General Mathematics Units 3 & 4

Chapter 6 — Modelling growth and decay with recurrence relations

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Contents

Chapter 6 Modelling growth and decay using recursion — 6A Sequences and recurrence relations

Theory

A **sequence** is an ordered list of numbers, for example

$$3, 7, 11, 15, 19, \ldots$$

Each number in the list is a **term**. The order matters: the sequence $3, 7, 11, \ldots$ is not the same as $11, 7, 3, \ldots$.

Naming the terms. It is convenient to label the terms with a single letter and a **subscript** that records each term's position. Writing the value with the letter V,

$$V_0, V_1, V_2, V_3, \ldots$$

the subscript is the **term number**. Here V_0 is the **starting term** (the *initial* value), V_1 is the next term, and in general V_n is the term reached after n steps through the list. For the sequence above, $V_0 = 3$, $V_1 = 7$, $V_2 = 11$ and $V_3 = 15$. Beginning the count at V_0 (rather than V_1) is the standard choice in this chapter, because V_0 will later stand for the amount of money you *start* with, before any time has passed.

Recurrence relations. Many sequences are built by a repeating rule: each new term is worked out from the term before it. A rule of this kind, together with a starting value, is called a **recurrence relation**. It always has two parts:

- a **starting value**, such as $V_0 = 3$; and
- a **rule** that produces the next term from the current one, written $V_{n+1} = f(V_n)$.

For example,

$$V_0 = 3, V_{n+1} = V_n + 4$$

says "start at 3, and to get the next term add 4 to the term you have". Because each term depends only on the **one** term immediately before it, this is called a **first-order** recurrence relation. The general first-order linear rule used throughout this chapter has the form

$$V_{n+1} = R V_n + D,$$

where R and D are constants.

Generating the terms by iteration. To list the terms you **iterate** the rule: substitute the current term, compute the next, then repeat. Starting from $V_0 = 3$ with $V_{n+1} = V_n + 4$:

Step	Substitute	New term
$V_1 = V_0 + 4$	$= 3 + 4$	$= 7$
$V_2 = V_1 + 4$	$= 7 + 4$	$= 11$
$V_3 = V_2 + 4$	$= 11 + 4$	$= 15$
$V_4 = V_3 + 4$	$= 15 + 4$	$= 19$

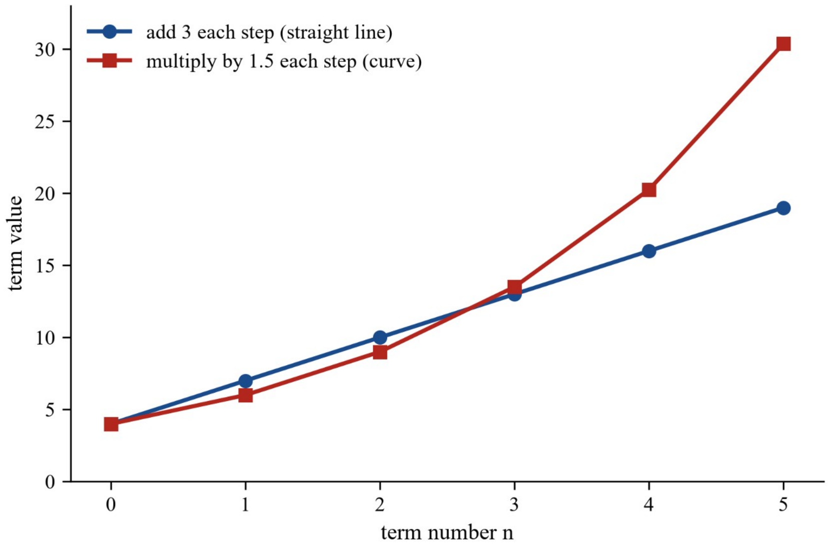
so the sequence is $3, 7, 11, 15, 19, \ldots$ as expected.

Two patterns to recognise. Almost every model in this chapter is built from one of two simple actions repeated over and over.

- **Add a constant.** The rule $V_{n+1} = V_n + D$ (this is the case $R = 1$) adds the **same amount** D at every step. If D is negative the terms decrease. The successive **differences** $V_{n+1} - V_n$ are all equal to D. Plotted against the term number, the terms lie on a **straight line** — a *linear* pattern.

- **Multiply by a constant.** The rule $V_{n+1}=RV_n$ (this is the case $D=0$) multiplies by the **same factor** R at every step. The successive **ratios** $V_{n+1}\div V_n$ are all equal to R. If $R>1$ the terms grow ever faster; if $0<R<1$ they shrink towards zero. Plotted against the term number, the terms follow a **curve** — a *geometric* pattern.

To decide which pattern a listed sequence follows, test the two things in turn: if the **differences** between successive terms are constant, it is *add a constant*; if instead the **ratios** of successive terms are constant, it is *multiply by a constant*. For instance, starting at 4 and adding 3 each time gives $4, 7, 10, 13, 16, 19$, while starting at 4 and multiplying by 1.5 each time gives $4, 6, 9, 13.5, 20.25, 30.375$. Both begin at 4, but the multiplying sequence soon pulls away, as the graph below shows.



