I\%$, $PV$, $PMT$ and $FV = 0$ and solve for $N$. The answer is almost never a whole number. Its **whole-number part** is the number of **full** payments; the leftover fraction means one more, **smaller** payment is still needed to finish. So **round $N$ up** to the next whole number to get the total number of payments — for example $N = 18.4$ means 18 full payments and a smaller 19th, so 19 payments in all. Dividing by the payments per year converts $N$ to a time in years (and months). A non-whole $N$ is exactly the "smaller final payment" met in the amortisation tables of earlier sections.

**Finding the interest rate $I\%$.** To answer "what interest rate makes these repayments clear the loan in this time?", enter $N$, $PV$, $PMT$ and $FV = 0$ and solve for $I\%$. The solver returns the **nominal annual** rate; divide by the number of compounds per year to get the rate **per period**, $r = \frac{I\%}{P/Y}$.

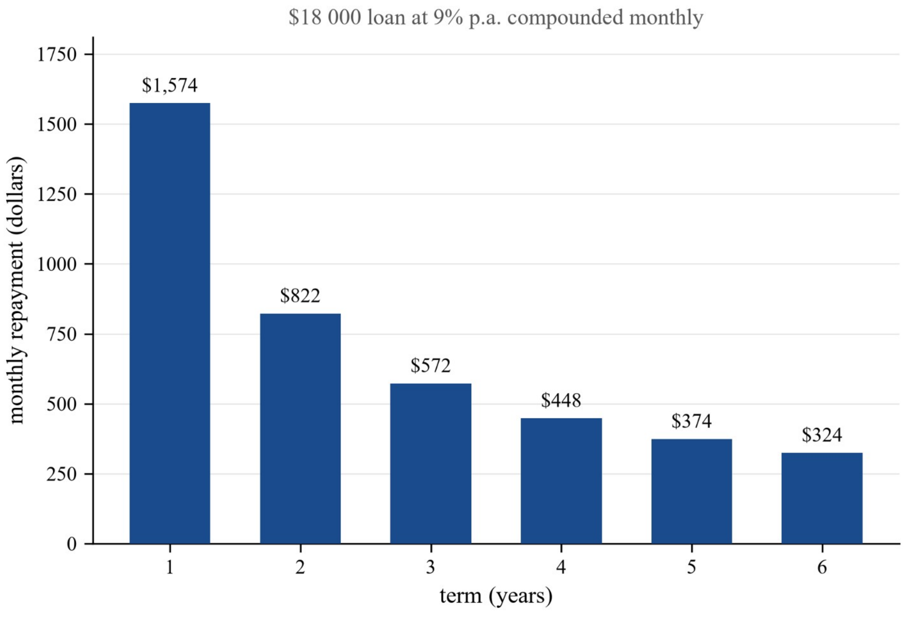
**Reading the answers against the recurrence.** The solver evaluates one formula and does **not** round period by period, whereas a by-hand amortisation table (following the chapter convention) rounds the interest to the nearest cent every period and carries the rounded balance forward. So a final payment, a total interest or a total cost read from a table can differ from the solver's own figures by a cent or two. This is not an error in either method — it is the rounding — and it is worth noting rather than hiding. Rebuilding the table with the solver's payment or rate is how you check the technology and interpret what it returns.

**Rounding convention (as throughout this chapter).** Each period the interest $\frac{r}{100} \times \left(\text{previous balance}\right)$ is rounded to the nearest cent before it is applied, and the rounded balance is carried forward, so every cell of an amortisation table is exact to the cent.

As a small illustration, suppose $\$1\,000$ is borrowed at $2\,\%$ per month and is to be repaid over three months. A finance solver ($N = 3$, $I\% = 24$, $PV = 1000$, $FV = 0$, $P/Y = C/Y = 12$) returns $PMT = -346.7547\ldots$, so the repayment rounds **up** to $\$346.76$. The amortisation table then clears the loan in three months with a slightly smaller final repayment:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$1\,000.00$ |
| 1 | $\$346.76$ | $\$20.00$ | $\$326.76$ | $\$673.24$ |
| 2 | $\$346.76$ | $\$13.46$ | $\$333.30$ | $\$339.94$ |
| 3 | $\$346.74$ | $\$6.80$ | $\$339.94$ | $\$0.00$ |

The figure below shows, for a loan of $\$18\,000$ at $9\,\%$ p.a. compounded monthly, the monthly repayment the solver returns for each choice of term. A longer term gives a **smaller** repayment, but — because interest is charged for longer — a **greater** total cost.




## Worked examples

### *Example 1 — scaffolded*

A loan of $\$18\,000$ is charged interest of $9\,\%$ p.a. compounded monthly and is to be repaid in full by equal monthly repayments over 4 years. Use a finance solver to find the monthly repayment, and hence the size of the final repayment and the total interest charged.

| Working | Comment |
|---|---|
| $r = \frac{9}{12} = 0.75\,\%$ per month; $N = 4 \times 12 = 48$ payments. | Monthly rate; four years of monthly payments. |
| Enter $N = 48$, $I\% = 9$, $PV = 18\,000$, $FV = 0$, | Loan: $PV$ positive (money received), $PMT$ negative |

| Working | Comment |
|---|---|
| $P/Y = C/Y = 12$; solve for $PMT$. | (money repaid). |
| Solver returns $PMT = -447.9308\ldots$, so the exact repayment is $\$447.9308\ldots$ | Not a whole number of cents. |
| Round **up** to the next cent: monthly repayment $= \$447.94$. | Rounding up guarantees the loan clears within the 48 months. |
| Amortise with $\$447.94$: the first 47 repayments are $\$447.94$ and the 48th is a smaller $\$447.41$. | The final row: balance $\$444.08$ plus its interest $\$3.33$ is repaid. |
| Total cost $= 47 \times 447.94 + 447.41 = \$21\,500.59$; total interest $= 21\,500.59 - 18\,000 = \$3\,500.59$. | Total paid, then subtract the amount borrowed. |

So the monthly repayment is $\$447.94$, the final repayment is $\$447.41$, and the total interest charged is $\$3\,500.59$. (Had the repayment been rounded *down* to $\$447.93$, a few cents would remain after 48 months and a 49th repayment would be needed.)

### *Example 2 — scaffolded*

A loan of $\$12\,000$ is charged $12\%$ p.a. compounded monthly and is repaid by $\$350$ at the end of each month. Use a finance solver to find how many repayments are needed and how long this takes, and state the size of the final repayment and the total interest charged.

| Working | Comment |
|---|---|
| $r = \frac{12}{12} = 1\%$ per month; the first month's interest is $0.01 \times 12\,000 = \$120 < \$350$. | The repayment exceeds the interest, so the balance falls. |
| Enter $I\% = 12$, $PV = 12\,000$, $PMT = -350$, $FV = 0$, $P/Y = C/Y = 12$; solve for $N$. | Loan signs: $PV$ positive, $PMT$ negative. |
| Solver returns $N = 42.19\ldots$ | Not a whole number of payments. |
| Interpret: 42 full repayments of $\$350$, then a smaller 43rd repayment. Round **up** to $N = 43$. | The .19 means the debt finishes partway through the 43rd month. |
| Time $= 43$ months $= 3$ years 7 months. | Divide by 12 payments per year. |
| Amortise with $\$350$: after 42 repayments $\$67.83$ is left, so the final repayment is $\$68.51$. | The balance $\$67.83$ plus its interest $\$0.68$ clears the loan. |
| Total cost $= 42 \times 350 + 68.51 = \$14\,768.51$; total interest $= 14\,768.51 - 12\,000 = \$2\,768.51$. | Total paid minus the amount borrowed. |

So 43 repayments are needed (3 years 7 months): 42 of $\$350$ and a final $\$68.51$, with total interest $\$2\,768.51$.

### *Example 3 — unscaffolded*

A car loan of $\$15\,000$ is repaid by equal monthly repayments of $\$352.28$ over 4 years, which clears it exactly. Use a finance solver to find the interest rate being charged, as both a nominal annual rate and a rate per month.

Here $N = 4 \times 12 = 48$ repayments clear the loan, so $FV = 0$. Entering $N = 48$, $PV = 15\,000$, $PMT = -352.28$, $FV = 0$ and $P/Y = C/Y = 12$, and solving for $I\%$, the solver returns $I\% = 6.0\ldots$

The nominal annual rate is therefore $6.0\%$ p.a. Because interest is compounded monthly, the rate per month is $\frac{6.0}{12} = 0.5\%$. (As a check, a $\$15\,000$ loan at $0.5\%$ per month repaid by $\$352.28$ per month does clear in 48 payments — the exact level repayment for that rate is $\$352.2754\ldots$, which rounds to the $\$352.28$ quoted.)

$\therefore$ the loan is charged $6.0\%$ p.a. compounded monthly, that is $0.5\%$ per month.

### *Example 4 — unscaffolded*

A retiree invests $\$200\,000$ in an annuity paying $5.4\,\%$ p.a. compounded monthly. (a) If they withdraw $\$1\,500$ at the end of each month, use a finance solver to find how long the fund lasts. (b) What monthly withdrawal would instead make the fund last exactly 20 years?

The monthly rate is $\frac{5.4}{12}=0.45\,\%$. For an annuity the money invested goes *from* the retiree ($PV=-200\,000$) and the withdrawals come *to* them ($PMT$ positive), with $FV=0$ when the fund is empty.

(a) Enter $I\%=5.4$, $PV=-200\,000$, $PMT=1500$, $FV=0$, $P/Y=C/Y=12$ and solve for $N$: the solver returns $N=204.08\ldots$ Rounding up, the fund provides **205** withdrawals — 204 full withdrawals of $\$1\,500$ and a smaller 205th. That is $205$ months, or **17 years 1 month**. (Amortising confirms the final withdrawal is $\$117.33$.)

(b) To last exactly 20 years the fund must supply $N=20\times 12=240$ withdrawals. Enter $N=240$, $I\%=5.4$, $PV=-200\,000$, $FV=0$ and solve for $PMT$: the solver returns $PMT=1364.50\ldots$, so the monthly withdrawal is $\$1\,364.50$.

The larger income of $\$1\,500$ empties the fund in about 17 years; taking the smaller income of $\$1\,364.50$ makes it last the full 20 years — the more you draw each month, the sooner the fund is exhausted.

$\therefore$ at $\$1\,500$ per month the fund lasts 17 years 1 month (205 withdrawals), and a monthly withdrawal of $\$1\,364.50$ makes it last exactly 20 years.

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## Practice questions

### *Scaffolded*

**Q1.** A loan of $\$9\,000$ is charged $8\,\%$ p.a. compounded quarterly and is to be repaid by equal quarterly repayments over 3 years. (a) State the interest rate per quarter and the number of repayments $N$. (b) Enter the known values in a finance solver and find the exact quarterly repayment. (c) State the repayment rounded up to the nearest cent, and the size of the final repayment. (d) Find the total interest charged.

**Q2.** A loan of $\$5\,000$ is charged $1.5\,\%$ interest per month and is repaid by $\$600$ at the end of each month. (a) Show that the first month's interest is less than $\$600$, so the balance falls. (b) Use a finance solver to find $N$, the number of payments. (c) Interpret the non-whole value of $N$ and state how many repayments are needed. (d) State the size of the final repayment and the total interest charged.

**Q3.** A loan of $\$8\,000$ is repaid by $\$254.40$ at the end of each month for 3 years, which clears it exactly. Use a finance solver to find (a) the nominal annual interest rate $I\,\%$, and (b) the interest rate per month.

**Q4.** A sum of $\$50\,000$ is invested in an annuity paying $6\,\%$ p.a. compounded monthly, to provide a regular monthly income for 4 years. (a) State the interest rate per month and the number of withdrawals $N$. (b) Use a finance solver to find the monthly withdrawal $PMT$. (c) State the income to the nearest cent. (d) State what the monthly income would be if the fund had to last 6 years instead, and comment.

**Q5.** A sum of $\$30\,000$ is invested in an annuity earning $2\,\%$ interest per quarter, from which $\$2\,500$ is withdrawn at the end of each quarter. (a) Use a finance solver to find $N$. (b) Interpret the non-whole value of $N$ and state the number of withdrawals. (c) Find the size of the final withdrawal and the total interest earned.

**Q6.** A loan of $\$20\,000$ is to be repaid by equal monthly repayments over 5 years. (a) Find the monthly repayment at $6\,\%$ p.a. compounded monthly. (b) Find the monthly repayment at $9\,\%$ p.a. compounded monthly. (c) State the effect of the higher rate on the monthly repayment and on the total interest charged.

### *Unscaffolded*

**Q7.** A loan of $\$25\,000$ is charged $9\,\%$ p.a. compounded monthly and is to be repaid by equal monthly repayments over 3 years. Find the monthly repayment, the size of the final repayment and the total interest charged.

**Q8.** A loan of $\$16\,000$ is charged $1\%$ interest per month and is repaid by $\$500$ at the end of each month. Find the number of repayments, the time in years and months, the size of the final repayment and the total interest charged.

**Q9.** A loan of $\$10\,000$ is repaid by $\$470.73$ at the end of each month for 2 years, which clears it exactly. Find the nominal annual interest rate and the interest rate per month.

**Q10.** A sum of $\$120\,000$ is invested at $4.8\%$ p.a. compounded monthly to provide a monthly income for 10 years. Find the monthly withdrawal, the size of the final withdrawal and the total interest earned.

**Q11.** A sum of $\$45\,000$ is invested in an annuity earning $0.6\%$ interest per month, from which $\$1\,000$ is withdrawn at the end of each month. Find how many withdrawals the fund provides, the time in years and months, the size of the final withdrawal and the total interest earned.

**Q12.** A sum of $\$40\,000$ is invested in an annuity from which $\$3\,409.61$ is withdrawn at the end of each quarter, and the fund lasts exactly 13 quarters. Find the nominal annual interest rate (compounded quarterly) and the interest rate per quarter.

**Q13.** A loan of $\$6\,000$ is charged $12\%$ p.a. compounded monthly and is to be repaid by equal monthly repayments over 1 year. (a) Find the exact monthly repayment. (b) Explain why it is rounded **up** to $\$533.10$ rather than to the nearest cent $\$533.09$. (c) For the $\$533.10$ plan, state the number of repayments, the final repayment and the total interest charged.

**Q14.** A home loan of $\$100\,000$ is charged $6\%$ p.a. compounded monthly and is repaid by $\$1\,200$ at the end of each month. Find the number of repayments, the time in years and months, the size of the final repayment and the total interest charged.

**Q15.** A loan of $\$24\,000$ is charged $9\%$ p.a. compounded monthly and is to be repaid by equal monthly repayments. (a) Find the monthly repayment over a 3-year term. (b) Find the monthly repayment over a 5-year term. (c) State the effect of the longer term on the monthly repayment and on the total interest charged.

**Q16.** A sum of $\$250\,000$ is invested to provide $\$1\,520.32$ per month for exactly 25 years. Find the nominal annual interest rate (compounded monthly) and the interest rate per month.

**Q17.** (a) Explain what a non-whole value of $N$ returned by a finance solver means, and how it is interpreted. (b) Explain why the regular repayment on a loan is rounded **up** to the next cent. (c) Explain why a total interest read from a by-hand amortisation table can differ by a cent or two from the finance solver's figure.

**Q18.** A mortgage of $\$320\,000$ is charged $5.4\%$ p.a. compounded monthly and is to be repaid by equal monthly repayments over 25 years. Find the monthly repayment, the total amount repaid and the total interest charged.

**Q19.** A loan of $\$40\,000$ is charged $9\%$ p.a. compounded monthly and is repaid by $\$900$ at the end of each month. Find the number of repayments, the time in years and months, the size of the final repayment and the total interest charged.

**Q20.** A loan of $\$12\,000$ is repaid by $\$551.53$ at the end of each month for 2 years, which clears it exactly. Find the nominal annual interest rate and the interest rate per month.

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### Answers

**Q1.** (a) $2\%$ per quarter, $N=12$; (b) $\$851.0364\ldots$; (c) repayment $\$851.04$, final repayment $\$851.01$; (d) $\$1212.45$. **Q2.** (a) interest $\$75.00<\$600$; (b) $N=8.97$; (c) 8 full repayments and a smaller 9th, so 9 repayments; (d) final $\$581.35$, total interest $\$381.35$. **Q3.** (a) $9.0\%$ p.a.; (b) $0.75\%$ per month. **Q4.** (a) $0.5\%$ per month, $N=48$; (b) and (c) $\$1174.25$ per month; (d) $\$828.64$ per month — a longer term gives a smaller income. **Q5.** (a) $N=13.86$; (b) 13 full withdrawals and a smaller 14th, so 14 withdrawals; (c) final $\$2149.52$, total interest $\$4649.52$. **Q6.** (a) $\$386.66$; (b) $\$415.17$; (c) the higher rate raises the repayment (by $\$28.51$ per month) and the total interest (from $\$3199.35$ to $\$4909.99$). **Q7.** repayment $\$795.00$; final repayment $\$794.72$; total interest $\$3619.72$. **Q8.** 39 repayments (3 years 3 months); final repayment $\$379.84$; total interest $\$3379.84$. **Q9.** $12.0\%$ p.a., that is $1\%$ per month. **Q10.** $\$1261.09$ per month; final withdrawal $\$1260.75$; total interest earned $\$31330.46$. **Q11.** 53 withdrawals (4 years 5 months); final withdrawal $\$609.71$; total interest earned $\$7609.71$. **Q12.** $6.0\%$ p.a., that is $1.5\%$ per quarter. **Q13.** (a) $\$533.0927\ldots$; (b) at $\$533.09$ a few cents remain after 12 months, forcing a 13th repayment, whereas rounding up to $\$533.10$ clears the loan within 12; (c) 12 repayments, final repayment $\$533.00$, total interest $\$397.10$. **Q14.** 109 repayments (9 years 1 month); final repayment $\$82.52$; total interest $\$29682.52$. **Q15.** (a) $\$763.20$; (b) $\$498.21$; (c) the longer term lowers the monthly repayment but raises the total interest (from $\$3474.96$ to $\$5891.90$). **Q16.** $5.4\%$ p.a., that is $0.45\%$ per month. **Q17.** (a) it means the balance reaches zero partway through a period: the whole-number part is the number of full payments and a smaller extra payment finishes the loan/annuity, so round $N$ up; (b) rounding up guarantees the debt is cleared within the term, with a slightly smaller final payment, rather than leaving a few cents owing; (c) the table rounds the interest to the cent every period while the solver rounds only once, so the accumulated roundings differ by a cent or two. **Q18.** monthly repayment $\$1946.02$; total repaid $\$583803.14$; total interest $\$263803.14$. **Q19.** 55 repayments (4 years 7 months); final repayment $\$238.70$; total interest $\$8838.70$. **Q20.** $9.6\%$ p.a., that is $0.8\%$ per month.

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### Fully-worked solutions

**Q1.** (a) $r=\frac{8}{4}=2\%$ per quarter, and $N=3\times4=12$ repayments. (b) Enter $N=12$, $I\%=8$, $PV=9000$, $FV=0$, $P/Y=C/Y=4$ and solve for $PMT$: the solver returns $PMT=-851.0364\ldots$, so the exact repayment is $\$851.0364\ldots$ (c) Rounded up to the next cent the repayment is $\$851.04$. Amortising at $2\%$ per quarter gives a smaller final repayment of $\$851.01$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9000.00$ |
| 1 | $\$851.04$ | $\$180.00$ | $\$671.04$ | $\$8328.96$ |
| 2 | $\$851.04$ | $\$166.58$ | $\$684.46$ | $\$7644.50$ |
| 3 | $\$851.04$ | $\$152.89$ | $\$698.15$ | $\$6946.35$ |
| 4 | $\$851.04$ | $\$138.93$ | $\$712.11$ | $\$6234.24$ |
| 5 | $\$851.04$ | $\$124.68$ | $\$726.36$ | $\$5507.88$ |
| 6 | $\$851.04$ | $\$110.16$ | $\$740.88$ | $\$4767.00$ |
| 7 | $\$851.04$ | $\$95.34$ | $\$755.70$ | $\$4011.30$ |
| 8 | $\$851.04$ | $\$80.23$ | $\$770.81$ | $\$3240.49$ |
| 9 | $\$851.04$ | $\$64.81$ | $\$786.23$ | $\$2454.26$ |
| 10 | $\$851.04$ | $\$49.09$ | $\$801.95$ | $\$1652.31$ |
| 11 | $\$851.04$ | $\$33.05$ | $\$817.99$ | $\$834.32$ |
| 12 | $\$851.01$ | $\$16.69$ | $\$834.32$ | $\$0.00$ |

(d) The total interest is the sum of the interest column, $\$1212.45$ (equivalently $11\times851.04+851.01-9000=10212.45-9000=\$1212.45$).