This is the key idea that the rest of the chapter develops: **adding a constant** produces the straight-line (linear) growth or decay behind *flat-rate* models, while **multiplying by a constant** produces the curved (geometric) growth or decay behind *compound-interest* and *reducing-balance* models. The financial detail is taken up in later sections of this chapter; here we only need to generate terms and recognise which pattern is at work.

Using CAS. General Mathematics is a technology-active, open-book course, and a calculator will iterate a recurrence for you. The quickest method uses the **previous-answer** key (often written **Ans**): type the starting value and press **enter**; then type the rule with **Ans** in place of the current term — for $V_{n+1}=V_n+4$ you enter `Ans + 4` — and press **enter** repeatedly. Each press displays the next term. A CAS also offers a recurrence or spreadsheet feature that lists many terms at once. Learn the by-hand iteration below so that you can set up the rule correctly and check that the calculator's output is reasonable.

Worked examples

Example 1 — scaffolded

A sequence is defined by the recurrence relation $V_0=5$, $V_{n+1}=V_n+3$. Write down the first five terms V_0, V_1, V_2, V_3, V_4.

Working	Comment
$V_0=5$.	The stated starting value.
$V_1=V_0+3=5+3=8$.	Apply the rule: add 3 to the current term.
$V_2=V_1+3=8+3=11$.	Iterate again using the new term.
$V_3=V_2+3=11+3=14$.	Each step adds the same constant, 3.
$V_4=V_3+3=14+3=17$.	The first five terms are $5, 8, 11, 14, 17$.

Example 2 — scaffolded

A sequence is defined by $V_0 = 2$, $V_{n+1} = 3V_n$. Write down the first five terms, and state whether the rule *adds a constant* or *multiplies by a constant*.

Working	Comment
$V_0 = 2$.	The starting value.
$V_1 = 3V_0 = 3 \times 2 = 6$.	Apply the rule: multiply the current term by 3.
$V_2 = 3V_1 = 3 \times 6 = 18$.	Iterate again.
$V_3 = 3V_2 = 3 \times 18 = 54$.	Each step multiplies by the same factor, 3.
$V_4 = 3V_3 = 3 \times 54 = 162$.	The first five terms are $2, 6, 18, 54, 162$.
Each term is 3 times the one before, so the rule **multiplies by a constant** (factor 3).	The successive ratios $6 \div 2, 18 \div 6, \ldots$ are all 3.

Example 3 — unscaffolded

A sequence begins $400, 200, 100, 50, \ldots$. Show that it can be generated by a recurrence relation of the form $V_{n+1} = RV_n$, state the starting value and the rule, and find the next term.

Test the successive **ratios**:

$$\frac{200}{400} = 0.5, \frac{100}{200} = 0.5, \frac{50}{100} = 0.5.$$

The ratio is the same each time, so the sequence multiplies by the constant factor $R = 0.5$. (The successive *differences* $-200, -100, -50$ are **not** constant, so it is not an add-a-constant sequence.) The starting value is $V_0 = 400$, so the recurrence relation is

$$V_0 = 400, V_{n+1} = 0.5V_n.$$

The next term is $V_4 = 0.5 \times 50 = 25$.

∴ the sequence is generated by $V_0 = 400$, $V_{n+1} = 0.5V_n$, and the next term is 25.

Example 4 — unscaffolded

A sequence is defined by $V_0 = 1000$, $V_{n+1} = 0.9V_n + 50$. Generate the terms V_1, V_2, V_3, V_4, and explain why the rule is neither a pure *add a constant* nor a pure *multiply by a constant* rule.

Iterating the rule (on a CAS, enter 1000, then `0.9 × Ans + 50` and press enter repeatedly):

$$V_1 = 0.9 \times 1000 + 50 = 950, V_2 = 0.9 \times 950 + 50 = 905,$$

$$V_3 = 0.9 \times 905 + 50 = 864.5, V_4 = 0.9 \times 864.5 + 50 = 828.05.$$

The successive differences are $-50, -45, -40.5, \ldots$, which are not constant, so it does not add a constant. The successive ratios are $950/1000 = 0.95$ and $905/950 \approx 0.9526$, which are not constant either, so it does not multiply by a constant. Because the rule both multiplies (by $R = 0.9$) **and** adds ($D = 50$), it mixes the two actions — exactly the form $V_{n+1} = RV_n + D$ that later models of loans and depreciation will use.

∴ the terms are $950, 905, 864.5, 828.05$, and the rule is of the general first-order linear form, neither purely adding nor purely multiplying.

Practice questions

Scaffolded

Q1. A sequence is $6, 10, 14, 18, 22$. (a) State V_0. (b) State V_2. (c) State V_4. (d) Which term (that is, what value of n) has $V_n = 14$?

Q2. For the recurrence relation $V_0 = 7$, $V_{n+1} = V_n + 5$: (a) find V_1; (b) find V_2; (c) find V_3; (d) list the first five terms V_0, V_1, V_2, V_3, V_4.

Q3. For the recurrence relation $V_0 = 3$, $V_{n+1} = 2V_n$: (a) find V_1; (b) find V_2; (c) find V_3; (d) list the first five terms.

Q4. For the recurrence relation $V_0 = 100$, $V_{n+1} = V_n - 12$, list the first five terms V_0, V_1, V_2, V_3, V_4.