**Q2.** (a) The first month's interest is $0.015 \times 5000 = \$75.00$, which is well below the $\$600$ repayment, so the balance falls each month. (b) Enter $I\% = 18$ (since $1.5\% \times 12 = 18\%$ p.a.), $PV = 5000$, $PMT = -600$, $FV = 0$, $P/Y = C/Y = 12$ and solve for $N$: the solver returns $N = 8.97\ldots$ (c) The whole-number part, 8, is the number of full $\$600$ repayments; the fraction means a smaller 9th repayment is still needed. Rounding up, **9 repayments** clear the loan. (d) Amortising at $1.5\%$ per month:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$5\,000.00$ |
| 1 | $\$600.00$ | $\$75.00$ | $\$525.00$ | $\$4\,475.00$ |
| 2 | $\$600.00$ | $\$67.13$ | $\$532.87$ | $\$3\,942.13$ |
| 3 | $\$600.00$ | $\$59.13$ | $\$540.87$ | $\$3\,401.26$ |
| 4 | $\$600.00$ | $\$51.02$ | $\$548.98$ | $\$2\,852.28$ |
| 5 | $\$600.00$ | $\$42.78$ | $\$557.22$ | $\$2\,295.06$ |
| 6 | $\$600.00$ | $\$34.43$ | $\$565.57$ | $\$1\,729.49$ |
| 7 | $\$600.00$ | $\$25.94$ | $\$574.06$ | $\$1\,155.43$ |
| 8 | $\$600.00$ | $\$17.33$ | $\$582.67$ | $\$572.76$ |
| 9 | $\$581.35$ | $\$8.59$ | $\$572.76$ | $\$0.00$ |

The final repayment is $\$581.35$ and the total interest is the sum of the interest column, $\$381.35$.

**Q3.** The $N = 3 \times 12 = 36$ repayments clear the loan, so $FV = 0$. Enter $N = 36$, $PV = 8000$, $PMT = -254.40$, $FV = 0$, $P/Y = C/Y = 12$ and solve for $I\%$: the solver returns $I\% = 9.0\ldots$ (a) The nominal annual rate is $9.0\%$ p.a. (b) Since interest is compounded monthly, the rate per month is $\frac{9.0}{12} = 0.75\%$.

**Q4.** (a) $r = \frac{6}{12} = 0.5\%$ per month, and $N = 4 \times 12 = 48$ withdrawals. (b) The invested sum goes from the investor, so $PV = -50\,000$; withdrawals come to them, so $PMT$ is positive. Enter $N = 48$, $I\% = 6$, $PV = -50\,000$, $FV = 0$, $P/Y = C/Y = 12$ and solve for $PMT$: the solver returns $PMT = 1174.25\ldots$ (c) The monthly income is $\$1\,174.25$. (d) For a 6-year term there are $N = 72$ withdrawals; solving for $PMT$ with $N = 72$ gives $\$828.64$ per month. Spreading the same $\$50\,000$ over a longer term gives a smaller monthly income.

**Q5.** (a) Enter $I\% = 8$ (since $2\% \times 4 = 8\%$ p.a.), $PV = -30\,000$, $PMT = 2500$, $FV = 0$, $P/Y = C/Y = 4$ and solve for $N$: the solver returns $N = 13.86\ldots$ (b) The whole-number part, 13, is the number of full $\$2\,500$ withdrawals; a smaller 14th withdrawal finishes the fund. Rounding up, the fund provides **14 withdrawals**. (c) Amortising at $2\%$ per quarter:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$30\,000.00$ |
| 1 | $\$2\,500.00$ | $\$600.00$ | $\$1\,900.00$ | $\$28\,100.00$ |
| 2 | $\$2\,500.00$ | $\$562.00$ | $\$1\,938.00$ | $\$26\,162.00$ |
| 3 | $\$2\,500.00$ | $\$523.24$ | $\$1\,976.76$ | $\$24\,185.24$ |
| 4 | $\$2\,500.00$ | $\$483.70$ | $\$2\,016.30$ | $\$22\,168.94$ |
| 5 | $\$2\,500.00$ | $\$443.38$ | $\$2\,056.62$ | $\$20\,112.32$ |
| 6 | $\$2\,500.00$ | $\$402.25$ | $\$2\,097.75$ | $\$18\,014.57$ |
| 7 | $\$2\,500.00$ | $\$360.29$ | $\$2\,139.71$ | $\$15\,874.86$ |
| 8 | $\$2\,500.00$ | $\$317.50$ | $\$2\,182.50$ | $\$13\,692.36$ |
| 9 | $\$2\,500.00$ | $\$273.85$ | $\$2\,226.15$ | $\$11\,466.21$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 10 | $\$2500.00$ | $\$229.32$ | $\$2270.68$ | $\$9195.53$ |
| 11 | $\$2500.00$ | $\$183.91$ | $\$2316.09$ | $\$6879.44$ |
| 12 | $\$2500.00$ | $\$137.59$ | $\$2362.41$ | $\$4517.03$ |
| 13 | $\$2500.00$ | $\$90.34$ | $\$2409.66$ | $\$2107.37$ |
| 14 | $\$2149.52$ | $\$42.15$ | $\$2107.37$ | $\$0.00$ |

The final withdrawal is $\$2149.52$ and the total interest earned is the sum of the interest column, $\$4649.52$.

**Q6.** In each case $N=5\times 12=60$, $PV=20000$, $FV=0$, $P/Y=C/Y=12$; solve for $PMT$ and round up. (a) At $I\%=6$ ($0.5\%$ per month) the solver returns $PMT=-386.66\ldots$, so the monthly repayment is $\$386.66$; amortising gives total interest $\$3199.35$. (b) At $I\%=9$ ($0.75\%$ per month) the solver returns $PMT=-415.17\ldots$, so the monthly repayment is $\$415.17$; amortising gives total interest $\$4909.99$. (c) The higher rate raises the monthly repayment by $415.17-386.66=\$28.51$ and raises the total interest from $\$3199.35$ to $\$4909.99$ — more interest is charged on the balance each month, so each repayment must be larger and more is paid overall.

**Q7.** $r=\dfrac{9}{12}=0.75\%$ per month and $N=3\times 12=36$. Enter $N=36$, $I\%=9$, $PV=25000$, $FV=0$, $P/Y=C/Y=12$ and solve for $PMT$: $PMT=-794.9933\ldots$, which rounds up to $\$795.00$. Amortising at $0.75\%$ per month, the first 35 repayments are $\$795.00$ and the 36th is a smaller $\$794.72$. The total cost is $35\times 795+794.72=\$28619.72$, so the total interest is $28619.72-25000=\$3619.72$.

**Q8.** Enter $I\%=12$ ($1\%$ per month), $PV=16000$, $PMT=-500$, $FV=0$, $P/Y=C/Y=12$ and solve for $N$: $N=38.76\ldots$ Rounding up, **39 repayments** are needed — 38 full repayments of $\$500$ and a smaller 39th. That is 39 months $=3$ years 3 months. Amortising, after 38 repayments $\$376.08$ remains, so the final repayment is $\$379.84$. The total cost is $38\times 500+379.84=\$19379.84$, so the total interest is $19379.84-16000=\$3379.84$.

**Q9.** The $N=2\times 12=24$ repayments clear the loan, so $FV=0$. Enter $N=24$, $PV=10000$, $PMT=-470.73$, $FV=0$, $P/Y=C/Y=12$ and solve for $I\%$: $I\%=12.0\ldots$ The nominal annual rate is $12.0\%$ p.a., so the monthly rate is $\dfrac{12.0}{12}=1\%$ per month.

**Q10.** $r=\dfrac{4.8}{12}=0.4\%$ per month and $N=10\times 12=120$. The invested sum gives $PV=-120000$ and withdrawals are positive. Enter $N=120$, $I\%=4.8$, $PV=-120000$, $FV=0$, $P/Y=C/Y=12$ and solve for $PMT$: $PMT=1261.09\ldots$, so the monthly income is $\$1261.09$. Amortising, the first 119 withdrawals are $\$1261.09$ and the 120th is a smaller $\$1260.75$. The total withdrawn is $119\times 1261.09+1260.75=\$151330.46$, so the total interest earned is $151330.46-120000=\$31330.46$.

**Q11.** Enter $I\%=7.2$ (since $0.6\%\times 12=7.2\%$ p.a.), $PV=-45000$, $PMT=1000$, $FV=0$, $P/Y=C/Y=12$ and solve for $N$: $N=52.61\ldots$ Rounding up, the fund provides **53 withdrawals** — 52 full withdrawals of $\$1000$ and a smaller 53rd. That is 53 months $=4$ years 5 months. Amortising, after 52 withdrawals $\$606.07$ remains, so the final withdrawal is $\$609.71$. The total withdrawn is $52\times 1000+609.71=\$52609.71$, so the total interest earned is $52609.71-45000=\$7609.71$.

**Q12.** The fund lasts exactly $N=13$ quarters, so $FV=0$. Enter $N=13$, $PV=-40000$, $PMT=3409.61$, $FV=0$, $P/Y=C/Y=4$ and solve for $I\%$: $I\%=6.0\ldots$ The nominal annual rate is $6.0\%$ p.a., so the rate per quarter is $\dfrac{6.0}{4}=1.5\%$.

**Q13.** (a) $r=\dfrac{12}{12}=1\%$ per month and $N=12$. Enter $N=12$, $I\%=12$, $PV=6000$, $FV=0$, $P/Y=C/Y=12$ and solve for $PMT$: $PMT=-533.0927\ldots$, the exact repayment $\$533.0927\ldots$ (b) If the repayment were rounded to the nearest cent, $\$533.09$, it would be a fraction of a cent *less* than the exact figure, so after 12 months $\$0.03$ would still be owing and a 13th repayment of $\$0.03$ would be needed. Rounding **up** to $\$533.10$ pays a fraction of a cent more each month, so the loan is cleared within the 12 months, with a slightly smaller final repayment. (c) Amortising at $1\%$ per month with $\$533.10$:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$6\,000.00$ |
| 1 | $\$533.10$ | $\$60.00$ | $\$473.10$ | $\$5\,526.90$ |
| 2 | $\$533.10$ | $\$55.27$ | $\$477.83$ | $\$5\,049.07$ |
| 3 | $\$533.10$ | $\$50.49$ | $\$482.61$ | $\$4\,566.46$ |
| 4 | $\$533.10$ | $\$45.66$ | $\$487.44$ | $\$4\,079.02$ |
| 5 | $\$533.10$ | $\$40.79$ | $\$492.31$ | $\$3\,586.71$ |
| 6 | $\$533.10$ | $\$35.87$ | $\$497.23$ | $\$3\,089.48$ |
| 7 | $\$533.10$ | $\$30.89$ | $\$502.21$ | $\$2\,587.27$ |
| 8 | $\$533.10$ | $\$25.87$ | $\$507.23$ | $\$2\,080.04$ |
| 9 | $\$533.10$ | $\$20.80$ | $\$512.30$ | $\$1\,567.74$ |
| 10 | $\$533.10$ | $\$15.68$ | $\$517.42$ | $\$1\,050.32$ |
| 11 | $\$533.10$ | $\$10.50$ | $\$522.60$ | $\$527.72$ |
| 12 | $\$533.00$ | $\$5.28$ | $\$527.72$ | $\$0.00$ |

So there are **12 repayments** (eleven of $\$533.10$ and a final $\$533.00$), with total interest $\$397.10$.

**Q14.** $r=\frac{6}{12}=0.5\%$ per month. Enter $I\%=6$, $PV=100\,000$, $PMT=-1200$, $FV=0$, $P/Y=C/Y=12$ and solve for $N$: $N=108.07\ldots$ Rounding up, **109 repayments** are needed — 108 full repayments of $\$1\,200$ and a smaller 109th. That is 109 months $=9$ years 1 month. Amortising, after 108 repayments $\$82.11$ remains, so the final repayment is $\$82.52$. The total cost is $108\times1200+82.52=\$129\,682.52$, so the total interest is $129\,682.52-100\,000=\$29\,682.52$.

**Q15.** In each case $r=\frac{9}{12}=0.75\%$ per month, $PV=24\,000$, $FV=0$, $P/Y=C/Y=12$; solve for $PMT$ and round up. (a) Over 3 years, $N=36$: $PMT=-763.1936\ldots$, so the monthly repayment is $\$763.20$; amortising gives total interest $\$3\,474.96$. (b) Over 5 years, $N=60$: $PMT=-498.2005\ldots$, so the monthly repayment is $\$498.21$; amortising gives total interest $\$5\,891.90$. (c) The longer term lowers the monthly repayment (from $\$763.20$ to $\$498.21$) but raises the total interest (from $\$3\,474.96$ to $\$5\,891.90$), because the debt is outstanding for longer, so more interest is charged in total.

**Q16.** The fund lasts exactly $N=25\times12=300$ months, so $FV=0$. Enter $N=300$, $PV=-250\,000$, $PMT=1520.32$, $FV=0$, $P/Y=C/Y=12$ and solve for $I\%$: $I\%=5.4\ldots$ The nominal annual rate is $5.4\%$ p.a., so the rate per month is $\frac{5.4}{12}=0.45\%$.

**Q17.** (a) A non-whole $N$ means the balance reaches zero **partway** through a period. The whole-number part is the number of full payments, and the fraction shows that one more, smaller payment is needed to finish; so $N$ is rounded **up** to give the total number of payments. (b) The solver's exact repayment is not a whole number of cents. Rounding it **up** to the next cent makes every repayment a fraction of a cent larger than needed, so the debt is certainly cleared within the agreed number of payments (with a slightly smaller final payment). Rounding down would leave a few cents owing and force a small extra payment. (c) The by-hand amortisation table rounds the interest to the nearest cent **every period** and carries the rounded balance forward, whereas the finance solver evaluates one formula and rounds only at the end. The many small period-by-period roundings accumulate, so a total read from the table can differ from the solver's figure by a cent or two.

**Q18.** $r = \frac{5.4}{12} = 0.45\,\%$ per month and $N = 25 \times 12 = 300$. Enter $N = 300$, $I\,\% = 5.4$, $P\,V = 320\,000$, $F\,V = 0$, $P\,/\,Y = C\,/\,Y = 12$ and solve for $P\,M\,T$: $P\,M\,T = -\,1946.0157\ldots$, which rounds up to $\$\,1\,946.02$. Amortising at $0.45\,\%$ per month, the first 299 repayments are $\$\,1\,946.02$ and the 300th is a smaller $\$\,1\,943.16$. The total repaid is $299 \times 1946.02 + 1943.16 = \$\,583\,803.14$, so the total interest is $583\,803.14 - 320\,000 = \$\,263\,803.14$.

**Q19.** $r = \frac{9}{12} = 0.75\,\%$ per month. Enter $I\,\% = 9$, $P\,V = 40\,000$, $P\,M\,T = -\,900$, $F\,V = 0$, $P\,/\,Y = C\,/\,Y = 12$ and solve for $N$: $N = 54.26\ldots$ Rounding up, **55 repayments** are needed — 54 full repayments of $\$\,900$ and a smaller 55th. That is 55 months $= 4$ years 7 months. Amortising, after 54 repayments $\$\,236.92$ remains, so the final repayment is $\$\,238.70$. The total cost is $54 \times 900 + 238.70 = \$\,48\,838.70$, so the total interest is $48\,838.70 - 40\,000 = \$\,8\,838.70$.

**Q20.** The $N = 2 \times 12 = 24$ repayments clear the loan, so $F\,V = 0$. Enter $N = 24$, $P\,V = 12\,000$, $P\,M\,T = -\,551.53$, $F\,V = 0$, $P\,/\,Y = C\,/\,Y = 12$ and solve for $I\,\%$: $I\,\% = 9.6\ldots$ The nominal annual rate is $9.6\,\%$ p.a., so the rate per month is $\frac{9.6}{12} = 0.8\,\%$.

## Chapter 7 Reducing-balance loans, annuities and investments — 7G Solving multi-stage financial problems

### Theory

Earlier sections built a separate recurrence for each kind of account: compound interest $V_{n+1} = R V_n$; a savings plan (compound interest with regular additions) $V_{n+1} = R V_n + D$; and a reducing-balance loan or annuity $V_{n+1} = R V_n - D$, together with its amortisation table. Real financial situations are rarely a single one of these running unchanged from start to finish. A loan's interest rate is revised partway through; a saver builds a fund and then lives off it; a borrower makes a one-off extra repayment; two competing offers must be compared. This section is about **multi-stage** problems, which are solved by breaking them into stages and using the tools already developed on each stage in turn.

**The key idea: carry the balance forward.** Split the situation into **stages**, where within a single stage the growth factor $R$ and the regular payment $D$ are both **fixed**. Each stage is then just one of the recurrences already met, solved by iterating one period at a time. The **closing balance of one stage becomes the opening value $V_0$ of the next stage**. Only the *balance* carries across the join; the period counter restarts at $0$ for each new stage, because the new stage has its own rule. Working through a multi-stage problem is therefore a sequence of familiar single-stage calculations, linked by passing the balance from one to the next.

**A change in the rate or the payment partway through.** Suppose a loan runs at one rate, then the rate is revised, or the borrower changes the size of the regular repayment. Iterate the first recurrence up to the moment of the change to find the balance there; then start a **second** recurrence from that balance, using the new growth factor $R$ (for a new rate) and/or the new payment $D$. For example, a loan at $1\%$ per month repaid by $\$1800$ becomes $V_{n+1} = 1.01 V_n - 1800$ for the first stage; if the rate then rises to $1.5\%$ per month the second stage is $V_{n+1} = 1.015 V_n - 1800$, started from the balance the first stage reached.

**Saving, then drawing down.** A very common two-stage plan is to **build** a fund and then **spend** it. Stage 1 is a savings plan, $V_{n+1} = R V_n + D$ with $D$ *added* each period, so the balance climbs. The value it reaches is the lump sum that starts Stage 2, an **annuity**, $V_{n+1} = R V_n - D$ with $D$ *withdrawn* each period, so the balance falls back towards zero. The balance therefore **rises during the saving stage and falls during the drawdown stage**. The two stages usually have different payments (a deposit while saving, a larger income while drawing down), and may even have different rates.

**A one-off (lump-sum) payment.** A single extra payment — an extra repayment off a loan, or an extra deposit into a savings plan — changes the balance **at one instant only**, with no interest attached to that instant. So a lump sum is applied by simply **subtracting it from (or adding it to) the current balance**, and then continuing the recurrence from the adjusted balance. A lump-sum extra repayment is normally made at the *end* of a period, straight after the regular payment; it cuts the balance immediately, so every later period's interest is charged on a smaller amount and the loan clears sooner and more cheaply.

**Comparing two options.** To compare two ways of financing the same purchase — two interest rates, two repayment amounts, or two loan terms — model **each option separately** to its payoff (or over a common length of time) and read off the figures that matter: the **time to repay**, the **total paid** (the total cost), and the **total interest**. The cheaper option is not always the one with the lower rate or the smaller repayment: a smaller monthly repayment at a lower rate may still cost less in total than a larger repayment at a higher rate, even though it takes longer. Setting the two amortisations side by side makes the trade-off clear.

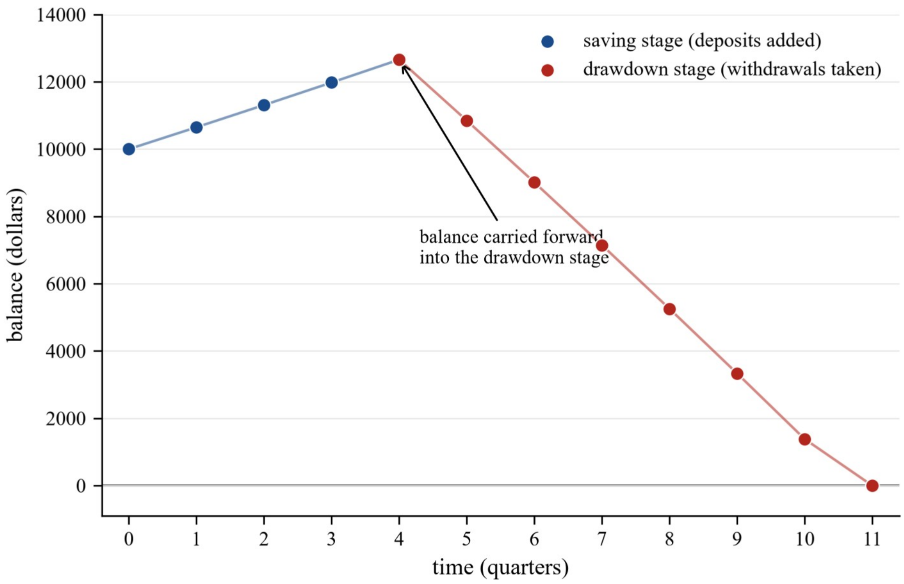
**Rounding convention.** As throughout this chapter, within every stage the interest is rounded to the nearest cent **each period** before it is applied, and the rounded balance is carried forward, so that every balance is exact to the cent and each row reconciles as

$$\text{new balance} = \text{previous balance} + \text{interest (to the nearest cent)} \pm \text{payment}.$$

The balance handed from one stage to the next is this rounded, exact-to-the-cent figure.

**Using technology.** A CAS **finance solver** (TVM) handles each stage in turn. Solve the first stage, read off the balance ($FV$) it reaches, then start the next stage with that value as the new $PV$ (entered with the correct sign) and $N$ reset for the new stage. Because the solver evaluates a single formula per stage and does **not** round period by period, a balance it reports can differ from a hand-built table by a cent or two; over several stages these small differences can add up, so where the solver and the table disagree it is worth noting, not hiding. Building each stage by hand is what lets you set up and check the technology.

The figure below shows a two-stage plan: a fund is built up by regular deposits (the balance rises), and is then drawn down by regular withdrawals as an annuity (the balance falls). The closing balance of the saving stage is exactly the opening balance of the drawdown stage — the single value carried across the join.




## Worked examples

### *Example 1 — scaffolded*

A loan of $\$20\,000$ is charged interest of $12\%$ p.a. compounded monthly and is repaid by $\$1\,800$ at the end of each month. After three months the rate rises to $18\%$ p.a. compounded monthly, while the repayment stays at $\$1\,800$. (a) Write a recurrence relation for the first stage. (b) Find the balance owing after three months. (c) Write a recurrence relation for the second stage. (d) Find the balance owing after a further two months (five months in all).

| Working | Comment |
|---|---|
| Stage 1: $r=\frac{12}{12}=1\%$ per month, so $R=1.01$. | Monthly rate = annual rate ÷ 12. |
| (a) $V_{n+1}=1.01V_n-1800,V_0=20\,000$. | Multiply by $R$ for the interest, then subtract the $\$1\,800$ repayment. |
| (b) Month 1: $20\,000+200-1800=18\,400.00$; month 2: $18\,400+184-1800=16\,784.00$; month 3: $16\,784+167.84-1800=15\,151.84$. | Iterate three times: the balance after three months is $\$15\,151.84$. |
| Stage 2: $r=\frac{18}{12}=1.5\%$ per month, so $R=1.015$. | The new rate gives a new growth factor. |
| (c) $V_{n+1}=1.015V_n-1800,V_0=15\,151.84$. | Start the new stage from the carried-forward balance. |
| (d) Month 4: interest $=0.015\times15\,151.84=227.28$, so $15\,151.84+227.28-1800=13\,579.12$; month 5: interest $=0.015\times13\,579.12=203.69$, so $13\,579.12+203.69-1800=11\,982.81$. | After five months $\$11\,982.81$ is owing. |

### *Example 2 — scaffolded*

An investor deposits $\$10\,000$ into an account earning $6\%$ p.a. compounded quarterly and adds $\$500$ at the end of each quarter for one year. She then stops depositing and instead withdraws $\$2\,000$ at the end of each quarter as income, at the same interest rate. (a) Construct a growth table for the saving year and state the value of the fund after one year. (b) Find the balance remaining after each of the first two withdrawals.

| Working | Comment |
|---|---|
| $r=\frac{6}{4}=1.5\%$ per quarter, so $R=1.015$. | Quarterly rate = annual rate $\div 4$. |
| Stage 1 (saving): $V_{n+1}=1.015V_n+500, V_0=10\,000$. | Add $1.5\%$ interest, then the $\$500$ deposit. |

Iterating the saving stage one quarter at a time gives the growth table.

| Quarter | Interest earned | Deposit | Balance |
|---|---|---|---|
| 0 | — | — | $\$10\,000.00$ |
| 1 | $\$150.00$ | $\$500.00$ | $\$10\,650.00$ |
| 2 | $\$159.75$ | $\$500.00$ | $\$11\,309.75$ |
| 3 | $\$169.65$ | $\$500.00$ | $\$11\,979.40$ |
| 4 | $\$179.69$ | $\$500.00$ | $\$12\,659.09$ |

(a) After one year the fund is worth $\$12\,659.09$ — the closing balance of the saving stage, which is carried forward as the opening balance of the drawdown stage.

Stage 2 (drawdown): $V_{n+1}=1.015V_n-2000$ with $V_0=12\,659.09$.

| Quarter | Interest earned | Withdrawal | Balance |
|---|---|---|---|
| 0 | — | — | $\$12\,659.09$ |
| 1 | $\$189.89$ | $\$2\,000.00$ | $\$10\,848.98$ |
| 2 | $\$162.73$ | $\$2\,000.00$ | $\$9\,011.71$ |

(b) After the first withdrawal $\$10\,848.98$ remains, and after the second $\$9\,011.71$ remains.

### *Example 3 — unscaffolded*

A loan of $\$15\,000$ is charged $9\%$ p.a. compounded monthly and is repaid by $\$1\,000$ at the end of each month. At the end of the third month, immediately after the regular repayment, the borrower pays an extra lump sum of $\$4\,000$ off the loan. Find the balance owing straight after that lump sum, and the balance owing two months later. How much better off is the borrower after those two months than if the lump sum had not been paid?

The monthly rate is $\frac{9}{12}=0.75\%$, so $R=1.0075$ and the first stage is $V_{n+1}=1.0075V_n-1000$ with $V_0=15\,000$. Iterating for three months (interest to the nearest cent):

$$V_1=15\,000+112.50-1000=14\,112.50,$$

$$V_2=14\,112.50+105.84-1000=13\,218.34,$$

$$V_3=13\,218.34+99.14-1000=12\,317.48.$$

The lump sum reduces the balance at that instant, with no interest attached, so straight after it the balance is $12\,317.48-4000=\$8\,317.48$.

The loan then continues at the same rate and repayment, starting from $\$8\,317.48$:

$$V_4=8\,317.48+62.38-1000=7\,379.86, V_5=7\,379.86+55.35-1000=6\,435.21.$$

So two months after the lump sum the balance is $\$6\,435.21$.

Had the lump sum **not** been paid, the balance would have carried on from $\$12\,317.48$:

$$12\,317.48 + 92.38 - 1000 = 11\,409.86,\ 11\,409.86 + 85.57 - 1000 = 10\,495.43.$$

The difference is $10\,495.43 - 6435.21 = \$4\,060.22$. Of this, $\$4\,000$ is the lump sum itself; the remaining $\$60.22$ is interest the borrower has **avoided**, because for those two months interest was charged on a balance smaller by the lump sum.

$\therefore$ the balance is $\$8\,317.48$ just after the lump sum and $\$6\,435.21$ two months later — $\$4\,060.22$ below where it would otherwise have been, a saving of $\$60.22$ in interest already.

### *Example 4 — unscaffolded*

A buyer needs to borrow $\$9\,500$ and is offered two loans. Option A charges $12\,\%$ p.a. compounded monthly and is repaid by $\$2\,000$ per month; Option B charges $6\,\%$ p.a. compounded monthly and is repaid by $\$1\,750$ per month. For each option find the number of repayments, the size of the final repayment and the total cost, and hence advise which loan is cheaper overall.

Option A has $r = \frac{12}{12} = 1\,\%$ per month; Option B has $r = \frac{6}{12} = 0.5\,\%$ per month. Building each amortisation table to payoff, with a smaller final repayment:

Option A ($1\,\%$ per month, $\$2\,000$ repayments):

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,500.00$ |
| 1 | $\$2\,000.00$ | $\$95.00$ | $\$1\,905.00$ | $\$7\,595.00$ |
| 2 | $\$2\,000.00$ | $\$75.95$ | $\$1\,924.05$ | $\$5\,670.95$ |
| 3 | $\$2\,000.00$ | $\$56.71$ | $\$1\,943.29$ | $\$3\,727.66$ |
| 4 | $\$2\,000.00$ | $\$37.28$ | $\$1\,962.72$ | $\$1\,764.94$ |
| 5 | $\$1\,782.59$ | $\$17.65$ | $\$1\,764.94$ | $\$0.00$ |

Option B ($0.5\,\%$ per month, $\$1\,750$ repayments):

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$9\,500.00$ |
| 1 | $\$1\,750.00$ | $\$47.50$ | $\$1\,702.50$ | $\$7\,797.50$ |
| 2 | $\$1\,750.00$ | $\$38.99$ | $\$1\,711.01$ | $\$6\,086.49$ |
| 3 | $\$1\,750.00$ | $\$30.43$ | $\$1\,719.57$ | $\$4\,366.92$ |
| 4 | $\$1\,750.00$ | $\$21.83$ | $\$1\,728.17$ | $\$2\,638.75$ |
| 5 | $\$1\,750.00$ | $\$13.19$ | $\$1\,736.81$ | $\$901.94$ |
| 6 | $\$906.45$ | $\$4.51$ | $\$901.94$ | $\$0.00$ |

Option A clears the debt in **five** repayments (four of $\$2\,000$ and a final $\$1\,782.59$), a total cost of $4 \times 2000 + 1782.59 = \$9\,782.59$, so the total interest is $9782.59 - 9500 = \$282.59$. Option B takes **six** repayments (five of $\$1\,750$ and a final $\$906.45$), a total cost of $5 \times 1750 + 906.45 = \$9\,656.45$, so the total interest is $\$156.45$. Option A repays the loan one month sooner, but Option B costs $9782.59 - 9656.45 = \$126.14$ less overall: its lower interest rate outweighs its smaller monthly repayment.

∴ Option B is the cheaper loan (total cost $\$9656.45$ against $\$9782.59$), even though Option A is repaid one month sooner — a trade-off of speed against total cost.

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## Practice questions

### *Scaffolded*

**Q1.** A loan of $\$12\,000$ is charged $12\%$ p.a. compounded monthly ($1\%$ per month) and is repaid by $\$1\,500$ at the end of each month. After two months the rate rises to $15\%$ p.a. compounded monthly ($1.25\%$ per month); the repayment is unchanged. (a) Write the recurrence relation for the first stage. (b) Find the balance owing after two months. (c) Write the recurrence relation for the second stage. (d) Find the balance owing after four months.

**Q2.** $\$8\,000$ is invested at $8\%$ p.a. compounded quarterly ($2\%$ per quarter), with $\$600$ added at the end of each quarter for three quarters. The deposits then stop and $\$1\,500$ is withdrawn at the end of each quarter at the same rate. (a) Write the recurrence for the saving stage and find the value of the fund after three quarters. (b) Write the recurrence for the drawdown stage. (c) Find the balance remaining after each of the first two withdrawals.

**Q3.** A loan of $\$10\,000$ is charged $1\%$ interest per month and is repaid by $\$900$ at the end of each month. At the end of the second month, straight after the regular repayment, an extra lump sum of $\$3\,000$ is paid off the loan. (a) Find the balance owing after two repayments. (b) Find the balance owing immediately after the lump sum is paid. (c) Find the balance owing after the next (third) repayment.

**Q4.** A loan of $\$15\,000$ is charged $1\%$ interest per month and is repaid by $\$2\,000$ at the end of each month for three months. The borrower then increases the repayment to $\$3\,000$ per month. (a) Find the balance owing after three months. (b) Write the recurrence relation for the second stage. (c) Find the balance owing after five months.

**Q5.** A loan of $\$6\,000$ is repaid by $\$1\,300$ at the end of each month. Compare Option A, which charges $1\%$ per month, with Option B, which charges $1.5\%$ per month, over the first three months. (a) Find the balance owing after three months under Option A. (b) Find the balance owing after three months under Option B. (c) State the difference between the two balances.

**Q6.** $\$10\,000$ is invested at $6\%$ p.a. compounded quarterly ($1.5\%$ per quarter), with $\$500$ added at the end of each quarter. After two quarters the rate falls to $4\%$ p.a. compounded quarterly ($1\%$ per quarter); the $\$500$ deposit continues. (a) Find the value of the investment after two quarters. (b) Write the recurrence relation for the second stage. (c) Find the value of the investment after four quarters.

### *Unscaffolded*

**Q7.** A loan of $\$25\,000$ is charged $6\%$ p.a. compounded monthly ($0.5\%$ per month) and is repaid by $\$2\,200$ per month. After four months the rate rises to $9\%$ p.a. compounded monthly ($0.75\%$ per month), the repayment unchanged. Find the balance owing after six months and the total interest charged over those six months.

**Q8.** $\$5\,000$ is invested at $12\%$ p.a. compounded monthly ($1\%$ per month), with $\$300$ added at the end of each month for six months. The deposits then stop and $\$1\,500$ is withdrawn at the end of each month at the same rate. Find the value of the fund after six months, and then the number of withdrawals the fund provides and the size of the final withdrawal.

**Q9.** A loan of $\$18\,000$ is charged $1\%$ interest per month and repaid by $\$1\,600$ per month. At the end of the third month, straight after the regular repayment, an extra lump sum of $\$5\,000$ is paid. Find the balance owing three months after the lump sum, and compare it with the balance that would have been owing had the lump sum not been paid. How much interest did the lump sum save over those three months?

**Q10.** A loan of $\$30\,000$ is charged $6\%$ p.a. compounded monthly ($0.5\%$ per month) and repaid by $\$2\,000$ per month for four months. The borrower then increases the repayment to $\$2\,500$ per month. Find the balance owing after eight months.

**Q11.** A buyer must borrow $\$12\,000$ and is offered two loans. Option A charges $1\%$ per month and is repaid by $\$2\,600$ per month; Option B charges $0.75\%$ per month and is repaid by $\$2\,500$ per month. For each option find the number of repayments, the size of the final repayment and the total cost, and state which loan costs less overall.

**Q12.** A retiree invests $\$40\,000$ in an annuity paying $8\%$ p.a. compounded quarterly ($2\%$ per quarter) and withdraws $\$3\,000$ at the end of each quarter. After three quarters the rate falls to $6\%$ p.a. compounded quarterly ($1.5\%$ per quarter); the withdrawal is unchanged. Find the balance remaining after five quarters.

**Q13.** $\$2\,000$ is invested at $6\%$ p.a. compounded monthly ($0.5\%$ per month), with $\$500$ added at the end of each month for one year. The deposits then stop and $\$800$ is withdrawn at the end of each month at the same rate, as income. Find the value of the fund after one year, and for how many months the fund can provide the income (including a smaller final withdrawal).

**Q14.** $\$6\,000$ is invested at $1\%$ interest per month, with $\$400$ added at the end of each month. At the end of the third month, straight after the regular deposit, an extra lump sum of $\$5\,000$ is deposited. Find the value of the investment after three months, immediately after the lump-sum deposit, and after six months.

**Q15.** A home loan of $\$50\,000$ is charged $6\%$ p.a. compounded monthly ($0.5\%$ per month) and repaid by $\$3\,000$ per month. After six months the rate rises to $9\%$ p.a. compounded monthly ($0.75\%$ per month), the repayment unchanged. (a) Find the balance owing after six months. (b) Find the balance owing after nine months. (c) Explain, in terms of the interest charged, why the balance falls more slowly after the rate rise.

**Q16.** A loan of $\$8\,000$ is charged $1\%$ interest per month and repaid by $\$1\,500$ per month. (a) Find the number of repayments and the total interest charged if the loan is repaid in the usual way. (b) Now suppose that, straight after the first repayment, an extra lump sum of $\$2\,000$ is paid. Find the new number of repayments and the new total interest charged. (c) State how much interest the lump sum saves.

**Q17.** A saver invests $\$9\,000$ at $1\%$ per month and adds $\$200$ at the end of each month for four months. The rate then rises to $1.5\%$ per month for a further three months, with the $\$200$ deposit continuing. Finally the saver stops depositing and withdraws $\$1\,000$ at the end of each month for two months at $1.5\%$ per month. Find the balance at the end of each of the three stages.

**Q18.** An annuity of $\$20\,000$ earns $1\%$ interest per month, and $\$1\,500$ is withdrawn at the end of each month for four months. The retiree then increases the withdrawal to $\$2\,500$ per month. Find the balance remaining after seven months.

**Q19.** In a multi-stage financial problem: (a) explain what is meant by "carrying the balance forward", and why only the balance — not the period number — passes from one stage to the next; (b) explain why a one-off lump-sum repayment is applied by subtracting it directly from the balance, with no interest attached to it, whereas a regular repayment has interest charged first.

**Q20.** A loan of $\$16\,000$ is charged $12\%$ p.a. compounded monthly ($1\%$ per month) and repaid by $\$2\,000$ per month for three months. The rate then rises to $15\%$ p.a. compounded monthly ($1.25\%$ per month), the repayment unchanged. (a) Write the recurrence relation for each stage. (b) Find the balance owing after six months. (c) Find the total interest charged over the six months. (d) A finance solver, run in two steps, returns a balance of $\$4\,745.13$ after six months. Explain why this differs by one cent from your answer to (b).

## Answers

**Q1.** (a) $V_{n+1}=1.01V_n-1500$, $V_0=12\,000$; (b) $\$9\,226.20$; (c) $V_{n+1}=1.0125V_n-1500$, $V_0=9\,226.20$; (d) $\$6\,439.55$. **Q2.** (a) $V_{n+1}=1.02V_n+600$, $V_0=8\,000$; $\$10\,325.90$; (b) $V_{n+1}=1.02V_n-1500$, $V_0=10\,325.90$; (c) $\$9\,032.42$, $\$7\,713.07$. **Q3.** (a) $\$8\,392.00$; (b) $\$5\,392.00$; (c) $\$4\,545.92$. **Q4.** (a) $\$9\,394.32$; (b) $V_{n+1}=1.01V_n-3000$, $V_0=9\,394.32$; (c) $\$3\,553.14$. **Q5.** (a) $\$2\,242.68$; (b) $\$2\,315.28$; (c) $\$72.60$. **Q6.** (a) $\$11\,309.75$; (b) $V_{n+1}=1.01V_n+500$, $V_0=11\,309.75$; (c) $\$12\,542.08$. **Q7.** Balance $\$12\,471.55$; total interest $\$671.55$. **Q8.** Fund after six months $\$7\,153.21$; five withdrawals (four of $\$1\,500$ and a final $\$1\,366.59$). **Q9.** With the lump sum $\$4\,112.63$; without it $\$9\,264.14$; the balance is $\$5\,151.51$ lower, so the lump sum ($\$5\,000$) saved $\$151.51$ in interest over the three months. **Q10.** $\$12\,923.34$. **Q11.** Option A: five repayments (final $\$1\,949.50$), total cost $\$12\,349.50$; Option B: five repayments (final $\$2\,267.89$), total cost $\$12\,267.89$; Option B costs less overall. **Q12.** $\$28\,227.62$. **Q13.** Fund after one year $\$8\,291.13$; the income lasts eleven months (ten withdrawals of $\$800$ and a final $\$535.38$). **Q14.** After three months $\$7\,393.85$; after the lump-sum deposit $\$12\,393.85$; after six months $\$13\,981.44$. **Q15.** (a) $\$33\,292.37$; (b) $\$24\,979.41$; (c) at the higher rate more of each $\$3\,000$ is taken by interest, so less reduces the principal and the balance falls more slowly. **Q16.** (a) six repayments, total interest $\$264.14$; (b) five repayments, total interest $\$175.37$; (c) $\$88.77$ saved. **Q17.** End of stage 1 $\$10\,177.52$; end of stage 2 $\$11\,251.45$; end of stage 3 $\$9\,576.52$. **Q18.** $\$7\,592.30$. **Q19.** (a) The closing balance of one stage becomes the opening value $V_0$ of the next; only the amount of money continues, while each stage has its own rule and so its own period count starting at $0$. (b) A lump sum changes the balance at a single instant with no period passing, so no interest is attached; a regular repayment is made at the end of a period, after that period's interest has been charged on the balance. **Q20.** (a) Stage 1 $V_{n+1}=1.01V_n-2000$, $V_0=16\,000$; stage 2 $V_{n+1}=1.0125V_n-2000$, $V_0=10\,424.62$; (b) $\$4\,745.14$; (c) $\$745.14$; (d) the solver evaluates each stage from a single formula without rounding to the cent each month, whereas the table rounds every month; those roundings accumulate to one cent, so the table finishes one cent above the solver.

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## Fully-worked solutions

**Q1.** Stage 1: $R=1.01$, so $V_{n+1}=1.01V_n-1500$ with $V_0=12\,000$. Month 1: interest $=0.01\times12\,000=120$, balance $12\,000+120-1500=10\,620.00$. Month 2: interest $=0.01\times10\,620=106.20$, balance $10\,620+106.20-1500=9\,226.20$. So (b) the balance after two months is $\$9\,226.20$. Stage 2: the new rate is $1.25\,\%$ per month, $R=1.0125$, so (c) $V_{n+1}=1.0125V_n-1500$ with $V_0=9\,226.20$. Month 3: interest $=0.0125\times9\,226.20=115.3275\to115.33$, balance $9\,226.20+115.33-1500=7\,841.53$. Month 4: interest $=0.0125\times7\,841.53=98.0191\to98.02$, balance $7\,841.53+98.02-1500=6\,439.55$. So (d) the balance after four months is $\$6\,439.55$.