Q5. For each recurrence rule, state whether it *adds a constant* or *multiplies by a constant*, and write down that constant. (a) $V_{n+1} = V_n + 6$ (b) $V_{n+1} = 4V_n$ (c) $V_{n+1} = V_n - 9$ (d) $V_{n+1} = 0.5V_n$

Q6. A sequence is $3, 12, 48, 192, \ldots$. (a) Find the ratios $\frac{12}{3}$, $\frac{48}{12}$ and $\frac{192}{48}$. (b) Is the sequence *add a constant* or *multiply by a constant*? (c) Write a recurrence relation (a starting value and a rule) that generates the sequence.

Unscaffolded

Q7. A sequence is defined by $V_0 = 9$, $V_{n+1} = V_n + 7$. Find V_1, V_2 and V_3.

Q8. A sequence is defined by $V_0 = 5$, $V_{n+1} = 3V_n$. Find V_4.

Q9. A sequence is defined by $V_0 = 250$, $V_{n+1} = V_n - 35$. Find V_5, and state which is the first term that is below 100.

Q10. A sequence is defined by $V_0 = 64$, $V_{n+1} = 0.5V_n$. List the terms V_0, V_1, V_2, V_3, V_4.

Q11. Write a recurrence relation (starting value and rule) that generates the sequence $11, 18, 25, 32, \ldots$.

Q12. Write a recurrence relation that generates the sequence $2, 6, 18, 54, \ldots$.

Q13. A sequence is $90, 78, 66, 54, \ldots$. (a) Write a recurrence relation that generates it. (b) Find the next two terms.

Q14. A sequence is $800, 400, 200, 100, \ldots$. (a) Write a recurrence relation that generates it. (b) Find the next term.

Q15. A sequence is defined by $V_0 = 4$, $V_{n+1} = 2V_n + 3$. (a) Find the first four terms V_0, V_1, V_2, V_3. (b) By checking the successive differences and ratios, state whether the rule adds a constant, multiplies by a constant, or does neither.

Q16. A tank holds 500 litres of water. Each hour a pump removes 45 litres. Let V_n be the volume (in litres) remaining after n hours. (a) Write a recurrence relation for V_n. (b) Find the volume remaining after 4 hours. (c) State whether this is an *add a constant* or a *multiply by a constant* model.

Q17. A colony starts with 20 bacteria and the number doubles every hour. Let V_n be the number of bacteria after n hours. (a) Write a recurrence relation for V_n. (b) Find the number of bacteria after 5 hours. (c) State whether this is an *add a constant* or a *multiply by a constant* model.

Q18. Two sequences both start at 5: sequence A is $5, 8, 11, 14, \ldots$ and sequence B is $5, 10, 20, 40, \ldots$. (a) State which sequence adds a constant and which multiplies by a constant. (b) Find the term V_6 of each sequence. (c) State which sequence is larger at V_6, and briefly explain why a multiplying sequence eventually overtakes an adding one.

Q19. A sequence is defined by $V_0 = 2$, $V_{n+1} = 1.5V_n$. (a) Using the previous-answer (**Ans**) method on a CAS, describe the keystrokes you would use to generate the terms. (b) Find V_5, correct to two decimal places.

Q20. A sequence is defined by $V_0 = 1000$, $V_{n+1} = 0.8V_n + 100$. (a) Find V_1, V_2 and V_3. (b) Explain why this rule is neither a pure *add a constant* nor a pure *multiply by a constant* rule, and state which general form it has.

Answers

Q1. (a) 6; (b) 14; (c) 22; (d) $n=2$. **Q2.** (a) 12; (b) 17; (c) 22; (d) $7,12,17,22,27$. **Q3.** (a) 6; (b) 12; (c) 24; (d) $3,6,12,24,48$. **Q4.** $100,88,76,64,52$. **Q5.** (a) adds a constant, 6; (b) multiplies by a constant, 4; (c) adds a constant, -9; (d) multiplies by a constant, 0.5. **Q6.** (a) all ratios equal 4; (b) multiply by a constant; (c) $V_0=3$, $V_{n+1}=4V_n$. **Q7.** $V_1=16$, $V_2=23$, $V_3=30$. **Q8.** $V_4=405$. **Q9.** $V_5=75$; the first term below 100 is $V_5=75$. **Q10.** $64,32,16,8,4$. **Q11.** $V_0=11$, $V_{n+1}=V_n+7$. **Q12.** $V_0=2$, $V_{n+1}=3V_n$. **Q13.** (a) $V_0=90$, $V_{n+1}=V_n-12$; (b) 42 and 30. **Q14.** (a) $V_0=800$, $V_{n+1}=0.5V_n$; (b) 50. **Q15.** (a) $4,11,25,53$; (b) neither (differences $7,14,28$ and ratios $2.75,2.27,\ldots$ are both non-constant). **Q16.** (a) $V_0=500$, $V_{n+1}=V_n-45$; (b) 320 litres; (c) adds a constant. **Q17.** (a) $V_0=20$, $V_{n+1}=2V_n$; (b) 640; (c) multiplies by a constant. **Q18.** (a) A adds a constant (3); B multiplies by a constant (2); (b) $A_6=23$, $B_6=320$; (c) B is larger, because repeated multiplying by a factor greater than 1 grows faster and faster, so it overtakes the steady straight-line increase of A. **Q19.** (a) enter 2, then enter `1.5 × Ans` and press enter repeatedly; (b) $V_5=15.19$ (exactly 15.1875). **Q20.** (a) $V_1=900$, $V_2=820$, $V_3=756$; (b) neither — the differences ($-100,-80,-64$) and the ratios ($0.9,0.911,\ldots$) are both non-constant; it has the general form $V_{n+1}=RV_n+D$ with $R=0.8$ and $D=100$.