**Q2.** (a) Saving stage: $R=1.02$, so $V_{n+1}=1.02V_n+600$ with $V_0=8\,000$. Quarter 1: interest 160, balance $8\,000+160+600=8\,760.00$. Quarter 2: interest $=0.02\times8\,760=175.20$, balance $8\,760+175.20+600=9\,535.20$. Quarter 3: interest $=0.02\times9\,535.20=190.704\to190.70$, balance $9\,535.20+190.70+600=10\,325.90$. The fund is worth $\$10\,325.90$ after three quarters. (b) Drawdown stage: $V_{n+1}=1.02V_n-1500$ with $V_0=10\,325.90$. (c) Quarter 1: interest $=0.02\times10\,325.90=206.518\to206.52$, balance $10\,325.90+206.52-1500=9\,032.42$. Quarter 2: interest $=0.02\times9\,032.42=180.6484\to180.65$, balance $9\,032.42+180.65-1500=7\,713.07$. So $\$9\,032.42$ remains after the first withdrawal and $\$7\,713.07$ after the second.

**Q3.** The loan runs at $1\,\%$ per month, repaid by $\$900$: $V_{n+1}=1.01V_n-900$, $V_0=10\,000$. Month 1: interest 100, balance $10\,000+100-900=9\,200.00$. Month 2: interest $=0.01\times9\,200=92$, balance $9\,200+92-900=8\,392.00$. So (a) after two repayments $\$8\,392.00$ is owing. (b) The lump sum reduces this at once, with no interest attached: $8\,392-3000=\$5\,392.00$. (c) Month 3 continues from $\$5\,392$: interest $=0.01\times5\,392=53.92$, balance $5\,392+53.92-900=\$4\,545.92$.

**Q4.** Stage 1: $V_{n+1}=1.01V_n-2000$ with $V_0=15\,000$. Month 1: interest 150, balance $15\,000+150-2000=13\,150.00$. Month 2: interest 131.50, balance $13\,150+131.50-2000=11\,281.50$. Month 3: interest $=0.01\times11\,281.50=112.815\to112.82$, balance $11\,281.50+112.82-2000=9\,394.32$. So (a) the balance after three months is $\$9394.32$. (b) With the larger repayment the second stage is $V_{n+1}=1.01V_n-3000$ with $V_0=9394.32$. Month 4: interest $=0.01\times9394.32=93.9432\to93.94$, balance $9394.32+93.94-3000=6488.26$. Month 5: interest $=0.01\times6488.26=64.8826\to64.88$, balance $6488.26+64.88-3000=3553.14$. So (c) the balance after five months is $\$3553.14$.

**Q5.** The repayment is $\$1300$ each month; the interest is the stated percentage of the previous balance. (a) Option A at $1\%$: $6000+60-1300=4760.00$; then $4760+47.60-1300=3507.60$; then interest $=0.01\times3507.60=35.076\to35.08$, balance $3507.60+35.08-1300=2242.68$. So $\$2242.68$ is owing after three months. (b) Option B at $1.5\%$: $6000+90-1300=4790.00$; then interest $=0.015\times4790=71.85$, balance $4790+71.85-1300=3561.85$; then interest $=0.015\times3561.85=53.42775\to53.43$, balance $3561.85+53.43-1300=2315.28$. So $\$2315.28$ is owing after three months. (c) The difference is $2315.28-2242.68=\$72.60$ — the higher rate leaves more owing after the same three repayments.

**Q6.** (a) Stage 1 at $1.5\%$ per quarter: $V_{n+1}=1.015V_n+500$ with $V_0=10000$. Quarter 1: interest $150$, balance $10000+150+500=10650.00$. Quarter 2: interest $=0.015\times10650=159.75$, balance $10650+159.75+500=11309.75$. So the investment is worth $\$11309.75$ after two quarters. (b) At the lower rate the second stage is $V_{n+1}=1.01V_n+500$ with $V_0=11309.75$. (c) Quarter 3: interest $=0.01\times11309.75=113.0975\to113.10$, balance $11309.75+113.10+500=11922.85$. Quarter 4: interest $=0.01\times11922.85=119.2285\to119.23$, balance $11922.85+119.23+500=12542.08$. So the investment is worth $\$12542.08$ after four quarters.

**Q7.** Stage 1 at $0.5\%$ per month: $V_{n+1}=1.005V_n-2200$ with $V_0=25000$. Month 1: interest $125$, balance $22925.00$. Month 2: interest $=0.005\times22925=114.625\to114.63$, balance $20839.63$. Month 3: interest $=0.005\times20839.63=104.198\to104.20$, balance $18743.83$. Month 4: interest $=0.005\times18743.83=93.719\to93.72$, balance $16637.55$. Stage 2 at $0.75\%$ per month: $V_{n+1}=1.0075V_n-2200$ with $V_0=16637.55$. Month 5: interest $=0.0075\times16637.55=124.7816\to124.78$, balance $14562.33$. Month 6: interest $=0.0075\times14562.33=109.2175\to109.22$, balance $12471.55$. So the balance after six months is $\$12471.55$. The total interest is the sum of the interest column, $125+114.63+104.20+93.72+124.78+109.22=\$671.55$ (check: six repayments total $\$13200$ and the balance fell by $25000-12471.55=12528.45$, and $13200-12528.45=\$671.55$).

**Q8.** Saving stage at $1\%$ per month: $V_{n+1}=1.01V_n+300$ with $V_0=5000$. Iterating for six months (interest to the nearest cent):

$$5350.00, 5703.50, 6060.54, 6421.15, 6785.36, 7153.21,$$

so the fund is worth $\$7153.21$ after six months. Drawdown stage at $1\%$ per month, withdrawing $\$1500$: $V_{n+1}=1.01V_n-1500$ with $V_0=7153.21$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 7 | $\$1500.00$ | $\$71.53$ | $\$1428.47$ | $\$5724.74$ |
| 8 | $\$1500.00$ | $\$57.25$ | $\$1442.75$ | $\$4281.99$ |
| 9 | $\$1500.00$ | $\$42.82$ | $\$1457.18$ | $\$2824.81$ |
| 10 | $\$1500.00$ | $\$28.25$ | $\$1471.75$ | $\$1353.06$ |
| 11 | $\$1366.59$ | $\$13.53$ | $\$1353.06$ | $\$0.00$ |

After four full withdrawals only $\$1353.06$ remains; its interest is $0.01\times1353.06=13.5306\to13.53$, so $1353.06+13.53=\$1366.59$ empties the fund. So the fund provides **five** withdrawals — four of $\$1500$ and a final withdrawal of $\$1366.59$.

**Q9.** Stage 1 at $1\%$ per month, $\$1600$ repayments: $V_{n+1}=1.01V_n-1600$, $V_0=18000$. Month 1: interest 180, balance $16580.00$. Month 2: interest $165.80$, balance $15145.80$. Month 3: interest

$=0.01\times 15\,145.80=151.458\rightarrow 151.46$, balance $13\,697.26$. The lump sum reduces this at once: $13\,697.26-5000=\$8\,697.26$. Continuing three more months from $\$8\,697.26$:

$$8\,697.26+86.97-1600=7\,184.23, 7\,184.23+71.84-1600=5\,656.07,$$

$$5\,656.07+56.56-1600=4\,112.63,$$

so the balance three months after the lump sum is $\$4\,112.63$. Had the lump sum not been paid, the loan would have continued from $\$13\,697.26$:

$$13\,697.26+136.97-1600=12\,234.23, 12\,234.23+122.34-1600=10\,756.57,$$

$$10\,756.57+107.57-1600=9\,264.14.$$

The balance is $9\,264.14-4\,112.63=\$5\,151.51$ lower with the lump sum. Of this, $\$5\,000$ is the lump sum itself, so the interest saved over the three months is $5\,151.51-5\,000=\$151.51$.

**Q10.** Stage 1 at $0.5\%$ per month, $\$2\,000$ repayments: $V_{n+1}=1.005\,V_n-2000$, $V_0=30\,000$. Month 1: interest 150, balance $28\,150.00$. Month 2: interest $=0.005\times 28\,150=140.75$, balance $26\,290.75$. Month 3: interest $=0.005\times 26\,290.75=131.4538\rightarrow 131.45$, balance $24\,422.20$. Month 4: interest $=0.005\times 24\,422.20=122.111\rightarrow 122.11$, balance $22\,544.31$. Stage 2 at $0.5\%$ per month, $\$2\,500$ repayments: $V_{n+1}=1.005\,V_n-2500$ with $V_0=22\,544.31$. Month 5: interest $=0.005\times 22\,544.31=112.7216\rightarrow 112.72$, balance $20\,157.03$. Month 6: interest $=0.005\times 20\,157.03=100.7852\rightarrow 100.79$, balance $17\,757.82$. Month 7: interest $=0.005\times 17\,757.82=88.7891\rightarrow 88.79$, balance $15\,346.61$. Month 8: interest $=0.005\times 15\,346.61=76.7331\rightarrow 76.73$, balance $12\,923.34$. So the balance owing after eight months is $\$12\,923.34$.

**Q11.** Option A at $1\%$ per month, $\$2\,600$ repayments:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$12\,000.00$ |
| 1 | $\$2\,600.00$ | $\$120.00$ | $\$2\,480.00$ | $\$9\,520.00$ |
| 2 | $\$2\,600.00$ | $\$95.20$ | $\$2\,504.80$ | $\$7\,015.20$ |
| 3 | $\$2\,600.00$ | $\$70.15$ | $\$2\,529.85$ | $\$4\,485.35$ |
| 4 | $\$2\,600.00$ | $\$44.85$ | $\$2\,555.15$ | $\$1\,930.20$ |
| 5 | $\$1\,949.50$ | $\$19.30$ | $\$1\,930.20$ | $\$0.00$ |

Option B at $0.75\%$ per month, $\$2\,500$ repayments:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$12\,000.00$ |
| 1 | $\$2\,500.00$ | $\$90.00$ | $\$2\,410.00$ | $\$9\,590.00$ |
| 2 | $\$2\,500.00$ | $\$71.93$ | $\$2\,428.07$ | $\$7\,161.93$ |
| 3 | $\$2\,500.00$ | $\$53.71$ | $\$2\,446.29$ | $\$4\,715.64$ |
| 4 | $\$2\,500.00$ | $\$35.37$ | $\$2\,464.63$ | $\$2\,251.01$ |
| 5 | $\$2\,267.89$ | $\$16.88$ | $\$2\,251.01$ | $\$0.00$ |

Each option takes **five** repayments. Option A's final repayment is $\$1\,949.50$, a total cost of $4\times 2600+1949.50=\$12\,349.50$ (total interest $\$349.50$). Option B's final repayment is $\$2\,267.89$, a total cost of $4\times 2500+2267.89=\$12\,267.89$ (total interest $\$267.89$). So **Option B costs less overall** — $12\,349.50-12\,267.89=\$81.61$ less — because its lower rate charges less interest, even though its monthly repayment is smaller.

**Q12.** Stage 1 at $2\%$ per quarter: $V_{n+1} = 1.02\,V_n - 3000$ with $V_0 = 40\,000$. Quarter 1: interest $800$, balance $37\,800.00$. Quarter 2: interest $756$, balance $35\,556.00$. Quarter 3: interest $= 0.02 \times 35\,556 = 711.12$, balance $33\,267.12$. Stage 2 at $1.5\%$ per quarter: $V_{n+1} = 1.015\,V_n - 3000$ with $V_0 = 33\,267.12$. Quarter 4: interest $= 0.015 \times 33\,267.12 = 499.0068 \rightarrow 499.01$, balance $30\,766.13$. Quarter 5: interest $= 0.015 \times 30\,766.13 = 461.492 \rightarrow 461.49$, balance $28\,227.62$. So the balance remaining after five quarters is $\$28\,227.62$.

**Q13.** Saving stage at $0.5\%$ per month: $V_{n+1} = 1.005\,V_n + 500$ with $V_0 = 2\,000$. Iterating for twelve months (interest to the nearest cent):

$$2\,510.00, 3\,022.55, 3\,537.66, 4\,055.35, 4\,575.63, 5\,098.51,$$

$$5\,624.00, 6\,152.12, 6\,682.88, 7\,216.29, 7\,752.37, 8\,291.13,$$

so the fund is worth $\$8\,291.13$ after one year. Drawdown stage at $0.5\%$ per month, withdrawing $\$800$: $V_{n+1} = 1.005\,V_n - 800$ with $V_0 = 8\,291.13$. Iterating gives the balances

$$7\,532.59, 6\,770.25, 6\,004.10, 5\,234.12, 4\,460.29, 3\,682.59,$$

$$2\,901.00, 2\,115.51, 1\,326.09, 532.72,$$

after ten withdrawals of $\$800$. Then only $\$532.72$ remains; its interest is $0.005 \times 532.72 = 2.6636 \rightarrow 2.66$, so $532.72 + 2.66 = \$535.38$ empties the fund. So the income lasts **eleven months** — ten withdrawals of $\$800$ and a final withdrawal of $\$535.38$.

**Q14.** Saving stage at $1\%$ per month: $V_{n+1} = 1.01\,V_n + 400$ with $V_0 = 6\,000$. Month 1: interest $60$, balance $6\,000 + 60 + 400 = 6\,460.00$. Month 2: interest $= 0.01 \times 6\,460 = 64.60$, balance $6\,460 + 64.60 + 400 = 6\,924.60$. Month 3: interest $= 0.01 \times 6\,924.60 = 69.246 \rightarrow 69.25$, balance $6\,924.60 + 69.25 + 400 = 7\,393.85$. So after three months the investment is worth $\$7\,393.85$. The lump-sum deposit adds to this at once: $7\,393.85 + 5\,000 = \$12\,393.85$. Continuing three more months from $\$12\,393.85$:

$$12\,393.85 + 123.94 + 400 = 12\,917.79, 12\,917.79 + 129.18 + 400 = 13\,446.97,$$

$$13\,446.97 + 134.47 + 400 = 13\,981.44,$$

so after six months the investment is worth $\$13\,981.44$.

**Q15.** (a) Stage 1 at $0.5\%$ per month: $V_{n+1} = 1.005\,V_n - 3000$ with $V_0 = 50\,000$. Month 1: interest $250$, balance $47\,250.00$. Month 2: interest $= 0.005 \times 47\,250 = 236.25$, balance $44\,486.25$. Month 3: interest $= 0.005 \times 44\,486.25 = 222.431 \rightarrow 222.43$, balance $41\,708.68$. Month 4: interest $= 0.005 \times 41\,708.68 = 208.5434 \rightarrow 208.54$, balance $38\,917.22$. Month 5: interest $= 0.005 \times 38\,917.22 = 194.5861 \rightarrow 194.59$, balance $36\,111.81$. Month 6: interest $= 0.005 \times 36\,111.81 = 180.5591 \rightarrow 180.56$, balance $33\,292.37$. So the balance after six months is $\$33\,292.37$. (b) Stage 2 at $0.75\%$ per month: $V_{n+1} = 1.0075\,V_n - 3000$ with $V_0 = 33\,292.37$. Month 7: interest $= 0.0075 \times 33\,292.37 = 249.6928 \rightarrow 249.69$, balance $30\,542.06$. Month 8: interest $= 0.0075 \times 30\,542.06 = 229.0655 \rightarrow 229.07$, balance $27\,771.13$. Month 9: interest $= 0.0075 \times 27\,771.13 = 208.2835 \rightarrow 208.28$, balance $24\,979.41$. So the balance after nine months is $\$24\,979.41$. (c) After the rise the interest is $0.75\%$ of the balance instead of $0.5\%$, so a larger part of each $\$3\,000$ repayment is used up paying interest and a smaller part reduces the principal; the balance therefore falls more slowly than before.

**Q16.** (a) At $1\%$ per month, $\$1\,500$ repayments: $V_{n+1} = 1.01\,V_n - 1500$, $V_0 = 8\,000$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$8\,000.00$ |
| 1 | $\$1\,500.00$ | $\$80.00$ | $\$1\,420.00$ | $\$6\,580.00$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 2 | $\$1500.00$ | $\$65.80$ | $\$1434.20$ | $\$5145.80$ |
| 3 | $\$1500.00$ | $\$51.46$ | $\$1448.54$ | $\$3697.26$ |
| 4 | $\$1500.00$ | $\$36.97$ | $\$1463.03$ | $\$2234.23$ |
| 5 | $\$1500.00$ | $\$22.34$ | $\$1477.66$ | $\$756.57$ |
| 6 | $\$764.14$ | $\$7.57$ | $\$756.57$ | $\$0.00$ |

So there are **six** repayments and the total interest is $80+65.80+51.46+36.97+22.34+7.57=\$264.14$. (b) With the lump sum, month 1 is unchanged (balance $\$6580.00$), then $\$2000$ is paid off at once: $6580-2000=\$4580.00$. Continuing at $\$1500$ per month:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 1 | $\$1500.00$ | $\$80.00$ | $\$1420.00$ | $\$6580.00$ |
| — | lump $\$2000.00$ | — | $\$2000.00$ | $\$4580.00$ |
| 2 | $\$1500.00$ | $\$45.80$ | $\$1454.20$ | $\$3125.80$ |
| 3 | $\$1500.00$ | $\$31.26$ | $\$1468.74$ | $\$1657.06$ |
| 4 | $\$1500.00$ | $\$16.57$ | $\$1483.43$ | $\$173.63$ |
| 5 | $\$175.37$ | $\$1.74$ | $\$173.63$ | $\$0.00$ |

So there are now **five** repayments and the total interest is $80+45.80+31.26+16.57+1.74=\$175.37$. (c) The lump sum saves $264.14-175.37=\$88.77$ in interest (and clears the loan one month sooner), because from month 2 on the interest is charged on a balance smaller by $\$2000$.

**Q17.** Stage 1 at $1\%$ per month, $\$200$ deposits: $V_{n+1}=1.01V_n+200$ with $V_0=9000$. Month 1: interest $90$, balance $9290.00$. Month 2: interest $=0.01\times9290=92.90$, balance $9582.90$. Month 3: interest $=0.01\times9582.90=95.829\to95.83$, balance $9878.73$. Month 4: interest $=0.01\times9878.73=98.7873\to98.79$, balance $10177.52$. So stage 1 ends at $\$10177.52$. Stage 2 at $1.5\%$ per month, $\$200$ deposits: $V_{n+1}=1.015V_n+200$ with $V_0=10177.52$. Month 5: interest $=0.015\times10177.52=152.6628\to152.66$, balance $10530.18$. Month 6: interest $=0.015\times10530.18=157.9527\to157.95$, balance $10888.13$. Month 7: interest $=0.015\times10888.13=163.3220\to163.32$, balance $11251.45$. So stage 2 ends at $\$11251.45$. Stage 3 at $1.5\%$ per month, $\$1000$ withdrawals: $V_{n+1}=1.015V_n-1000$ with $V_0=11251.45$. Month 8: interest $=0.015\times11251.45=168.7718\to168.77$, balance $10420.22$. Month 9: interest $=0.015\times10420.22=156.3033\to156.30$, balance $9576.52$. So stage 3 ends at $\$9576.52$. The three closing balances are $\$10177.52$, $\$11251.45$ and $\$9576.52$.

**Q18.** Stage 1 at $1\%$ per month, $\$1500$ withdrawals: $V_{n+1}=1.01V_n-1500$ with $V_0=20000$. Month 1: interest $200$, balance $18700.00$. Month 2: interest $187$, balance $17387.00$. Month 3: interest $=0.01\times17387=173.87$, balance $16060.87$. Month 4: interest $=0.01\times16060.87=160.6087\to160.61$, balance $14721.48$. Stage 2 at $1\%$ per month, $\$2500$ withdrawals: $V_{n+1}=1.01V_n-2500$ with $V_0=14721.48$. Month 5: interest $=0.01\times14721.48=147.2148\to147.21$, balance $12368.69$. Month 6: interest $=0.01\times12368.69=123.6869\to123.69$, balance $9992.38$. Month 7: interest $=0.01\times9992.38=99.9238\to99.92$, balance $7592.30$. So the balance remaining after seven months is $\$7592.30$.

**Q19.** (a) A multi-stage problem is split into stages, and within each stage a single recurrence (with a fixed rate and payment) is iterated. “Carrying the balance forward” means taking the closing balance of one stage and using it as the opening value $V_0$ of the next stage. Only the balance passes across the join because that is the amount of money actually in the account; the new stage has a **different rule** — a different rate, payment or direction — so it is a fresh recurrence with its period count starting again at $0$. Counting periods straight through would wrongly suggest the new rule had been running from the start. (b) A regular repayment is made at the **end** of a period, so a full period of interest is charged on the balance first and only then is the payment subtracted. A one-off lump sum is not tied to a period at all: it is paid at a single instant, during which no time passes and so no interest accrues. It is therefore applied by subtracting it directly from the current balance, after which the recurrence resumes and the next period's interest is charged on the reduced balance.

**Q20.** (a) Stage 1: $r = 1\,\%$ per month, $R = 1.01$, so $V_{n+1} = 1.01\,V_n - 2000$ with $V_0 = 16\,000$. Stage 2: $r = 1.25\,\%$ per month, $R = 1.0125$, so $V_{n+1} = 1.0125\,V_n - 2000$ with $V_0$ the balance carried forward. (b) Stage 1: Month 1: interest $160$, balance $14\,160.00$. Month 2: interest $141.60$, balance $12\,301.60$. Month 3: interest $= 0.01 \times 12\,301.60 = 123.016 \rightarrow 123.02$, balance $10\,424.62$. Stage 2 from $\$10\,424.62$: Month 4: interest $= 0.0125 \times 10\,424.62 = 130.30775 \rightarrow 130.31$, balance $8\,554.93$. Month 5: interest $= 0.0125 \times 8\,554.93 = 106.9366 \rightarrow 106.94$, balance $6\,661.87$. Month 6: interest $= 0.0125 \times 6\,661.87 = 83.2734 \rightarrow 83.27$, balance $4\,745.14$. So the balance owing after six months is $\$4\,745.14$. (c) The total interest is $160 + 141.60 + 123.02 + 130.31 + 106.94 + 83.27 = \$745.14$ (check: six repayments total $\$12\,000$ and the balance fell by $16\,000 - 4\,745.14 = 11\,254.86$, and $12\,000 - 11\,254.86 = \$745.14$). (d) The finance solver works each stage from a single formula and rounds only at the end of the stage, whereas the hand table rounds the interest to the nearest cent every month. Those small monthly roundings accumulate to one cent across the six months, so the table's $\$4\,745.14$ is one cent above the solver's $\$4\,745.13$.

### Theory

An earlier section modelled a **reducing-balance loan** with the recurrence

$$V_{n+1} = R V_n - D, V_0 = \text{amount borrowed},$$

where $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r\,\%$ is the interest rate per period) and $D$ is the fixed repayment made at the end of each period. There each repayment was **larger** than the interest charged, so part of it went towards the interest and the rest reduced the balance owing — the debt fell period by period. This section looks at the special case where the repayment is made as small as it can be while the lender is still paid: each payment covers **exactly the interest**, and nothing more.

**The interest-only loan.** In an *interest-only* loan the borrower pays only the interest charged each period; none of the payment goes towards the amount borrowed. Because the principal is never touched, the interest charged is the same every period, so the payment is the same every period:

$$D = \frac{r}{100} \times V_0,$$

the interest on the original amount borrowed. Since the payment exactly cancels the interest, the balance owing stays fixed at $V_0$.

**Why the balance never reduces.** Follow one period through. The balance owing is $V_0$. Interest of $\frac{r}{100} V_0$ is added, taking the amount owed up to $V_0 + \frac{r}{100} V_0$. The payment $D = \frac{r}{100} V_0$ is then made, bringing the balance back down to exactly $V_0$. The payment has done no more than pay off the interest that was just added, so the debt is exactly as big as it was before. This happens every period, so the balance never changes:

$$V_{n+1} = V_n = V_0 \text{ for every } n.$$

**The recurrence.** The interest-only loan is still the reducing-balance recurrence $V_{n+1} = R V_n - D$ — it is just the case in which $D$ has been chosen to equal the interest, $D = (R-1) V_0 = \frac{r}{100} V_0$. Substituting $V_n = V_0$,

$$V_{n+1} = R V_0 - (R-1) V_0 = V_0,$$

so iterating the recurrence returns the same number every time. In practice it is simplest to read it as: *interest is charged, an equal payment removes it, and the balance is unchanged.*

**Rounding convention.** Following the standard used throughout this chapter, the interest is rounded to the nearest cent each period. The interest-only payment is set to that rounded interest, so each period reconciles as

$$\text{new balance} = \text{previous balance} + \text{interest (to the nearest cent)} - \text{payment} = \text{previous balance},$$

and the balance is exactly $V_0$ after every payment. (When the exact interest is not a whole number of cents, the payment is the rounded figure — for example an exact interest of $\$62.625$ is paid as $\$62.63$.)

**Total interest, and the cost of the loan.** Every payment is interest, so over $n$ periods the **total interest paid** is simply

$$\text{total interest} = n \times D.$$

Because none of the principal has been repaid, the full $V_0$ is **still owed** at the end of the interest-only term. It must be repaid then — as a single lump sum, by refinancing, or by switching to a reducing-balance arrangement (below). The **total cost** of borrowing this way is therefore the interest plus the principal that must still be cleared,

$$\text{total cost} = n \times D + V_0.$$

**Comparing interest-only and reducing-balance loans.** For the same loan at the same rate, the interest-only payment $\frac{r}{100}V_0$ is **smaller** than a reducing-balance repayment, because a reducing-balance repayment must cover the interest *and* chip away at the principal. So interest-only payments are easier to meet in the short term. But there are two costs to this:

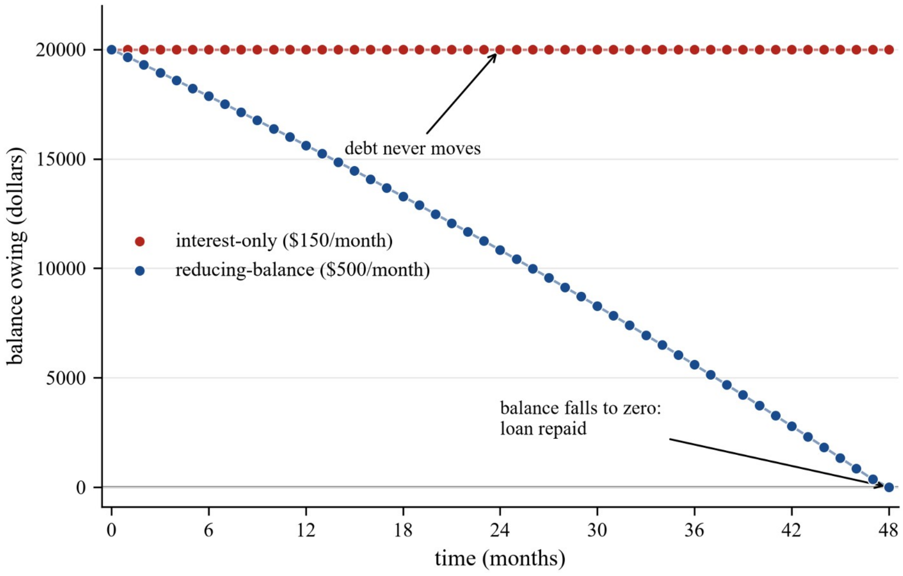
- the debt makes **no progress** — after any number of interest-only payments the amount owed is still the whole $V_0$, so no equity is built up;
- over the same term the interest is charged on the **full** principal every period, whereas on a reducing-balance loan the balance falls, so the interest charged falls with it. An interest-only loan therefore runs up **more** total interest than a reducing-balance loan over the same time.

**Switching to a reducing-balance loan.** Interest-only loans are often arranged for a fixed introductory period, after which the loan reverts to a normal reducing-balance loan. At the moment of the switch the balance owing is still the original $V_0$ (nothing has been repaid). From then on the borrower pays an amount **larger** than the interest, so each payment now splits into an interest part and a principal-reduction part and the balance at last begins to fall, exactly as in a reducing-balance loan.

**A boundary case.** An earlier section noted the matching situation for an *investment*: if the amount withdrawn each period equals exactly the interest earned, the balance never changes, giving a perpetuity that pays a fixed income forever. The interest-only loan is the same boundary case seen from the borrower's side — the payment is set to exactly the interest, so the balance holds steady. If the payment were made even slightly larger than the interest the balance would slowly fall (a reducing-balance loan); slightly smaller, and the unpaid interest would be added on and the debt would slowly **grow**.

**Using technology.** A CAS **finance solver** (TVM) uses $N$ (number of periods), $I\%$ (nominal annual rate), $PV$ (present value), $PMT$ (payment) and $FV$ (future value), with $P/Y = C/Y$ periods per year. For an interest-only loan you receive the principal, so $PV$ is positive; you pay the interest each period, so $PMT$ is negative; and because the whole principal is still owed at the end, $FV = -PV$ (you finish owing what you borrowed). Setting $PMT$ to the value that makes $FV = -PV$ returns the interest-only payment. You should still be able to reason directly: $D = \frac{r}{100}V_0$, and the balance is unchanged.

The figure below compares the same $\$20\,000$ debt under two loans at $9\%$ p.a. compounded monthly. The interest-only balance (upper points) is a flat line — every payment clears just the interest, so the debt never moves. The reducing-balance loan (lower points) is repaid at a larger fixed amount, so its balance falls, slowly at first and then faster, until the debt is cleared.




## Worked examples

### *Example 1 — scaffolded*

Rana borrows $\$24\,000$ at $6\,\%$ p.a. compounded monthly on an interest-only basis, paying only the interest at the end of each month. (a) Find the monthly interest-only payment and write a recurrence relation for the balance $V_n$ owing after $n$ months. (b) Find the balance owing after each of the first three months. (c) Find the total interest paid in the first year.

| Working | Comment |
|---|---|
| $r=\frac{6}{12}=0.5\,\%$ per month, so $R=1+\frac{0.5}{100}=1.005$. | Monthly rate = annual rate ÷ 12; growth factor $R=1+\frac{r}{100}$. |
| (a) Payment $D=0.005\times 24\,000=120$, the interest on $\$24\,000$. | Interest-only, so each payment equals the interest charged: $\$120.00$. |
| Recurrence: $V_{n+1}=1.005\,V_n-120\,,V_0=24\,000$. | Multiply by $R$ for the interest, then subtract the $\$120$ payment. |
| (b) Month 1: interest $=0.005\times 24\,000=120$, so $V_1=24\,000+120-120=24\,000$. | The payment clears the interest exactly; the balance is unchanged: $\$24\,000.00$. |
| Month 2: interest $=0.005\times 24\,000=120$, so $V_2=24\,000+120-120=24\,000$. | Same balance ⇒ same interest ⇒ same payment. |
| Month 3: $V_3=24\,000$, as before. | The balance owing stays at $\$24\,000.00$ every month. |
| (c) Total interest $=12\times 120=1440$. | Twelve equal payments, each entirely interest: $\$1\,440.00$. |

### *Example 2 — scaffolded*

A business borrows $\$15\,000$ at $7.2\,\%$ p.a. compounded monthly, interest-only, for four years, after which the full amount borrowed is repaid in one lump sum. Show the first three months in a table, and find the total interest paid over the four years and the total cost of the loan.

| Working | Comment |
|---|---|
| $r=\frac{7.2}{12}=0.6\,\%$ per month, so the payment is $D=0.006\times 15\,000=90$. | Interest-only payment = interest on $\$15\,000$: $\$90.00$ per month. |
| Recurrence: $V_{n+1}=1.006\,V_n-90\,,V_0=15\,000$. | Each month add $0.6\,\%$ interest, then subtract the $\$90$ |

| Working | Comment |
|---|---|
| | payment. |

Iterating the recurrence gives the balance table (only the first three months are shown; every later month is identical).

| Month | Interest | Payment | Balance |
|---|---|---|---|
| 0 | — | — | $\$15\,000.00$ |
| 1 | $\$90.00$ | $\$90.00$ | $\$15\,000.00$ |
| 2 | $\$90.00$ | $\$90.00$ | $\$15\,000.00$ |
| 3 | $\$90.00$ | $\$90.00$ | $\$15\,000.00$ |

The balance never changes, so the interest is $\$90$ every month for the whole four years. Four years is $4\times12=48$ months, so the total interest is $48\times90=\$4\,320.00$. At the end the $\$15\,000$ borrowed is still owed and is repaid as a lump sum, so the total cost of the loan is $4320+15\,000=\$19\,320.00$.

### *Example 3 — unscaffolded*

A debt of $\$20\,000$ is charged $9\,\%$ p.a. compounded monthly. Compare, over the first six months, paying the loan interest-only with repaying it at $\$500$ at the end of each month. How much interest does each approach pay, and how much of the debt does each clear?

The monthly rate is $\frac{9}{12}=0.75\,\%$, so $R=1.0075$.

**Interest-only.** The payment is $0.0075\times20\,000=\$150$ per month and the recurrence is $V_{n+1}=1.0075\,V_n-150$. Because the payment equals the interest, the balance stays at $\$20\,000$ for all six months. The interest paid is $6\times150=\$900$, and none of the debt is cleared — after six months $\$20\,000$ is still owed.

**Reducing-balance at $\$500$ per month.** Here $V_{n+1}=1.0075\,V_n-500$. Iterating, and rounding the interest to the nearest cent each month:

$$19\,650.00\,,19\,297.38\,,18\,942.11\,,18\,584.18\,,18\,223.56\,,17\,860.24\,,$$

so after six months $\$17\,860.24$ is owed. The interest charged is the sum of the interest column, $150+147.38+144.73+142.07+139.38+136.68=\$860.24$, and the debt has fallen by $20\,000-17\,860.24=\$2\,139.76$.

The interest-only payments are much smaller ($\$150$ against $\$500$), but they buy no progress: the $\$20\,000$ is untouched. The $\$500$ repayments cost more each month, yet they clear $\$2\,139.76$ of the debt and — because the balance, and so the interest, falls each month — actually run up **less** interest ($\$860.24$ against $\$900$) over the six months.

$\therefore$ interest-only pays $\$900$ interest and clears none of the $\$20\,000$; the $\$500$ repayments pay $\$860.24$ interest and reduce the debt to $\$17\,860.24$.

### *Example 4 — unscaffolded*

Tom borrows $\$16\,000$ at $9\,\%$ p.a. compounded monthly. For the first three months the loan is interest-only; from the fourth month it becomes a reducing-balance loan repaid at $\$600$ at the end of each month. Find the balance owing at the end of each of the first six months, and the interest paid in each phase.

The monthly rate is $\frac{9}{12}=0.75\,\%$, so $R=1.0075$ throughout.

**Interest-only phase (months 1–3).** The payment is $0.0075\times16\,000=\$120$ per month, which equals the interest, so the balance stays at $\$16\,000$ for the first three months. The interest paid is $3\times120=\$360$, and nothing is taken off the principal.

**Reducing-balance phase (from month 4).** The balance at the switch is still $\$16\,000$, and now $\$600$ is repaid each month, which is more than the interest, so the balance begins to fall. The first three reducing-balance months form an amortisation table:

| Month | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 3 | — | — | — | $\$16\,000.00$ |
| 4 | $\$600.00$ | $\$120.00$ | $\$480.00$ | $\$15\,520.00$ |
| 5 | $\$600.00$ | $\$116.40$ | $\$483.60$ | $\$15\,036.40$ |
| 6 | $\$600.00$ | $\$112.77$ | $\$487.23$ | $\$14\,549.17$ |

Once the $\$600$ repayments start, each payment splits into interest and a principal reduction, and the balance falls — from $\$16\,000$ to $\$14\,549.17$ over three months. The interest paid in this phase (months 4–6) is $120+116.40+112.77=\$349.17$.

$\therefore$ the balances are $\$16\,000$ for months 1–3, then $\$15\,520.00$, $\$15\,036.40$ and $\$14\,549.17$; the interest is $\$360$ over the interest-only phase and $\$349.17$ over the first three reducing-balance months.

## Practice questions

### *Scaffolded*

**Q1.** A company borrows $\$30\,000$ at $8\,\%$ p.a. compounded quarterly on an interest-only basis. (a) Find the quarterly interest rate and the interest charged each quarter. (b) Write a recurrence relation for the balance $V_n$ owing after $n$ quarters. (c) Find $V_1$ and $V_2$. (d) Find the total interest paid in the first year.

**Q2.** An interest-only loan is modelled by $V_{n+1}=1.005\,V_n-250$ with $V_0=50\,000$. (a) State the interest rate per period and the payment. (b) Show that the payment equals the interest charged in the first period. (c) Find $V_1$ and $V_2$. (d) Explain why the balance owing never changes.

**Q3.** A loan of $\$12\,000$ is charged $6\,\%$ p.a. compounded monthly and is paid interest-only. (a) Copy and complete the balance table for the first four months.

| Month | Interest | Payment | Balance |
|---|---|---|---|
| 0 | — | — | $\$12\,000.00$ |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

(b) State the balance owing after four months.
(c) Find the total interest paid in the first four months.
(d) State what must be repaid at the end of the interest-only term.

**Q4.** A loan of $\$8\,350$ is charged $9\,\%$ p.a. compounded monthly and is paid interest-only. (a) Show that the exact monthly interest is $\$62.625$. (b) State the monthly payment, rounded to the nearest cent. (c) Write a recurrence relation for the balance owing. (d) Find the balance owing after two months.

**Q5.** A debt of $\$10\,000$ is charged $12\,\%$ p.a. compounded monthly. (a) Find the interest-only monthly payment, and the total interest paid over three months if the loan is paid interest-only. (b) If instead $\$1\,000$ is repaid at the end of each month, find the balance owing after three months. (c) State how much of the $\$10\,000$ each approach has cleared after three months.

**Q6.** A loan of $\$14\,000$ is charged $6\%$ p.a. compounded monthly. For the first two months it is interest-only; from the third month it is repaid at $\$800$ per month. (a) Find the interest-only monthly payment. (b) State the balance owing after the first two months. (c) Find the balance owing after each of the next two months (the first two reducing-balance months).

### *Unscaffolded*

**Q7.** A loan of $\$45\,000$ is charged $6\%$ p.a. compounded monthly and is paid interest-only. Write a recurrence relation for the balance owing, state the balance owing after three months, and find the total interest paid in the first year.

**Q8.** An interest-only loan is charged $0.6\%$ interest per month, and the monthly payment is $\$108$. Find the amount borrowed.

**Q9.** A property investor borrows $\$250\,000$ at $5.4\%$ p.a. compounded monthly, interest-only, for three years, after which the principal is repaid in full. Find the monthly payment, the total interest paid over the three years, and the total cost of the loan.

**Q10.** A loan of $\$13\,450$ is charged $9\%$ p.a. compounded monthly and is paid interest-only. (a) Find the exact monthly interest and hence the monthly payment, correct to the nearest cent. (b) State the balance owing after six months. (c) Find the total interest paid in the first six months.

**Q11.** A debt of $\$8\,000$ is charged $12\%$ p.a. compounded monthly. Over the first four months, compare paying it interest-only with repaying it at $\$700$ per month: for each, find the balance owing after four months and the total interest paid.

**Q12.** A home buyer borrows $\$180\,000$ at $6\%$ p.a. compounded monthly. For the first year the loan is interest-only. (a) Find the monthly interest-only payment. (b) Find the total interest paid during the interest-only year. (c) State the balance owing when the interest-only year ends. (d) From the next month the loan is repaid at $\$1\,200$ per month. Find the interest, the principal reduction and the new balance for that first reducing-balance payment.

**Q13.** Explain, in terms of the recurrence $V_{n+1}=RV_n-D$, why an interest-only loan and a perpetuity (an investment that pays a fixed income forever) are described by the *same* boundary case, and state what $V_0$ and $D$ represent in each.

**Q14.** A loan of $\$60\,000$ is charged $8\%$ p.a. compounded quarterly and is paid interest-only for five years. (a) Find the quarterly payment. (b) Find the total interest paid over the five years. (c) Find the total cost of the loan, given the principal is repaid at the end.

**Q15.** Two loans of the same size are taken out at the same interest rate for the same number of periods. One is paid interest-only (with the principal repaid at the end); the other is a reducing-balance loan. Explain why the interest-only loan pays **more** total interest, even though its regular payments are smaller.

**Q16.** An interest-only loan is charged $0.7\%$ interest per month. The monthly payment is $\$84$, and the loan runs interest-only for two years before the principal is repaid. (a) Find the amount borrowed. (b) Find the total interest paid over the two years. (c) Find the total cost of the loan.

**Q17.** A loan of $\$22\,000$ is charged $9\%$ p.a. compounded monthly. It is interest-only for the first three months, then repaid at $\$900$ per month. (a) Find the interest-only monthly payment and the balance owing after three months. (b) Construct an amortisation table for the first three reducing-balance payments, showing the interest, principal reduction and balance.

**Q18.** Two investors each borrow $\$100\,000$ interest-only. Investor A is charged $0.5\%$ interest per month; investor B is charged $0.4\%$ interest per month. (a) Find each investor's monthly payment. (b) State, with a reason, whose debt is smaller after one year.

**Q19.** A debt of $\$20\,000$ is charged $6\%$ p.a. compounded monthly, so the interest is $\$100$ in the first month. (a) Find the balance owing after three months if the payment is exactly $\$100$ per month. (b) Find the balance owing after three months if the payment is $\$150$ per month. (c) Find the balance owing after three months if the payment is only $\$50$ per month, and describe what is happening to the debt.

**Q20.** An interest-only loan of $\$100\,000$ is charged $6\%$ p.a. compounded monthly. (a) Find the monthly payment. (b) On a CAS finance solver a student enters $N = 12$, $I\% = 6$, $PV = 100\,000$, $PMT = -500$ and $P/Y = C/Y = 12$. State the future value $FV$ the solver returns, and explain what it means. (c) Explain why an interest-only borrower still owes the full $\$100\,000$ after one year.