Fully-worked solutions

Q1. Writing the terms with their subscripts, $V_0=6$, $V_1=10$, $V_2=14$, $V_3=18$, $V_4=22$. So (a) $V_0=6$, (b) $V_2=14$, (c) $V_4=22$. For (d), the term equal to 14 is V_2, so $n=2$.

Q2. Start at $V_0=7$ and add 5 each time. $V_1=7+5=12$; $V_2=12+5=17$; $V_3=17+5=22$; $V_4=22+5=27$. The first five terms are $7,12,17,22,27$.

Q3. Start at $V_0=3$ and multiply by 2 each time. $V_1=2\times3=6$; $V_2=2\times6=12$; $V_3=2\times12=24$; $V_4=2\times24=48$. The first five terms are $3,6,12,24,48$.

Q4. Start at $V_0=100$ and subtract 12 each time (the rule adds the constant -12). $V_1=100-12=88$; $V_2=88-12=76$; $V_3=76-12=64$; $V_4=64-12=52$. The terms are $100,88,76,64,52$.

Q5. (a) The rule adds the fixed amount 6, so it **adds a constant**, 6. (b) The rule multiplies by the fixed factor 4, so it **multiplies by a constant**, 4. (c) Subtracting 9 is adding -9, so it **adds a constant**, -9. (d) The rule multiplies by the fixed factor 0.5, so it **multiplies by a constant**, 0.5.

Q6. (a) $\frac{12}{3}=4$, $\frac{48}{12}=4$, $\frac{192}{48}=4$. (b) The ratios are all equal, so the sequence **multiplies by a constant** (and the differences $9,36,144$ are not constant, confirming it is not an adding sequence). (c) The starting value is $V_0=3$ and the factor is 4, so the recurrence relation is $V_0=3$, $V_{n+1}=4V_n$.

Q7. Start at $V_0=9$ and add 7 each time. $V_1=9+7=16$; $V_2=16+7=23$; $V_3=23+7=30$.

Q8. Start at $V_0=5$ and multiply by 3 each time. $V_1=15$, $V_2=45$, $V_3=135$, $V_4=3\times135=405$.

Q9. Subtract 35 each time from $V_0=250$: $V_1=215$, $V_2=180$, $V_3=145$, $V_4=110$, $V_5=75$. Since $V_4=110\geq100$ but $V_5=75<100$, the first term below 100 is $V_5=75$.

Q10. Halve each time from $V_0=64$: $V_1=32$, $V_2=16$, $V_3=8$, $V_4=4$. The terms are $64,32,16,8,4$.

Q11. The differences are $18-11=7$, $25-18=7$, $32-25=7$: a constant difference of 7, so the sequence adds the constant 7. With $V_0=11$ the recurrence relation is $V_0=11$, $V_{n+1}=V_n+7$.

Q12. The ratios are $\frac{6}{2}=3$, $\frac{18}{6}=3$, $\frac{54}{18}=3$: a constant ratio of 3, so the sequence multiplies by 3. With $V_0=2$ the recurrence relation is $V_0=2$, $V_{n+1}=3V_n$.

Q13. (a) The differences are $78-90=-12$, $66-78=-12$, $54-66=-12$: a constant difference of -12. With $V_0=90$ the recurrence relation is $V_0=90$, $V_{n+1}=V_n-12$. (b) Continuing, $54-12=42$ and $42-12=30$, so the next two terms are 42 and 30.

Q14. (a) The ratios are $\frac{400}{800}=0.5$, $\frac{200}{400}=0.5$, $\frac{100}{200}=0.5$: a constant ratio of 0.5. With $V_0=800$ the recurrence relation is $V_0=800$, $V_{n+1}=0.5V_n$. (b) The next term is $0.5\times100=50$.

Q15. (a) $V_0=4$; $V_1=2\times4+3=11$; $V_2=2\times11+3=25$; $V_3=2\times25+3=53$. So the first four terms are $4,11,25,53$. (b) The differences are $7,14,28$ (not constant), so it does not add a constant; the ratios are $11/4=2.75$ and $25/11\approx2.27$ (not constant), so it does not multiply by a constant. The rule does **neither** — it multiplies by 2 **and** adds 3.

Q16. (a) Each hour the volume falls by the fixed amount 45, starting from 500, so $V_0=500$, $V_{n+1}=V_n-45$. (b) Iterating: $V_1=455$, $V_2=410$, $V_3=365$, $V_4=320$, so 320 litres remain after 4 hours. (c) The same amount is removed each hour, so it **adds a constant** (here the constant is -45).

Q17. (a) Each hour the number is multiplied by 2, starting from 20, so $V_0=20$, $V_{n+1}=2V_n$. (b) Iterating: $V_1=40$, $V_2=80$, $V_3=160$, $V_4=320$, $V_5=640$, so there are 640 bacteria after 5 hours. (c) The number is multiplied by the same factor each hour, so it **multiplies by a constant**.

Q18. (a) Sequence A has constant difference 3 ($8-5=11-8=3$), so it **adds a constant**; sequence B has constant ratio 2 ($10\div5=20\div10=2$), so it **multiplies by a constant**. (b) For A, add 3 six times from 5: $5,8,11,14,17,20,23$, so $A_6=23$. For B, multiply by 2 six times from 5: $5,10,20,40,80,160,320$, so $B_6=320$. (c) B is larger. Adding a constant increases the terms by the same amount each step (a straight line), but multiplying by a factor greater than 1 increases them by an ever-larger amount each step, so the multiplying sequence eventually grows past the adding one.

Q19. (a) Type the starting value 2 and press **enter**. Then type `1.5 × Ans` (so that each new term is 1.5 times the previous answer) and press **enter**; each further press of **enter** applies the rule again and shows the next term. (b) Iterating: $V_1=3$, $V_2=4.5$, $V_3=6.75$, $V_4=10.125$, $V_5=15.1875$. To two decimal places, $V_5=15.19$.

Q20. (a) $V_1=0.8\times1000+100=900$; $V_2=0.8\times900+100=820$; $V_3=0.8\times820+100=756$. (b) The differences are $-100,-80,-64$ (not constant), so the rule does not add a constant; the ratios are $900/1000=0.9$ and $820/900\approx0.911$ (not constant), so it does not multiply by a constant. Because it multiplies (by $R=0.8$) and adds ($D=100$) at each step, it has the general first-order linear form $V_{n+1}=RV_n+D$.

Theory

Many financial quantities change by the **same amount** in every time period. A simple-interest investment gains a fixed number of dollars each year; an asset losing value by flat-rate depreciation drops by a fixed number of dollars each year. A pattern of *repeated addition of a constant* is described by a **first-order linear recurrence relation**.

The recurrence. Write V_n for the value after n periods and V_0 for the starting value. The general first-order linear recurrence has the form

$$V_0 = a, V_{n+1} = R V_n + D,$$

where a, R and D are constants. This section studies the purely **additive** case $R = 1$,

$$V_0 = a, V_{n+1} = V_n + D,$$

in which the same constant D is added at every step. If $D > 0$ the value **grows**; if $D < 0$ the value **decays** (decreases). Multiplying by a constant $R \neq 1$ — compound interest and reducing-balance depreciation — is treated later in this chapter.

Generating the sequence. A recurrence is used by *iterating*: start at V_0 and apply the rule repeatedly, each new term built from the one before,

$$V_1 = V_0 + D, V_2 = V_1 + D, V_3 = V_2 + D, \ldots$$

Each line uses the previous answer, so the terms are generated one after another. On a CAS you can enter the starting value, then the rule $\text{ans} + D$, and press enter repeatedly to list the terms, or use the calculator's built-in sequence facility. General Mathematics is a technology-active, open-book course; learn the by-hand iteration below so you understand what the technology is producing.

The constant-difference signature. Because the same amount is added each time,

$$V_{n+1} - V_n = D \text{ for every } n,$$

so the sequence has a **constant first difference**. When the values are plotted against time, the points are equally spaced vertically and lie exactly on a **straight line** of slope D: rising for growth ($D > 0$), falling for decay ($D < 0$). A constant difference and a straight-line graph are the signature of **linear** growth and decay, and they are what distinguish it from the *curved* graphs produced when a value is repeatedly **multiplied** by a constant (studied later in this chapter). A rule for jumping straight to V_n without listing every term is developed in the next section; here we generate the terms by iterating.

Money: compute exactly, present rounded. Amounts of money are worked out exactly and then written to the nearest cent, for example $\$1250.00$. Where an interest amount is not a whole number of cents, only the *displayed* value is rounded.

Simple interest. Under **simple interest** the interest earned each period is calculated on the **original principal** P only — earlier interest does not itself earn interest. At a rate of $r\,\%$ per period the interest per period is the constant

$$D = \frac{r}{100} \times P,$$

so the balance follows $V_0 = P, V_{n+1} = V_n + D$: linear growth. When interest is instead added to the balance and then earns interest itself, the growth is *compound* — a multiplying rule, studied later in this chapter. A nominal annual rate applied k times a year gives a per-period rate of $\frac{r}{k}\%$; for example $9\,\%$ per annum paid monthly is $0.75\,\%$ per month.

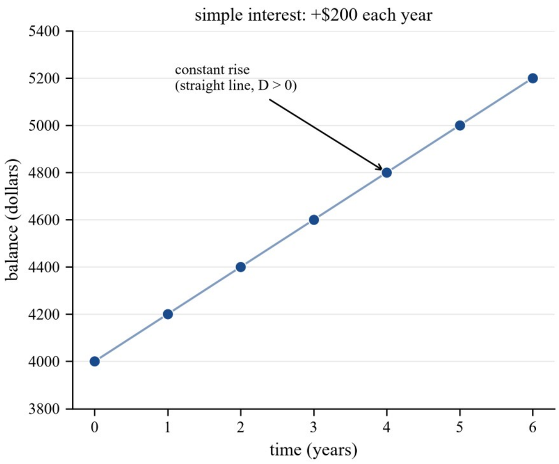
Flat-rate (straight-line) depreciation. An asset depreciates by **flat rate** (also called **straight-line**) when it loses the *same dollar amount* of value each period. If it depreciates at r% of its **purchase price** P per year, the annual loss is the constant $D=\frac{r}{100}\times P$, and the value follows $V_0=P, V_{n+1}=V_n-D$: linear decay.