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## Answers

**Q1.** (a) $2\%$ per quarter, interest $\$600.00$; (b) $V_{n+1}=1.02V_n-600$, $V_0=30000$; (c) $V_1=V_2=\$30000.00$; (d) $\$2400.00$. **Q2.** (a) $0.5\%$ per period, payment $\$250$; (b) interest $=0.005\times50000=\$250$, equal to the payment; (c) $V_1=V_2=\$50000.00$; (d) each payment removes exactly the interest just added, so nothing comes off the principal. **Q3.** (a) every row: interest $\$60.00$, payment $\$60.00$, balance $\$12000.00$; (b) $\$12000.00$; (c) $\$240.00$; (d) the full $\$12000$ borrowed. **Q4.** (a) $0.0075\times8350=\$62.625$; (b) $\$62.63$; (c) $V_{n+1}=1.0075V_n-62.63$, $V_0=8350$; (d) $\$8350.00$. **Q5.** (a) payment $\$100.00$, total interest $\$300.00$; (b) $\$7272.91$; (c) interest-only clears $\$0$; the $\$1000$ repayments clear $10000-7272.91=\$2727.09$. **Q6.** (a) $\$70.00$; (b) $\$14000.00$; (c) $\$13270.00$ then $\$12536.35$. **Q7.** $V_{n+1}=1.005V_n-225$, $V_0=45000$; balance $\$45000.00$; total interest $\$2700.00$. **Q8.** $\$18000$. **Q9.** Payment $\$1125.00$; total interest $\$40500.00$; total cost $\$290500.00$. **Q10.** (a) exact interest $\$100.875$, payment $\$100.88$; (b) $\$13450.00$; (c) $\$605.28$. **Q11.** Interest-only: balance $\$8000.00$, interest $\$320.00$. At $\$700$/month: balance $\$5482.55$, interest $\$282.55$. **Q12.** (a) $\$900.00$; (b) $\$10800.00$; (c) $\$180000.00$; (d) interest $\$900.00$, principal reduction $\$300.00$, balance $\$179700.00$. **Q13.** In both, $D$ is chosen to equal the interest $\frac{r}{100}V_0$, so $V_{n+1}=RV_0-(R-1)V_0=V_0$ and the balance holds steady. For a loan $V_0$ is the amount borrowed and $D$ is the interest-only repayment; for a perpetuity $V_0$ is the amount invested and $D$ is the fixed income withdrawn. **Q14.** (a) $\$1200.00$; (b) $\$24000.00$; (c) $\$84000.00$. **Q15.** On the reducing-balance loan the debt falls each period, so the interest (charged on the balance) falls too; the interest-only loan is charged interest on the full principal every period, so it accumulates more interest over the same term. **Q16.** (a) $\$12000$; (b) $\$2016.00$; (c) $\$14016.00$. **Q17.** (a) payment $\$165.00$, balance $\$22000.00$; (b) balances $\$21265.00$, $\$20524.49$, $\$19778.42$ (see table in solutions). **Q18.** (a) A $\$500.00$, B $\$400.00$; (b) neither — both debts are still $\$100000$ after a year, because interest-only payments never reduce the principal. **Q19.** (a) $\$20000.00$; (b) $\$19849.25$; (c) $\$20150.75$ — the $\$50$ payment is less than the $\$100$ interest, so the unpaid interest is added on and the debt grows. **Q20.** (a) $\$500.00$; (b) $FV=-\$100000.00$; you still owe the full $\$100000$ at the end of the year. (c) every payment covers only that month's interest, so nothing is taken off the principal and the whole $\$100000$ remains owing.

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## Fully-worked solutions

**Q1.** $r=\frac{8}{4}=2\%$ per quarter, so the interest each quarter is $0.02\times30000=\$600.00$; this is the interest-only payment. The recurrence is $V_{n+1}=1.02V_n-600$ with $V_0=30000$. Quarter 1: $V_1=30000+600-600=30000$; quarter 2 is identical, so $V_1=V_2=\$30000.00$. The balance never changes, so each quarter's interest is $\$600$; over the four quarters of the first year the total interest is $4\times600=\$2400.00$.

**Q2.** The factor $1.005$ is an interest rate of $0.5\%$ per period, and the subtracted constant is the payment $\$250$. (b) The interest in the first period is $0.005\times50000=\$250$, exactly equal to the payment. (c) Since the payment removes exactly the interest, $V_1=50000+250-250=50000$ and likewise $V_2=\$50000.00$. (d) Each period the payment pays off only the interest that was just added, so no part of it reduces the principal; the balance therefore stays at $\$50000$ forever.

**Q3.** $r=\frac{6}{12}=0.5\%$ per month, so the interest each month is $0.005\times12000=\$60.00$, which is also the payment.

| Month | Interest | Payment | Balance |
|---|---|---|---|
| 0 | — | — | $\$12000.00$ |
| 1 | $\$60.00$ | $\$60.00$ | $\$12000.00$ |
| 2 | $\$60.00$ | $\$60.00$ | $\$12000.00$ |
| 3 | $\$60.00$ | $\$60.00$ | $\$12000.00$ |
| 4 | $\$60.00$ | $\$60.00$ | $\$12000.00$ |

(b) The balance owing after four months is $\$12\,000.00$. (c) The total interest is $4 \times 60 = \$240.00$. (d) At the end of the interest-only term the full $\$12\,000$ borrowed is still owed and must be repaid.

**Q4.** (a) $r = \frac{9}{12} = 0.75\,\%$ per month, so the exact monthly interest is $0.0075 \times 8350 = \$62.625$. (b) Rounded to the nearest cent the payment is $\$62.63$. (c) With $R = 1.0075$ and the payment $\$62.63$, the recurrence is $V_{n+1} = 1.0075\,V_n - 62.63$ with $V_0 = 8350$. (d) The payment equals the rounded interest each month, so the balance is unchanged: $V_1 = V_2 = \$8\,350.00$.

**Q5.** $r = \frac{12}{12} = 1\,\%$ per month, $R = 1.01$. (a) Interest-only, the payment is $0.01 \times 10\,000 = \$100.00$ per month; the balance stays at $\$10\,000$, so over three months the interest is $3 \times 100 = \$300.00$. (b) Repaying $\$1\,000$ per month, $V_{n+1} = 1.01\,V_n - 1000$: $V_1 = 10\,000 + 100 - 1000 = 9100.00$; $V_2 = 9100 + 91 - 1000 = 8191.00$; $V_3 = 8191 + 81.91 - 1000 = 7272.91$. So $\$7\,272.91$ is owed after three months. (c) Interest-only has cleared $\$0$ of the debt; the $\$1\,000$ repayments have cleared $10\,000 - 7272.91 = \$2\,727.09$.

**Q6.** $r = \frac{6}{12} = 0.5\,\%$ per month, $R = 1.005$. (a) Interest-only payment $= 0.005 \times 14\,000 = \$70.00$. (b) For the first two months the balance is unchanged at $\$14\,000.00$. (c) From the third month $\$800$ is repaid. Month 3: interest $= 0.005 \times 14\,000 = 70$, principal reduction $= 800 - 70 = 730$, balance $= 14\,000 - 730 = 13\,270.00$. Month 4: interest $= 0.005 \times 13\,270 = 66.35$, principal reduction $= 800 - 66.35 = 733.65$, balance $= 13\,270 - 733.65 = 12\,536.35$. So the balances are $\$13\,270.00$ and $\$12\,536.35$.

**Q7.** $r = \frac{6}{12} = 0.5\,\%$ per month, so the interest-only payment is $0.005 \times 45\,000 = \$225$ and the recurrence is $V_{n+1} = 1.005\,V_n - 225$ with $V_0 = 45\,000$. Each payment equals the interest, so the balance stays at $\$45\,000.00$ — in particular after three months. Over the twelve months of the first year the interest is $12 \times 225 = \$2\,700.00$.

**Q8.** The payment equals the interest on the amount borrowed: $0.006 \times V_0 = 108$, so $V_0 = \frac{108}{0.006} = \$18\,000$.

**Q9.** $r = \frac{5.4}{12} = 0.45\,\%$ per month, so the monthly payment is $0.0045 \times 250\,000 = \$1\,125.00$. Three years is $3 \times 12 = 36$ months, so the total interest is $36 \times 1125 = \$40\,500.00$. The $\$250\,000$ principal is still owed at the end, so the total cost is $40\,500 + 250\,000 = \$290\,500.00$.

**Q10.** $r = \frac{9}{12} = 0.75\,\%$ per month. (a) The exact monthly interest is $0.0075 \times 13\,450 = \$100.875$, so the payment is $\$100.88$ to the nearest cent. (b) The payment equals the rounded interest each month, so the balance is unchanged at $\$13\,450.00$ after six months. (c) The total interest is $6 \times 100.88 = \$605.28$.

**Q11.** $r = \frac{12}{12} = 1\,\%$ per month, $R = 1.01$, $V_0 = 8000$. Interest-only: the payment is $0.01 \times 8000 = \$80$ per month, the balance stays at $\$8\,000.00$, and the interest over four months is $4 \times 80 = \$320.00$. At $\$700$ per month, $V_{n+1} = 1.01\,V_n - 700$: $V_1 = 8000 + 80 - 700 = 7380.00$; $V_2 = 7380 + 73.80 - 700 = 6753.80$; $V_3 = 6753.80 + 67.54 - 700 = 6121.34$ (interest $67.538 \to 67.54$); $V_4 = 6121.34 + 61.21 - 700 = 5482.55$ (interest $61.2134 \to 61.21$). So $\$5\,482.55$ is owed, and the interest is $80 + 73.80 + 67.54 + 61.21 = \$282.55$. The reducing-balance loan pays less interest and, unlike the interest-only loan, has cut the debt.

**Q12.** $r = \frac{6}{12} = 0.5\,\%$ per month, $R = 1.005$. (a) Interest-only payment $= 0.005 \times 180\,000 = \$900.00$. (b) The balance stays at $\$180\,000$, so twelve months of interest is $12 \times 900 = \$10\,800.00$. (c) Nothing has been repaid, so the balance owing when the interest-only year ends is $\$180\,000.00$. (d) With a $\$1\,200$ payment: interest $= 0.005 \times 180\,000 = 900$, principal reduction $= 1200 - 900 = 300$, and the new balance is $180\,000 - 300 = \$179\,700.00$. The debt has at last begun to fall.

**Q13.** Both are the recurrence $V_{n+1}=RV_n-D$ with $D$ chosen to equal the interest, $D=\frac{r}{100}V_0=(R-1)V_0$. Substituting the starting value, $V_{n+1}=RV_0-(R-1)V_0=V_0$, so the balance never changes — the boundary between a falling balance ($D$ larger) and a rising one ($D$ smaller). For an **interest-only loan**, $V_0$ is the amount borrowed and $D$ is the interest-only repayment; for a **perpetuity**, $V_0$ is the amount invested and $D$ is the fixed income withdrawn each period. The arithmetic is identical; only the story differs.

**Q14.** $r=\frac{8}{4}=2\%$ per quarter. (a) The quarterly payment is $0.02\times 60\,000=\$1\,200.00$. (b) Five years is $5\times 4=20$ quarters, so the total interest is $20\times 1200=\$24\,000.00$. (c) The $\$60\,000$ principal is repaid at the end, so the total cost is $24\,000+60\,000=\$84\,000.00$.

**Q15.** On a reducing-balance loan each repayment is larger than the interest, so part of it reduces the principal; the balance falls period by period, and because the interest is charged on the balance, the interest charged falls too. On an interest-only loan the balance stays at the full principal, so every period is charged interest on the whole amount borrowed — the largest possible interest. Over the same number of periods the interest-only loan therefore runs up more total interest, even though each of its payments (interest only) is smaller than a reducing-balance repayment.

**Q16.** (a) The payment equals the interest on the amount borrowed: $0.007\times V_0=84$, so $V_0=\frac{84}{0.007}=\$12\,000$. (b) Two years is $24$ months, so the total interest is $24\times 84=\$2\,016.00$. (c) The $\$12\,000$ principal is repaid at the end, so the total cost is $2016+12\,000=\$14\,016.00$.

**Q17.** $r=\frac{9}{12}=0.75\%$ per month, $R=1.0075$. (a) Interest-only payment $=0.0075\times 22\,000=\$165.00$; the balance is unchanged at $\$22\,000.00$ after three months. (b) From the fourth month $\$900$ is repaid; each payment splits into interest and principal reduction.

| Month | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 3 | — | — | — | $\$22\,000.00$ |
| 4 | $\$900.00$ | $\$165.00$ | $\$735.00$ | $\$21\,265.00$ |
| 5 | $\$900.00$ | $\$159.49$ | $\$740.51$ | $\$20\,524.49$ |
| 6 | $\$900.00$ | $\$153.93$ | $\$746.07$ | $\$19\,778.42$ |

Month 4: interest $0.0075\times 22\,000=165$, reduction $900-165=735$, balance $22\,000-735=21\,265.00$. Month 5: interest $0.0075\times 21\,265=159.4875\rightarrow 159.49$, reduction 740.51, balance 20 524.49. Month 6: interest $0.0075\times 20\,524.49=153.93\ldots\rightarrow 153.93$, reduction 746.07, balance 19 778.42.

**Q18.** (a) Investor A: $0.005\times 100\,000=\$500.00$ per month; investor B: $0.004\times 100\,000=\$400.00$ per month. (b) Neither debt is smaller: on an interest-only loan every payment is exactly the interest, so no part of it reduces the principal. After one year both investors still owe the full $\$100\,000$. (B simply pays less interest each month because of the lower rate.)

**Q19.** $r=\frac{6}{12}=0.5\%$ per month, so the interest in the first month is $0.005\times 20\,000=\$100$. (a) Paying exactly $\$100$ (interest-only) leaves the balance unchanged: $\$20\,000.00$ after three months. (b) Paying $\$150$ ($V_{n+1}=1.005V_n-150$): $V_1=20\,000+100-150=19\,950.00$; $V_2=19\,950+99.75-150=19\,899.75$; $V_3=19\,899.75+99.50-150=19\,849.25$ (interest 99.49875 → 99.50). So $\$19\,849.25$ is owed — paying more than the interest makes the debt fall. (c) Paying only $\$50$ ($V_{n+1}=1.005V_n-50$): $V_1=20\,000+100-50=20\,050.00$; $V_2=20\,050+100.25-50=20\,100.25$; $V_3=20\,100.25+100.50-50=20\,150.75$. The $\$50$ payment is less than the $\$100$ interest, so the unpaid interest is added to the debt and the balance **grows**, reaching $\$20\,150.75$.

**Q20.** $r=\frac{6}{12}=0.5\,\%$ per month. (a) The monthly payment is $0.005\times 100\,000=\$\,500.00$. (b) The solver returns $F\,V=-\,\$\,100\,000.00$. The sign convention makes money you owe negative, so after twelve $\$\,500$ payments you still owe $\$\,100\,000$ — the future value confirms that an interest-only loan leaves the balance unchanged. (c) Each $\$\,500$ payment covers exactly that month's interest ($0.005\times 100\,000=500$), so no part of any payment goes towards the principal; the whole $\$\,100\,000$ borrowed is still owed at the end of the year.

### Theory

An earlier section modelled an **annuity** — a lump sum invested and then run down by a fixed regular withdrawal — with the recurrence

$$V_{n+1} = R V_n - D, V_0 = \text{amount invested},$$

where $R = 1 + \frac{r}{100}$ is the growth factor for one period ($r$% is the interest rate per period) and $D$ is the withdrawal each period. In an ordinary annuity the withdrawal is **larger** than the interest earned, so a little of the capital is drawn down every period and the fund eventually empties. This section looks at the special case in which the fund never empties — a **perpetuity**.

**What a perpetuity is.** A **perpetuity** is an annuity that lasts **indefinitely**: the same payment is made every period, forever, and the capital in the fund is never used up. This is achieved by withdrawing **exactly** the interest earned each period and no more. If the fund earns interest of $\frac{r}{100} V_0$ in a period and precisely that amount is withdrawn, the balance is returned to $V_0$, and the following period earns the same interest again. The withdrawal that keeps the balance constant is therefore

$$D = \frac{r}{100} \times V_0,$$

the interest on the starting capital.

**Why the balance stays constant.** Substitute $D = \frac{r}{100} V_0$ into the annuity recurrence. With $R = 1 + \frac{r}{100}$,

$$V_{n+1} = R V_0 - D = \left(1 + \frac{r}{100}\right) V_0 - \frac{r}{100} V_0 = V_0 + \frac{r}{100} V_0 - \frac{r}{100} V_0 = V_0.$$

So $V_1 = V_0$, and because the balance is unchanged the next period repeats the same arithmetic: $V_2 = V_0$, $V_3 = V_0$, and so on. Every term of the sequence equals the starting value,

$$V_{n+1} = V_n = V_0 \text{ for all } n,$$

a **constant** sequence. The recurrence still has the annuity form $V_{n+1} = R V_n - D$; it is the special choice $D = \frac{r}{100} V_0$ that flattens it.

**The perpetuity relation.** The single equation

$$\text{payment} = \frac{r}{100} \times \text{principal}$$

ties together the three quantities of a perpetuity. Rearranging it answers the two questions that arise in practice:

- the **principal** needed to fund a given perpetual payment is
$$\text{principal} = \frac{\text{payment} \times 100}{r};$$
- the **payment** a given principal can sustain forever is $\text{payment} = \frac{r}{100} \times \text{principal}$; and the **rate** required is $r = \frac{\text{payment} \times 100}{\text{principal}}$.

Here $r\%$ is always the rate **per period** and the payment is made **each period**: an annual payment uses the annual rate, a monthly payment uses the monthly rate $r = \frac{\text{annual rate}}{12}$, and so on.

**Rounding convention.** As throughout this chapter, the interest is rounded to the nearest cent each period before it is used. In every perpetuity here the interest works out to an exact number of cents, so the withdrawal removes precisely that interest and the balance returns exactly to $V_0$. Set out in an amortisation table, a perpetuity has a **principal-reduction column that is $\$0.00$ in every row** — nothing is ever taken off the capital — and a **balance column that repeats $V_0$ forever**:

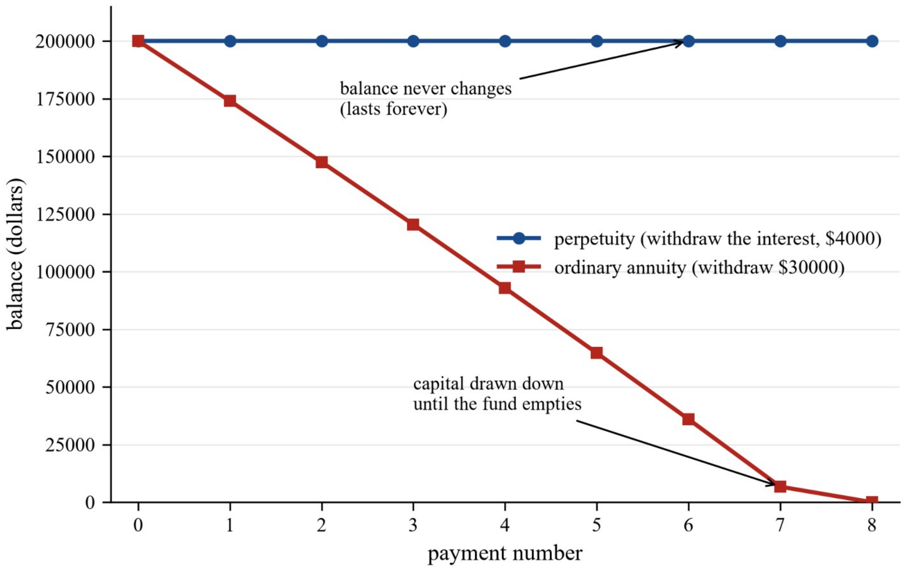
$$\text{principal reduction} = \text{payment} - \text{interest} = 0\,, \text{balance} = V_0 \,\text{each period}\,.$$

**If the withdrawal is not exactly the interest.** The balance stays constant *only* when the withdrawal equals the interest. If a fund pays out **more** than the interest, the extra comes out of the capital: the balance falls a little each period and the perpetuity becomes an ordinary annuity that will one day be exhausted. If it pays out **less** than the interest, the unspent interest is left to compound and the balance slowly grows. Only the exact balance $D = \frac{r}{100}V_0$ keeps a payment going forever without touching the capital.

**Contexts.** Perpetuities model **endowments**, **scholarships**, **bursaries** and **prizes** — any arrangement where a sum is set aside so that a fixed payment can be made from its interest “in perpetuity”. A benefactor who wants a scholarship of a certain size paid every year forever must invest the principal given by the relation above; a fund of a known size can support a prize equal to the interest it earns.

**Using technology.** A CAS **finance solver** (TVM) handles a perpetuity through its usual variables $N$, $I\%$, $PV$, $PMT$, $FV$ and $P/Y = C/Y$. Because the balance never changes, the future value has the **same size as the amount invested**: entering $PV = -V_0$ and $PMT = D = \frac{r}{100}V_0$ returns an $FV$ equal to the $V_0$ you started with, whatever the number of periods $N$. The solver is convenient for finding the principal or the payment directly, but the one-line perpetuity relation is usually quicker; keep the recurrence in mind so you can see *why* the capital is preserved.