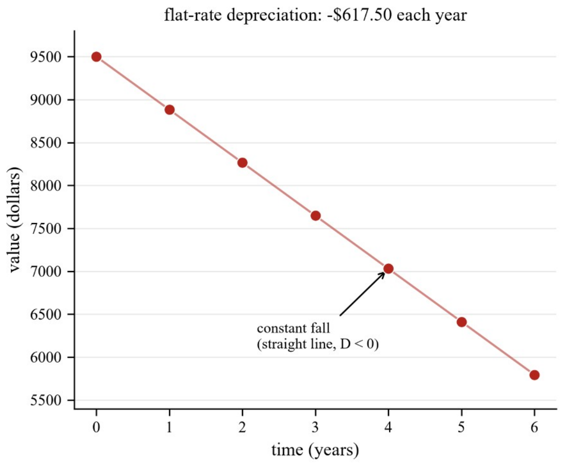
Unit-cost depreciation. An asset depreciates by **unit cost** when it loses a fixed amount of value for each **unit of use** — each kilometre driven, item produced, page printed or hour run. If the asset loses $\$c$ per unit and is used for u units in a period, the loss that period is the constant $D=c\times u$, and again $V_{n+1}=V_n-D$: linear decay driven by usage rather than by age.

In every case the change per period is a fixed amount D, so all three models use the same additive recurrence and all three produce a straight-line value–time graph.

Model	Each period	Constant D	Recurrence
Simple interest	add interest on the principal	$D=\frac{r}{100}P$	$V_{n+1}=V_n+D$
Flat-rate (straight-line) depreciation	subtract a fixed % of purchase price	$D=\frac{r}{100}P$	$V_{n+1}=V_n-D$
Unit-cost depreciation	subtract cost per unit × units used	$D=cu$	$V_{n+1}=V_n-D$

The graphs below show the signature: a simple-interest balance rising along a straight line (constant amount added each year) and a flat-rate depreciation value falling along a straight line (constant amount lost each year).





Worked examples

Example 1 — scaffolded

\$4,000 is invested at 5% per annum simple interest. Generate the balance at the end of each of the first four years, and state the balance after four years.

Working	Comment
$D=\frac{5}{100}\times 4000=\$200.00$.	Simple interest is charged on the principal only, so the yearly interest is constant.
$V_0=4000, V_{n+1}=V_n+200$.	The additive recurrence: add \$200.00 each year ($D>0$, growth).
$V_1=4000+200=\$4\,200.00$.	End of year 1.
$V_2=4200+200=\$4\,400.00$.	End of year 2 (build each term from the previous balance).

Working	Comment
$V_3 = 4400 + 200 = \$4\,600.00$.	End of year 3.
$V_4 = 4600 + 200 = \$4\,800.00$.	End of year 4.

So after four years the balance is \$4,800.00. The four increases are equal (\$200.00 each), so the balances lie on a straight line.

Example 2 — scaffolded

A machine bought for \$9,500 depreciates by flat rate at 6.5% of its purchase price each year. Write the recurrence, generate the value at the end of each of the first three years, and find its value after three years.

Working	Comment
$D = \frac{6.5}{100} \times 9500 = \$617.50$.	Flat rate: the same fraction of the *purchase price* is lost every year.
$V_0 = 9500, V_{n+1} = V_n - 617.50$.	Additive recurrence with $D < 0$ (decay).
$V_1 = 9500 - 617.50 = \$8\,882.50$.	End of year 1.
$V_2 = 8882.50 - 617.50 = \$8\,265.00$.	End of year 2.
$V_3 = 8265.00 - 617.50 = \$7\,647.50$.	End of year 3.

So after three years the machine is worth \$7,647.50. The value falls by the same \$617.50 each year, so its value–time graph is a straight line sloping down.

Example 3 — unscaffolded

A delivery van is bought for \$45,000. It depreciates by unit cost at \$0.25 for each kilometre travelled, and it travels 20,000 km per year. Write a recurrence for its value at the end of each year, and find its value after two years.

Because the van travels a fixed 20,000 km each year, the value lost per year is the constant

$$D = 0.25 \times 20\,000 = \$5\,000.00.$$

Taking V_n as the value in dollars after n years, with $V_0 = 45\,000$,

$$V_{n+1} = V_n - 5000.$$

Iterating,

$$V_1 = 45\,000 - 5000 = \$40\,000.00, V_2 = 40\,000 - 5000 = \$35\,000.00.$$

∴ after two years the van is worth \$35,000.00.

Example 4 — unscaffolded

The value of a car is modelled by $V_0 = 25\,000$, $V_{n+1} = V_n - 2000$, where V_n is its value in dollars after n years. State the purchase price and the annual depreciation, say whether this is growth or decay, find the car's value after five years, and explain why its value–time graph is a straight line.

The starting term is the purchase price, \$25,000.00, and each year \$2,000.00 is subtracted, so the annual (flat-rate) depreciation is \$2,000.00. Because $D = -2000 < 0$ the value is decaying. Iterating for five years,

$$25\,000 \to 23\,000 \to 21\,000 \to 19\,000 \to 17\,000 \to 15\,000,$$

so $V_5 = \$15\,000.00$. The value falls by exactly \$2,000.00 every year — a constant first difference — so the plotted points are equally spaced and lie on a straight line of slope -2000.

∴ purchase price \$25,000.00, depreciation \$2,000.00 per year (decay), value after five years \$15,000.00, and the constant yearly drop gives a straight-line graph.

Practice questions

Scaffolded

Q1. \$2,000 is invested at 6% per annum simple interest. (a) Find the interest D added each year. (b) Write the recurrence relation for the balance V_n. (c) Generate V_1, V_2, V_3, V_4. (d) State the balance after four years.

Q2. A laptop bought for \$2,400 depreciates by flat rate at \$300 each year. (a) Write the recurrence relation for its value V_n. (b) Generate V_1, V_2, V_3, V_4. (c) State its value after four years. (d) Is this an example of growth or of decay?

Q3. A taxi bought for \$36,000 depreciates by unit cost at \$0.40 for each kilometre driven, and it is driven 25,000 km each year. (a) Find the depreciation D for one year. (b) Write the recurrence relation for its value at the end of each year. (c) Find its value after three years.

Q4. An investment earning simple interest is modelled by $V_0 = 8000$, $V_{n+1} = V_n + 300$, where V_n is the balance in dollars after n years. (a) State the principal. (b) State the interest earned each year. (c) Find the annual simple interest rate, as a percentage of the principal. (d) Generate V_1 to V_5. (e) State the balance after five years.

Q5. \$3,000 is invested at 9% per annum simple interest, with interest paid monthly. (a) Find the interest D added each month, to the nearest cent. (b) Write the recurrence relation for the balance V_n after n months. (c) Find the balance after six months.

Q6. The value of a machine at the end of each year is recorded below.

Year n	0	1	2	3
Value (\$)	20,000	17,600	15,200	12,800

(a) Show that the value has a constant first difference.
(b) State the annual depreciation.
(c) Write the recurrence relation for the value V_n.
(d) Find the value after five years.

Unscaffolded

Q7. \$5,000 is invested at 4.5% per annum simple interest. Write the recurrence relation for the balance, and find the balance after six years.

Q8. A car bought for \$32,000 depreciates by flat rate at 12.5% of its purchase price each year. Write the recurrence relation for its value, and find its value after four years.

Q9. A 3D printer bought for \$7,200 depreciates by unit cost at \$0.05 for each gram of filament used, and it uses 8,000 g each year. Write the recurrence relation for its value at the end of each year, and find its value after three years.

Q10. An investment earns \$156 in interest each year under simple interest at a rate of 5.2% per annum. Find the principal invested, and the balance after four years.

Q11. \$8,000 is invested at 7% per annum simple interest. Find the total interest earned after five years, and the balance at that time.

Q12. A machine's value falls along a straight line from \$15,000 when new to \$9,000 after four years, by flat-rate depreciation. Find the annual depreciation, write the recurrence relation for its value, and find its value after six years.

Q13. A generator bought for \$8,500 depreciates by unit cost at \$1.20 for each hour it runs, and it runs 500 hours each year. Write the recurrence relation for its value at the end of each year, and find its value after five years.

Q14. \$4,500 is invested at 6.2% per annum simple interest, with interest paid monthly. Find the monthly interest, write the recurrence relation for the balance after n months, and find the balance after eight months.

Q15. Two machines each cost $12,000. Machine A depreciates by flat rate at $1,000 each year. Machine B depreciates by unit cost at $0.50 per kilometre and is driven 3,000 km each year. Determine which machine is worth more after four years, and by how much.

Q16. An asset depreciates by flat rate at $850 each year and is worth $5,750 after five years. Find its purchase price.

Q17. Under simple interest, $6,000 grows to $7,440 after four years. Find the annual simple interest rate, and write the recurrence relation for the balance.

Q18. A van bought for $40,000 depreciates by unit cost at $0.30 for each kilometre driven. After how many kilometres is it worth $25,000?

Q19. Equipment bought for $21,000 depreciates by flat rate at $3,500 each year. Find its value after four years, and the number of years until it is fully written off (worth $0).

Q20. A car's value is modelled by $V_0 = 13\,000$, $V_{n+1} = V_n - 1625$, where V_n is its value in dollars after n years, by flat-rate depreciation. (a) State the purchase price. (b) State the annual depreciation. (c) Express the annual depreciation as a percentage of the purchase price. (d) Find the value after four years. (e) Explain why the value–time graph is a straight line, and how it would differ if the value were instead multiplied by a constant each year.