The graph below compares two funds that each start with the same capital and earn the same rate. The **perpetuity** withdraws exactly the interest each period, so its balance stays level forever (the flat line). The **ordinary annuity** withdraws more than the interest, so its balance steps down each period until the fund is exhausted.



### Worked examples

#### *Example 1 — scaffolded*

A retiree invests $\$250\,000$ in a perpetuity paying $4.8\%$ p.a. compounded monthly, and withdraws the interest at the end of each month so that the capital is preserved. (a) Find the monthly interest rate. (b) Find the monthly payment that keeps the balance constant. (c) Write the recurrence relation, and show the balance is unchanged over the first three months.

| Working | Comment |
|---|---|
| (a) $r=\frac{4.8}{12}=0.4\%$ per month, so $R=1.004$. | Monthly rate = annual rate $\div 12$; growth factor $R=1+\frac{r}{100}$. |
| (b) $D=\frac{0.4}{100}\times 250\,000=0.004\times 250\,000=\$1\,000.00$. | The perpetual payment is the interest on the capital, $\frac{r}{100}\times V_0$. |
| (c) $V_{n+1}=1.004V_n-1000, V_0=250\,000$. | The annuity recurrence with the withdrawal set equal to the interest. |

Each month the interest is $0.004\times 250\,000=\$1\,000.00$ and exactly $\$1\,000.00$ is withdrawn, so nothing is taken off the capital and the balance returns to $\$250\,000.00$.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$250\,000.00$ |
| 1 | $\$1\,000.00$ | $\$1\,000.00$ | $\$0.00$ | $\$250\,000.00$ |
| 2 | $\$1\,000.00$ | $\$1\,000.00$ | $\$0.00$ | $\$250\,000.00$ |
| 3 | $\$1\,000.00$ | $\$1\,000.00$ | $\$0.00$ | $\$250\,000.00$ |

The balance is $\$250\,000.00$ at the end of every month — the perpetuity can pay $\$1\,000$ a month forever.

#### *Example 2 — scaffolded*

A donor wants to establish a scholarship that pays $\$6\,000$ at the end of each year forever. The money will be invested in a fund earning $5\%$ p.a. Find the amount that must be invested, and confirm that the interest in the first year equals the scholarship payment.

| Working | Comment |
|---|---|
| Perpetuity relation: $\text{payment}=\frac{r}{100}\times\text{principal}$. | The payment is the interest on the capital. |
| Rearrange for the principal: $\text{principal}=\frac{\text{payment}\times 100}{r}=\frac{6000\times 100}{5}$. | Solve the relation for the unknown principal. |
| $\text{principal}=\frac{600\,000}{5}=\$120\,000.00$. | The amount that must be invested. |
| Check: interest in year 1 $=0.05\times 120\,000=\$6\,000.00$. | Equal to the scholarship, so the capital is untouched. |

Investing $\$120\,000$ at $5\%$ p.a. earns $\$6\,000$ interest in the first year; paying that out as the scholarship leaves the $\$120\,000$ intact to earn another $\$6\,000$ the next year, and so on forever.

#### *Example 3 — unscaffolded*

An endowment fund of $\$480\,000$ earns $6\%$ p.a. compounded quarterly and pays a fixed grant at the end of each quarter, leaving the capital intact. (a) Find the quarterly grant. (b) The trustees of a second fund want to pay a grant of $\$9\,000$ per quarter forever at the same rate. Find the capital that second fund needs.

The quarterly interest rate is $r = \frac{6}{4} = 1.5\,\%$.

(a) The perpetual grant is the interest earned each quarter on the capital:

$$D = \frac{1.5}{100} \times 480\,000 = 0.015 \times 480\,000 = \$\,7\,200.00.$$

Withdrawing $\$\,7\,200$ each quarter removes exactly the interest, so the $\$\,480\,000$ is preserved and the grant can be paid indefinitely.

(b) Rearranging the perpetuity relation for the principal,

$$\text{principal} = \frac{\text{payment} \times 100}{r} = \frac{9000 \times 100}{1.5} = \frac{900\,000}{1.5} = \$\,600\,000.00.$$

∴ the first fund pays a quarterly grant of $\$\,7\,200$, and a perpetual grant of $\$\,9\,000$ per quarter at $1.5\,\%$ per quarter requires capital of $\$\,600\,000$.

### *Example 4 — unscaffolded*

A benefactor leaves $\$\,180\,000$ to fund an annual prize, invested at $5\,\%$ p.a. paid annually. (a) Find the largest annual prize that can be paid forever. (b) Explain what would happen to the fund if a prize of $\$\,10\,000$ were paid each year instead, and find the balance after the first two years.

(a) The largest prize that leaves the capital intact is the interest earned each year:

$$D = \frac{5}{100} \times 180\,000 = 0.05 \times 180\,000 = \$\,9\,000.00.$$

A prize of $\$\,9\,000$ exactly matches the interest, so the $\$\,180\,000$ is never drawn down and the prize is paid in perpetuity.

(b) A prize of $\$\,10\,000$ is more than the $\$\,9\,000$ interest, so each year $\$\,1\,000$ of the extra comes out of the capital and the balance falls. Iterating $V_{n+1} = 1.05\,V_n - 10\,000$:

$$V_1 = 180\,000 + 9000 - 10\,000 = 179\,000,$$

$$V_2 = 179\,000 + (0.05 \times 179\,000) - 10\,000 = 179\,000 + 8950 - 10\,000 = 177\,950.$$

The balance drops to $\$\,179\,000$ after one year and $\$\,177\,950$ after two, and it keeps falling. This is no longer a perpetuity but an ordinary annuity, which will eventually be exhausted.

∴ the fund can pay a perpetual prize of $\$\,9\,000$ a year; paying $\$\,10\,000$ instead draws down the capital ($\$\,179\,000$ after one year, $\$\,177\,950$ after two) and the fund will not last forever.

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## Practice questions

### *Scaffolded*

**Q1.** A perpetuity of $\$\,400\,000$ earns $3\,\%$ per year, paid annually, with the interest withdrawn each year so the capital is preserved. (a) Find the interest earned in the first year. (b) State the withdrawal each year that keeps the balance constant. (c) Write the recurrence relation for the balance. (d) State the balance after the first withdrawal and after the second.

**Q2.** A scholarship of $\$\,8\,000$ is to be paid at the end of each year forever from a fund earning $4\,\%$ p.a. (a) Write the perpetuity relation connecting the payment, the rate and the principal. (b) Find the principal that must be invested. (c) Verify that the interest earned in the first year equals $\$\,8\,000$.

**Q3.** $\$150\,000$ is invested at $6\%$ p.a. compounded monthly to provide a monthly income forever. (a) Find the monthly interest rate. (b) Find the monthly payment that leaves the capital unchanged. (c) Copy and complete the perpetuity table for the first three months.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$150\,000.00$ |
| 1 | | | $\$0.00$ | |
| 2 | | | $\$0.00$ | |
| 3 | | | $\$0.00$ | |

**Q4.** A council wants to award a grant of $\$3\,000$ at the end of each quarter forever from a fund earning $8\%$ p.a. compounded quarterly. (a) Find the quarterly interest rate. (b) Find the amount that must be invested. (c) Find the total paid out over the first year (four quarters), and state the balance at the end of the year.

**Q5.** The perpetuity table below is for a fund of $\$90\,000$ earning $1\%$ interest per month, from which the interest is withdrawn each month so the balance stays constant.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$90\,000.00$ |
| 1 | $\$900.00$ | $\$900.00$ | $\$0.00$ | $A$ |
| 2 | $B$ | $C$ | $\$0.00$ | $D$ |

(a) Find $A$, $B$, $C$ and $D$.
(b) State the total amount withdrawn over the two months.

**Q6.** Two perpetuities each pay $\$5\,000$ at the end of each year forever. Fund A earns $5\%$ p.a. and Fund B earns $4\%$ p.a. (a) Find the principal needed for Fund A. (b) Find the principal needed for Fund B. (c) State which fund needs more capital, and explain why.

### *Unscaffolded*

**Q7.** $\$320\,000$ is invested at $2.5\%$ per quarter to provide a perpetual quarterly income. Find the quarterly payment, and state the balance of the fund after 10 years.

**Q8.** A university endowment must pay $\$45\,000$ at the end of each year forever from a fund earning $6\%$ p.a. Find the amount that must be invested.

**Q9.** A perpetuity pays $\$1\,200$ at the end of each month forever from a fund earning $4.8\%$ p.a. compounded monthly. Find the amount invested.

**Q10.** $\$600\,000$ is placed in a perpetuity earning $5\%$ p.a. paid annually. (a) Find the annual payment. (b) The rate then rises to $6\%$ p.a. Find the new annual payment the same capital can sustain forever.

**Q11.** A benefactor donates $\$250\,000$ to fund an annual bursary forever, invested at $4\%$ p.a. (a) Find the annual bursary. (b) If the bursary were instead set at $\$12\,000$, explain what would happen to the fund, and find the balance after the first two years.

**Q12.** Find the interest rate per annum (paid annually) at which a fund of $\$180\,000$ provides a perpetual annual income of $\$10\,800$.

**Q13.** A fund of $\$500\,000$ earns $1.5\%$ per quarter and pays a perpetual quarterly grant. Construct the perpetuity table for the first three quarters, and state the total grant paid over those three quarters.

**Q14.** A perpetuity is to pay $\$600$ at the end of each month forever. Compare the capital required if the fund earns (a) $6\%$ p.a. compounded monthly with (b) $3.6\%$ p.a. compounded monthly, and (c) state which rate needs more capital and why.

**Q15.** A charity invests $\$840\,000$ at $5\%$ p.a. paid annually, planning to pay a perpetual grant equal to the interest. (a) Find the annual grant. (b) In one year the charity withdraws $\$50\,000$ instead of the perpetual amount. Find the balance at the end of that year, and state whether the fund can still pay the original grant the next year.

**Q16.** A perpetuity is modelled by the recurrence $V_{n+1}=RV_n-D$, where $R=1+\frac{r}{100}$. Using this recurrence, explain why setting $D=\frac{r}{100}V_0$ makes the balance the same in every period.

**Q17.** A perpetuity fund earns $2\%$ per quarter, and a quarterly payment of $\$4\,000$ is required forever. (a) Find the capital needed. (b) A finance solver is set with $PV=-200\,000$, $PMT=4000$, $I\%=8$ and $P/Y=C/Y=4$. Explain what value it returns for $FV$, and why.

**Q18.** A retiree has $\$480\,000$ and wants a monthly income forever from a fund earning $6\%$ p.a. compounded monthly. (a) Find the monthly income. (b) The retiree decides they need $\$2\,600$ per month instead. Find the extra capital that must be added to the fund to sustain $\$2\,600$ per month forever at the same rate.

**Q19.** A town endowment pays $\$18\,000$ at the end of each year forever and earns $4.5\%$ p.a. (a) Find the capital in the fund. (b) The trustees want to double the annual payment to $\$36\,000$, still forever, at the same rate. State the new capital required, and by what factor the capital changes.

**Q20.** A perpetuity of $\$360\,000$ earns $5\%$ p.a. paid annually. (a) Find the annual payment. (b) By iterating the recurrence for two years, show that the balance is unchanged. (c) Explain why a perpetuity can be regarded as an annuity with an unlimited number of payments, and what distinguishes it from the ordinary annuities studied earlier in this chapter.

## Answers

**Q1.** (a) $\$12\,000.00$; (b) $\$12\,000.00$; (c) $V_{n+1}=1.03V_n-12\,000,\ V_0=400\,000$; (d) $\$400\,000.00$ after each withdrawal. **Q2.** (a) $\text{payment}=\frac{r}{100}\times\text{principal}$; (b) $\$200\,000.00$; (c) $0.04\times200\,000=\$8\,000$. **Q3.** (a) $0.5\%$ per month; (b) $\$750.00$; (c) interest and payment $\$750.00$ each row, balance $\$150\,000.00$ each row. **Q4.** (a) $2\%$ per quarter; (b) $\$150\,000.00$; (c) total paid $\$12\,000.00$, balance $\$150\,000.00$. **Q5.** (a) $A=\$90\,000.00$, $B=\$900.00$, $C=\$900.00$, $D=\$90\,000.00$; (b) $\$1\,800.00$. **Q6.** (a) $\$100\,000.00$; (b) $\$125\,000.00$; (c) Fund B, because a lower interest rate earns less on each dollar, so more capital is needed to generate the same $\$5\,000$. **Q7.** $\$8\,000.00$ per quarter; balance still $\$320\,000.00$. **Q8.** $\$750\,000.00$. **Q9.** $\$300\,000.00$. **Q10.** (a) $\$30\,000.00$; (b) $\$36\,000.00$. **Q11.** (a) $\$10\,000.00$; (b) $\$12\,000$ exceeds the $\$10\,000$ interest, so the capital is drawn down: $\$248\,000.00$ after one year and $\$245\,920.00$ after two — the fund would eventually be exhausted. **Q12.** $6\%$ p.a. **Q13.** Grant $\$7\,500.00$ each quarter, balance $\$500\,000.00$ each row; total over three quarters $\$22\,500.00$. **Q14.** (a) $\$120\,000.00$; (b) $\$200\,000.00$; (c) the $3.6\%$ fund needs more capital, because a lower rate earns less per dollar. **Q15.** (a) $\$42\,000.00$; (b) balance $\$832\,000.00$; no — the interest next year is only $0.05\times832\,000=\$41\,600$, less than the $\$42\,000$ grant. **Q16.** With $D=\frac{r}{100}V_0$, $V_{n+1}=\left(1+\frac{r}{100}\right)V_0-\frac{r}{100}V_0=V_0$; the balance returns to $V_0$, so the same holds every period and the balance is constant. **Q17.** (a) $\$200\,000.00$; (b) $FV$ has the same size as the $\$200\,000$ invested — the balance never changes, so the future value equals the capital. **Q18.** (a) $\$2\,400.00$; (b) capital for $\$2\,600$ is $\$520\,000$, so an extra $\$40\,000.00$ is needed. **Q19.** (a) $\$400\,000.00$; (b) $\$800\,000.00$, a factor of 2 (capital is proportional to the payment at a fixed rate). **Q20.** (a) $\$18\,000.00$; (b) $V_1=360\,000+18\,000-18\,000=360\,000$, $V_2=360\,000$; (c) the payment continues without limit because only the interest is withdrawn, so unlike an ordinary annuity the capital is never drawn down and the fund never empties.

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## Fully-worked solutions

**Q1.** (a) The interest in the first year is $\frac{3}{100}\times400\,000=0.03\times400\,000=\$12\,000.00$. (b) To keep the balance constant the withdrawal equals that interest, $\$12\,000.00$. (c) The recurrence is the annuity form with $D=12\,000$: $V_{n+1}=1.03V_n-12\,000$, $V_0=400\,000$. (d) Each year the interest ($\$12\,000$) is exactly withdrawn, so $V_1=400\,000+12\,000-12\,000=\$400\,000.00$ and likewise $V_2=\$400\,000.00$ — the balance is unchanged.

**Q2.** (a) The perpetual payment is the interest on the capital, so $\text{payment}=\frac{r}{100}\times\text{principal}$. (b) Rearranging for the principal, $\text{principal}=\frac{\text{payment}\times100}{r}=\frac{8000\times100}{4}=\frac{800\,000}{4}=\$200\,000.00$. (c) The interest earned in the first year is $0.04\times200\,000=\$8\,000.00$, which is exactly the scholarship, so the $\$200\,000$ is left intact to repeat every year.

**Q3.** (a) The monthly rate is $r=\frac{6}{12}=0.5\%$. (b) The payment that preserves the capital is $D=0.005\times150\,000=\$750.00$. (c) Each month the interest is $0.005\times150\,000=\$750.00$ and $\$750.00$ is withdrawn, so the principal reduction is $\$0.00$ and the balance returns to $\$150\,000.00$:

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$150\,000.00$ |
| 1 | $\$750.00$ | $\$750.00$ | $\$0.00$ | $\$150\,000.00$ |
| 2 | $\$750.00$ | $\$750.00$ | $\$0.00$ | $\$150\,000.00$ |
| 3 | $\$750.00$ | $\$750.00$ | $\$0.00$ | $\$150\,000.00$ |

**Q4.** (a) The quarterly rate is $r=\frac{8}{4}=2\%$. (b) Rearranging the perpetuity relation, $\text{principal}=\frac{3000\times100}{2}=\frac{300\,000}{2}=\$150\,000.00$. (c) The grant is the interest, $0.02\times150\,000=\$3\,000$ each quarter; over four quarters the total paid is $4\times3000=\$12\,000.00$, and since only the interest is withdrawn the balance at the end of the year is still $\$150\,000.00$.

**Q5.** Each month the interest is $0.01\times90\,000=\$900.00$ and exactly $\$900.00$ is withdrawn, so the balance returns to $\$90\,000.00$ every month. Reading the table, $A=\$90\,000.00$ (balance after month 1); month 2 repeats month 1, so the payment $B=\$900.00$, the interest $C=\$900.00$ and the balance $D=\$90\,000.00$. (b) The total withdrawn over the two months is $900+900=\$1\,800.00$.

**Q6.** Using $\text{principal}=\frac{\text{payment}\times100}{r}$ with a payment of $\$5\,000$: (a) Fund A at $5\%$: $\frac{5000\times100}{5}=\frac{500\,000}{5}=\$100\,000.00$. (b) Fund B at $4\%$: $\frac{5000\times100}{4}=\frac{500\,000}{4}=\$125\,000.00$. (c) Fund B needs more capital ($\$125\,000$ against $\$100\,000$). A lower interest rate earns less on each dollar invested, so a larger principal is required to generate the same $\$5\,000$ of interest each year.

**Q7.** The perpetual payment is the interest each quarter, $D=\frac{2.5}{100}\times320\,000=0.025\times320\,000=\$8\,000.00$. Because exactly the interest is withdrawn, the capital is never touched, so after 10 years (or any length of time) the balance is still $\$320\,000.00$.

**Q8.** Rearranging the perpetuity relation for the principal, $\text{principal}=\frac{45\,000\times100}{6}=\frac{4\,500\,000}{6}=\$750\,000.00$. (Check: $0.06\times750\,000=\$45\,000$, the annual payment.)

**Q9.** The monthly rate is $r=\frac{4.8}{12}=0.4\%$. The capital is $\text{principal}=\frac{1200\times100}{0.4}=\frac{120\,000}{0.4}=\$300\,000.00$. (Check: $0.004\times300\,000=\$1\,200$, the monthly payment.)

**Q10.** (a) At $5\%$ the payment is $0.05\times600\,000=\$30\,000.00$ per year. (b) At the higher rate the same $\$600\,000$ earns $0.06\times600\,000=\$36\,000.00$ per year, so it can pay a perpetual payment of $\$36\,000$.

**Q11.** (a) The perpetual bursary is the interest, $0.04\times250\,000=\$10\,000.00$. (b) A bursary of $\$12\,000$ is $\$2\,000$ more than the $\$10\,000$ interest, so $\$2\,000$ of capital is drawn down each year. Iterating $V_{n+1}=1.04\,V_n-12\,000$:
$V_1=250\,000+10\,000-12\,000=\$248\,000.00$;
$V_2=248\,000+(0.04\times248\,000)-12\,000=248\,000+9\,920-12\,000=\$245\,920.00$. The balance keeps falling, so the fund is no longer a perpetuity and would eventually be exhausted.

**Q12.** Rearranging the relation for the rate, $r=\frac{\text{payment}\times100}{\text{principal}}=\frac{10\,800\times100}{180\,000}=\frac{1\,080\,000}{180\,000}=6$, so the rate is $6\%$ p.a. (Check: $0.06\times180\,000=\$10\,800$.)

**Q13.** The quarterly grant is the interest, $0.015\times500\,000=\$7\,500.00$; withdrawing exactly that leaves the balance at $\$500\,000.00$ each quarter.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$500\,000.00$ |
| 1 | $\$7\,500.00$ | $\$7\,500.00$ | $\$0.00$ | $\$500\,000.00$ |
| 2 | $\$7\,500.00$ | $\$7\,500.00$ | $\$0.00$ | $\$500\,000.00$ |
| 3 | $\$7\,500.00$ | $\$7\,500.00$ | $\$0.00$ | $\$500\,000.00$ |

The total grant paid over the three quarters is $3 \times 7\,500 = \$\,22\,500.00$.

**Q14.** The payment is $\$\,600$ per month. (a) At $6\,\%$ p.a. the monthly rate is $\frac{6}{12} = 0.5\,\%$, so the capital is $\frac{600 \times 100}{0.5} = \frac{60\,000}{0.5} = \$\,120\,000.00$. (b) At $3.6\,\%$ p.a. the monthly rate is $\frac{3.6}{12} = 0.3\,\%$, so the capital is $\frac{600 \times 100}{0.3} = \frac{60\,000}{0.3} = \$\,200\,000.00$. (c) The $3.6\,\%$ fund needs more capital ($\$\,200\,000$ against $\$\,120\,000$), because a lower rate earns less interest on each dollar, so a larger principal is required to produce the same $\$\,600$ a month.

**Q15.** (a) The perpetual grant is the interest, $0.05 \times 840\,000 = \$\,42\,000.00$. (b) Withdrawing $\$\,50\,000$ that year: the interest is $\$\,42\,000$, so $V = 840\,000 + 42\,000 - 50\,000 = \$\,832\,000.00$. The $\$\,8\,000$ overspend has come out of the capital, leaving $\$\,832\,000$. The interest the next year would be only $0.05 \times 832\,000 = \$\,41\,600$, which is less than the $\$\,42\,000$ grant, so the fund can no longer sustain the original perpetual grant.

**Q16.** A perpetuity withdraws exactly the interest on the starting capital, $D = \frac{r}{100} V_0$. Substituting into the recurrence with $R = 1 + \frac{r}{100}$,

$$V_{n+1} = \left(1 + \frac{r}{100}\right) V_0 - \frac{r}{100} V_0 = V_0 + \frac{r}{100} V_0 - \frac{r}{100} V_0 = V_0.$$

So after one period the balance is back to $V_0$. Because the balance is unchanged, the next period earns the same interest and the same argument gives $V_0$ again, and so on — the balance equals $V_0$ in every period, a constant sequence.

**Q17.** (a) With a quarterly rate of $2\,\%$ and a payment of $\$\,4\,000$, $\text{principal} = \frac{4000 \times 100}{2} = \frac{400\,000}{2} = \$\,200\,000.00$. (b) With $PV = -200\,000$ (the capital invested) and $PMT = 4000$ (the interest, withdrawn each quarter), the balance never changes, so the solver returns a future value of the same size as the $\$\,200\,000$ invested — the capital is intact whatever the number of periods $N$.

**Q18.** (a) The monthly rate is $\frac{6}{12} = 0.5\,\%$, so the monthly income is $0.005 \times 480\,000 = \$\,2\,400.00$. (b) To pay $\$\,2\,600$ per month forever at $0.5\,\%$ per month the capital required is $\frac{2600 \times 100}{0.5} = \frac{260\,000}{0.5} = \$\,520\,000.00$. Since the retiree already has $\$\,480\,000$, the extra capital needed is $520\,000 - 480\,000 = \$\,40\,000.00$.

**Q19.** (a) The capital is $\text{principal} = \frac{18\,000 \times 100}{4.5} = \frac{1\,800\,000}{4.5} = \$\,400\,000.00$. (b) To pay $\$\,36\,000$ forever at $4.5\,\%$ the capital is $\frac{36\,000 \times 100}{4.5} = \frac{3\,600\,000}{4.5} = \$\,800\,000.00$. Doubling the payment doubles the capital (a factor of 2): at a fixed rate the required capital is proportional to the payment.

**Q20.** (a) The annual payment is $0.05 \times 360\,000 = \$\,18\,000.00$. (b) Iterating $V_{n+1} = 1.05\,V_n - 18\,000$: $V_1 = 360\,000 + 18\,000 - 18\,000 = \$\,360\,000.00$ and $V_2 = 360\,000 + 18\,000 - 18\,000 = \$\,360\,000.00$, so the balance is unchanged. (c) Each period only the interest is withdrawn, so the capital is never reduced and the same payment can be made without limit — an annuity with an unlimited number of payments. This is what distinguishes it from the ordinary annuities studied earlier: those withdraw more than the interest, so their capital is drawn down each period and the fund is eventually exhausted, whereas a perpetuity lasts forever.

## Topic 2 Test A — Multiple-choice

*One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 3 minutes. Writing time: 20 minutes. 12 multiple-choice questions, 12 marks. Choose the response that is **correct** for the question.*

**Question 1**

A sequence is defined by the recurrence relation $V_0 = 5$, $V_{n+1} = 3V_n - 4$. The value of $V_2$ is

**A.** 29

**B.** 11

**C.** 27

**D.** 83

**Question 2**

$\$6\,500$ is invested at $4\%$ per annum simple interest, with the interest added at the end of each year. A recurrence relation for the value $V_n$ after $n$ years is

**A.** $V_{n+1} = 1.04V_n, V_0 = 6\,500$

**B.** $V_{n+1} = V_n + 260, V_0 = 6\,500$

**C.** $V_{n+1} = 1.04V_n + 260, V_0 = 6\,500$

**D.** $V_{n+1} = V_n + 6\,500, V_0 = 260$

**Question 3**

A courier motorcycle bought for $\$13\,500$ is depreciated by the unit-cost method at 18 cents for each kilometre ridden, and it is ridden $10\,000$ km each year. Its book value after 4 years is

**A.** $\$11\,700.00$

**B.** $\$8\,100.00$

**C.** $\$6\,300.00$

**D.** $\$3\,780.00$

**Question 4**

$\$2\,000$ is invested at $8\%$ per annum compounded quarterly for 3 years. The interest rate per compounding period and the number of compounding periods are

**A.** $2\%$ per quarter, 12 periods

**B.** $8\%$ per year, 3 periods

**C.** $8\%$ per quarter, 12 periods

**D.** $0.6\overline{6}\%$ per month, 36 periods

**Question 5**

$\$5\,000$ is invested at $4\,\%$ per annum compounded annually. Its value after $3$ years, to the nearest cent, is

**A.** $\$5\,600.00$

**B.** $\$5\,612.00$

**C.** $\$5\,849.29$

**D.** $\$5\,624.32$

**Question 6**

A boat bought for $\$34\,000$ depreciates by $14\,\%$ per year by the reducing-balance method. Its value after $4$ years, to the nearest cent, is

**A.** $\$14\,960.00$

**B.** $\$18\,598.28$

**C.** $\$21\,625.90$

**D.** $\$15\,994.52$

**Question 7**

Interest of $9\,\%$ per annum is compounded quarterly. The effective annual interest rate, correct to two decimal places, is

**A.** $9.00\,\%$

**B.** $9.38\,\%$

**C.** $9.31\,\%$

**D.** $9.41\,\%$

**Question 8**

An account opens with $\$3\,000$, pays $1\,\%$ interest per month, and has $\$500$ deposited at the end of each month. The balance at the end of the second month is

**A.** $\$4\,000.00$

**B.** $\$4\,060.30$

**C.** $\$4\,095.30$

**D.** $\$4\,065.30$

**Question 9**

A loan of $\$8\,400$ is charged $1.5\,\%$ interest per month and is repaid by $\$1\,450$ at the end of each month. The balance owing after the first repayment is

**A.** $\$7\,076.00$

**B.** $\$6\,950.00$

**C.** $\$6\,824.00$

**D.** $\$7\,034.00$

**Question 10**

An annuity is entered into a CAS finance solver, which returns $N = 26.4$ for the number of monthly payments it can make. This means the fund is exhausted by

**A.** $26$ equal payments

**B.** $26$ full payments and a smaller $27$th payment

**C.** $27$ full payments of equal size

**D.** $264$ payments

**Question 11**

A loan of $\$46\,000$ is charged $9\,\%$ per annum compounded monthly and is paid interest-only. The monthly payment is

**A.** $\$4\,140.00$

**B.** $\$690.00$

**C.** $\$345.00$

**D.** $\$230.00$

**Question 12**

A perpetuity is to pay $\$10\,500$ at the end of each year forever from a fund earning $6\,\%$ per annum. The amount that must be invested is

**A.** $\$63\,000.00$

**B.** $\$105\,000.00$

**C.** $\$630\,000.00$

**D.** $\$175\,000.00$

*End of Topic 2 Test A*

***Test A — answers and solutions***

| Q | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Answer | A | B | C | A | D | B | C | D | A | B | C | D |

**Q1.** $V_1 = 3(5) - 4 = 11$, then $V_2 = 3(11) - 4 = 29$. **Q2.** Simple interest adds a fixed $\frac{4}{100} \times 6\,500 = \$260$ each year, so $V_{n+1} = V_n + 260$ with $V_0 = 6\,500$. **Q3.** Unit cost: $d = 0.18 \times 10\,000 = 1\,800$ per year, so $V_4 = 13\,500 - 4 \times 1\,800 = \$6\,300.00$. **Q4.** Quarterly means $k = 4$, so $r = \frac{8}{4} = 2\,\%$ per quarter and $n = 4 \times 3 = 12$ periods. **Q5.** $V_3 = 5\,000 \times 1.04^3 = 5\,000 \times 1.124864 = \$5\,624.32$. **Q6.** $R = 0.86$, so $V_4 = 34\,000 \times 0.86^4 = 34\,000 \times 0.54700816 = \$18\,598.28$. **Q7.** $r = \frac{9}{4} = 2.25\,\%$, so $r_{\text{eff}} = (1.0225^4 - 1) \times 100\,\% = 9.31\,\%$. **Q8.** Month 1: $1.01 \times 3\,000 + 500 = 3\,530.00$; month 2: $1.01 \times 3\,530 + 500 = 4\,065.30$. **Q9.** Interest $= 0.015 \times 8\,400 = 126$; principal reduction $= 1\,450 - 126 = 1\,324$; balance $= 8\,400 - 1\,324 = \$7\,076.00$. **Q10.** A non-whole $N$ means the fund runs out partway through a period: 26 full payments and a smaller $27$th. **Q11.** Interest-only: $D = \frac{r}{100} \times V_0 = 0.0075 \times 46\,000 = \$345.00$ (with $r = \frac{9}{12} = 0.75\,\%$ per month). **Q12.** $\text{principal} = \frac{\text{payment} \times 100}{r} = \frac{10\,500 \times 100}{6} = \$175\,000.00$.

## Topic 2 Test B — Short-answer

*One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 5 minutes. Writing time: 40 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.*

**Question 1** (6 marks)

A delivery van is bought for $\$40\,000$.

**a.** Under the flat-rate (straight-line) method it is depreciated by $9\%$ of its purchase price each year. Write a recurrence relation and an explicit rule for its book value $V_n$ after $n$ years. 2 marks

**b.** Find its book value after $6$ years under the flat-rate method. 1 mark

**c.** Instead, the van is depreciated by the reducing-balance method, losing $9\%$ of its value each year. Write an explicit rule for its value $V_n$ after $n$ years, and find its value after $6$ years, correct to the nearest cent. 2 marks

**d.** State which method gives the greater book value after $6$ years. 1 mark

**Question 2** (7 marks)

$\$8\,000$ is invested at $6\%$ per annum compounded quarterly.

**a.** Find the value of the investment after $3$ years, correct to the nearest cent. 2 marks

**b.** Find the effective annual interest rate, correct to two decimal places. 2 marks

**c.** Using technology, find the number of whole quarters for the investment to first exceed $\$12\,000$, and express this time in years. 3 marks

**Question 3** (7 marks)

On the first day, $\$10\,000$ is deposited into an account paying $4.8\%$ per annum compounded monthly. At the end of each month a further $\$400$ is deposited. Throughout, round the interest to the nearest cent each month.

**a.** Write a recurrence relation for the balance $V_n$ after $n$ months. 1 mark

**b.** Construct a growth table showing the interest earned, the deposit and the balance for each of the first four months. 3 marks

**c.** Find the total interest earned over the first four months. 1 mark

**d.** Find the first month at the end of which the balance exceeds $\$12\,000$. 2 marks

**Question 4** (8 marks)

A loan of $\$7\,600$ is charged $8.4\%$ per annum compounded monthly and is repaid by $\$1\,300$ at the end of each month. Round the interest to the nearest cent each month.

**a.** State the interest rate per month and write a recurrence relation for the balance owing $V_n$ after $n$ months. 1 mark

**b.** Construct the full amortisation table until the loan is repaid, showing the payment, interest, principal reduction and balance for each month. 4 marks

**c.** State the size of the final repayment and find the total interest charged over the life of the loan. 2 marks

**d.** Explain why the interest part of each repayment decreases while the principal-reduction part increases. 1 mark

**Question 5** (6 marks)

A loan of $\$20\,000$ is charged $7.2\%$ per annum compounded monthly.

**a.** The loan is to be repaid by equal monthly repayments over $4$ years. Use a finance solver to find the exact monthly repayment, and state it rounded to the appropriate number of cents. 2 marks

**b.** Explain why the repayment in part **a.** is rounded up rather than to the nearest cent, and state the size of the final repayment. 2 marks

**c.** If the loan were instead repaid by $\$400$ at the end of each month, use a finance solver to find how many repayments would be needed, and express this time in years and months. 2 marks

**Question 6** (6 marks)

**a.** A benefactor wants to fund a bursary that pays $\$12\,600$ at the end of each year forever, from a fund earning $4.5\%$ per annum. Find the amount that must be invested. 2 marks

**b.** Separately, a business borrows $\$45\,000$ at $8\%$ per annum compounded quarterly on an interest-only basis. Find the quarterly payment and the total interest paid over $3$ years. 2 marks

**c.** Explain why, for the interest-only loan in part **b.**, the full $\$45\,000$ is still owed after $3$ years. 2 marks

*End of Topic 2 Test B*

*Test B — answers and solutions*

**Q1.** (a) Flat-rate depreciation writes off a fixed $9\%$ of the purchase price each year, $d=\frac{9}{100}\times 40\,000=3\,600$. The recurrence is $V_{n+1}=V_n-3\,600$ with $V_0=40\,000$, and the explicit rule is $V_n=40\,000-3\,600\,n$. (b) $V_6=40\,000-3\,600\times 6=40\,000-21\,600=\$18\,400.00$. (c) Reducing-balance depreciation multiplies by $R=1-\frac{9}{100}=0.91$ each year, so $V_n=40\,000\times 0.91^n$. Then $V_6=40\,000\times 0.91^6=40\,000\times 0.56786925\ldots=\$22\,714.77$. (d) The reducing-balance value ($\$22\,714.77$) is greater than the flat-rate value ($\$18\,400.00$) after $6$ years.

**Q2.** (a) The quarterly rate is $r=\frac{6}{4}=1.5\%$, so $R=1.015$, and 3 years is $n=4\times 3=12$ quarters. Then

$$V_{12}=8\,000\times 1.015^{12}=8\,000\times 1.1956182\ldots=\$9\,564.95.$$

(b) One year is four quarters, so $r_{\text{eff}}=\left(1.015^4-1\right)\times 100\%=6.136355\ldots\%\approx 6.14\%$. (c) Require $8\,000\times 1.015^n>12\,000$, that is $1.015^n>1.5$. Solving $1.015^n=1.5$ with the CAS `solve` command gives $n\approx 27.23$; equivalently, entering $I\%=6$, $PV=-8\,000$, $PMT=0$, $FV=12\,000$, $P/Y=C/Y=4$ in a finance solver and solving for $N$ returns $N\approx 27.23$. Interest is only added at the end of a whole quarter and the balance is rising, so round up to $n=28$ quarters. (Check: $V_{27}=8\,000\times 1.015^{27}\approx\$11\,958.40<12\,000$ and $V_{28}\approx\$12\,137.78>12\,000$.) Since $28\div 4=7$, the investment first exceeds $\$12\,000$ after 28 quarters, that is 7 years.

**Q3.** (a) The monthly rate is $r=\frac{4.8}{12}=0.4\%$, so $R=1.004$ and the recurrence is $V_{n+1}=1.004\,V_n+400$ with $V_0=10\,000$. (b) Each month the interest is $0.4\%$ of the previous balance (to the nearest cent), and $\$400$ is added.

| Month | Interest earned | Deposit | Balance |
|---|---|---|---|
| 0 | — | — | $\$10\,000.00$ |
| 1 | $\$40.00$ | $\$400.00$ | $\$10\,440.00$ |
| 2 | $\$41.76$ | $\$400.00$ | $\$10\,881.76$ |
| 3 | $\$43.53$ | $\$400.00$ | $\$11\,325.29$ |
| 4 | $\$45.30$ | $\$400.00$ | $\$11\,770.59$ |

(c) The amount contributed is $10\,000+4\times 400=11\,600$, so the total interest is $11\,770.59-11\,600=\$170.59$ — the same as the sum of the interest column, $40+41.76+43.53+45.30=\$170.59$.

(d) Continuing the iteration, month 5 earns interest $0.004\times 11\,770.59=47.08236\to 47.08$, giving a balance of $11\,770.59+47.08+400=\$12\,217.67$. At the end of month 4 the balance is $\$11\,770.59$ (below $\$12\,000$) and at the end of month 5 it is $\$12\,217.67$, so the balance first exceeds $\$12\,000$ at the end of **month 5**.

**Q4.** (a) The monthly rate is $r=\frac{8.4}{12}=0.7\%$, so $R=1.007$ and the recurrence is $V_{n+1}=1.007\,V_n-1\,300$ with $V_0=7\,600$. (b) Each month the interest is $0.7\%$ of the previous balance (to the nearest cent); the principal reduction is $1\,300-\text{interest}$; and the new balance is the previous balance less the principal reduction. The last month owes only $\$1\,278.10$, whose interest is $0.007\times 1\,278.10=8.9467\to 8.95$, so a smaller final repayment of $1\,278.10+8.95=\$1\,287.05$ clears the loan.

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | $\$7\,600.00$ |
| 1 | $\$1\,300.00$ | $\$53.20$ | $\$1\,246.80$ | $\$6\,353.20$ |
| 2 | $\$1\,300.00$ | $\$44.47$ | $\$1\,255.53$ | $\$5\,097.67$ |

| Payment number | Payment | Interest | Principal reduction | Balance |
|---|---|---|---|---|
| 3 | $\$1\,300.00$ | $\$35.68$ | $\$1\,264.32$ | $\$3\,833.35$ |
| 4 | $\$1\,300.00$ | $\$26.83$ | $\$1\,273.17$ | $\$2\,560.18$ |
| 5 | $\$1\,300.00$ | $\$17.92$ | $\$1\,282.08$ | $\$1\,278.10$ |
| 6 | $\$1\,287.05$ | $\$8.95$ | $\$1\,278.10$ | $\$0.00$ |

(c) The final repayment is $\$1\,287.05$. The total interest is the sum of the interest column, $53.20+44.47+35.68+26.83+17.92+8.95=\$187.05$ (check: $5\times 1\,300+1\,287.05-7\,600=\$187.05$).

(d) With each repayment the balance owing falls, and the interest for a period is a fixed percentage of that balance, so the interest charged gets smaller each month. Because the repayments are equal, the part of each that is not interest — the principal reduction — must correspondingly get larger.

**Q5.** (a) The monthly rate is $r=\dfrac{7.2}{12}=0.6\,\%$, and 4 years is $N=4\times 12=48$ repayments. Entering $N=48$, $I\,\%=7.2$, $PV=20\,000$, $FV=0$ and $P/Y=C/Y=12$ and solving for $PMT$, the solver returns $PMT=-480.7828\ldots$, so the exact repayment is $\$480.7828\ldots$, which rounds **up** to a monthly repayment of $\$480.79$. (b) Rounding up to the next cent makes every repayment a fraction of a cent larger than the exact figure, which guarantees the loan is cleared within the agreed $48$ months (rounding down would leave a few cents owing and force a $49$th repayment). To size the last repayment, return to the solver with $N=47$, $I\,\%=7.2$, $PV=20\,000$, $PMT=-480.79$, $P/Y=C/Y=12$ and solve for $FV$: it returns $FV=-477.5299\ldots$, so $\$477.53$ is still owing after the $47$th repayment, and the $48$th repayment is that balance plus its month's interest, $477.5299\ldots\times 1.006=\$480.40$. (A hand-built table that rounds the interest to the nearest cent every month instead leaves $\$477.55$ owing and a final repayment of $\$480.42$. The solver does not round period by period, so the two standard methods differ by two cents here; either value is accepted.) (c) With a $\$400$ repayment, enter $I\,\%=7.2$, $PV=20\,000$, $PMT=-400$, $FV=0$, $P/Y=C/Y=12$ and solve for $N$: the solver returns $N=59.62\ldots$ The whole-number part, 59, is the number of full $\$400$ repayments, and the fraction means a smaller $60$th repayment is still needed, so rounding up gives $60$ repayments in all. Since $60\div 12=5$, this takes $5$ years 0 months.

**Q6.** (a) The perpetual payment is the interest on the capital, $\text{payment}=\dfrac{r}{100}\times\text{principal}$. Rearranging for the principal,

$$\text{principal}=\frac{\text{payment}\times 100}{r}=\frac{12\,600\times 100}{4.5}=\$280\,000.00.$$

(Check: $0.045\times 280\,000=\$12\,600$, the bursary.) (b) The quarterly rate is $r=\dfrac{8}{4}=2\,\%$, so the interest-only payment is $D=0.02\times 45\,000=\$900.00$ each quarter. Three years is $3\times 4=12$ quarters, so the total interest paid is $12\times 900=\$10\,800.00$. (c) On an interest-only loan each payment covers exactly that quarter's interest and no more, so no part of any payment reduces the principal. The balance therefore stays at $\$45\,000$ throughout, and the full $\$45\,000$ borrowed is still owed at the end of the three years.