Answers

Q1. (a) \$120.00; (b) $V_0 = 2000, V_{n+1} = V_n + 120$; (c) \$2,120.00, \$2,240.00, \$2,360.00, \$2,480.00; (d) \$2,480.00. **Q2.** (a) $V_0 = 2400, V_{n+1} = V_n - 300$; (b) \$2,100.00, \$1,800.00, \$1,500.00, \$1,200.00; (c) \$1,200.00; (d) decay. **Q3.** (a) \$10,000.00; (b) $V_0 = 36\,000, V_{n+1} = V_n - 10\,000$; (c) \$6,000.00. **Q4.** (a) \$8,000.00; (b) \$300.00; (c) 3.75%; (d) \$8,300.00, \$8,600.00, \$8,900.00, \$9,200.00, \$9,500.00; (e) \$9,500.00. **Q5.** (a) \$22.50; (b) $V_0 = 3000, V_{n+1} = V_n + 22.50$; (c) \$3,135.00. **Q6.** (a) each difference is $-\$2\,400.00$; (b) \$2,400.00 per year; (c) $V_0 = 20\,000, V_{n+1} = V_n - 2400$; (d) \$8,000.00. **Q7.** $V_0 = 5000, V_{n+1} = V_n + 225$; balance after six years \$6,350.00. **Q8.** $V_0 = 32\,000, V_{n+1} = V_n - 4000$; value after four years \$16,000.00. **Q9.** $V_0 = 7200, V_{n+1} = V_n - 400$; value after three years \$6,000.00. **Q10.** Principal \$3,000.00; balance after four years \$3,624.00. **Q11.** Total interest \$2,800.00; balance \$10,800.00. **Q12.** Depreciation \$1,500.00 per year; $V_0 = 15\,000, V_{n+1} = V_n - 1500$; value after six years \$6,000.00. **Q13.** $V_0 = 8500, V_{n+1} = V_n - 600$; value after five years \$5,500.00. **Q14.** \$23.25 per month; $V_0 = 4500, V_{n+1} = V_n + 23.25$; balance after eight months \$4,686.00. **Q15.** Machine A \$8,000.00, machine B \$6,000.00; A is worth \$2,000.00 more. **Q16.** \$10,000.00. **Q17.** 6% per annum; $V_0 = 6000, V_{n+1} = V_n + 360$. **Q18.** 50,000 km. **Q19.** Value after four years \$7,000.00; fully written off after 6 years. **Q20.** (a) \$13,000.00; (b) \$1,625.00; (c) 12.5%; (d) \$6,500.00; (e) the value falls by the same \$1,625.00 each year (constant first difference), so the points lie on a straight line; a constant *multiplier* would change the value by a different amount each year and curve the graph.

Fully-worked solutions

Q1. The yearly interest is $D = \frac{6}{100} \times 2000 = \$120.00$, charged on the principal only, so the recurrence is $V_0 = 2000$, $V_{n+1} = V_n + 120$. Iterating, $V_1 = 2000 + 120 = \$2\,120.00$, $V_2 = 2120 + 120 = \$2\,240.00$, $V_3 = 2240 + 120 = \$2\,360.00$, $V_4 = 2360 + 120 = \$2\,480.00$. After four years the balance is \$2,480.00.

Q2. The value drops by a fixed \$300 each year, so $V_0 = 2400$, $V_{n+1} = V_n - 300$. Iterating, $V_1 = 2400 - 300 = \$2\,100.00$, $V_2 = 2100 - 300 = \$1\,800.00$, $V_3 = 1800 - 300 = \$1\,500.00$, $V_4 = 1500 - 300 = \$1\,200.00$. After four years the laptop is worth \$1,200.00. Since a constant amount is *subtracted* each year ($D < 0$), this is decay.

Q3. Driving 25,000 km at \$0.40 per km loses $D = 0.40 \times 25\,000 = \$10\,000.00$ of value each year, so $V_0 = 36\,000$, $V_{n+1} = V_n - 10\,000$. Iterating, $V_1 = 36\,000 - 10\,000 = \$26\,000.00$, $V_2 = 26\,000 - 10\,000 = \$16\,000.00$, $V_3 = 16\,000 - 10\,000 = \$6\,000.00$. After three years the taxi is worth \$6,000.00.

Q4. The starting term $V_0 = 8000$ is the principal, \$8,000.00, and \$300.00 is added each year, so the yearly interest is \$300.00. As a percentage of the principal the rate is $\frac{300}{8000} \times 100 = 3.75\,\%$ per annum. Iterating, $V_1 = 8300$, $V_2 = 8600$, $V_3 = 8900$, $V_4 = 9200$, $V_5 = \$9\,500.00$. After five years the balance is \$9,500.00.

Q5. Interest is paid monthly, so the monthly rate is $\frac{9}{12} = 0.75\,\%$ and the monthly interest is $D = \frac{0.75}{100} \times 3000 = \$22.50$. The recurrence, with n in months, is $V_0 = 3000$, $V_{n+1} = V_n + 22.50$. After six months, $V_6 = 3000 + 6 \times 22.50 = 3000 + 135 = \$3\,135.00$.

Q6. The first differences are $17\,600 - 20\,000 = -2400$, $15\,200 - 17\,600 = -2400$ and $12\,800 - 15\,200 = -2400$, all equal, so the value has a constant first difference of $-\$2\,400.00$. The value therefore falls by \$2,400.00 each year, and the recurrence is $V_0 = 20\,000$, $V_{n+1} = V_n - 2400$. Continuing, $V_4 = 12\,800 - 2400 = \$10\,400.00$ and $V_5 = 10\,400 - 2400 = \$8\,000.00$, so after five years the machine is worth \$8,000.00.

Q7. The yearly interest is $D=\frac{4.5}{100}\times 5000=\$225.00$, so $V_0=5000$, $V_{n+1}=V_n+225$. After six years, $V_6=5000+6\times 225=5000+1350=\$6\,350.00$.

Q8. Flat rate at 12.5% of the purchase price gives $D=\frac{12.5}{100}\times 32\,000=\$4\,000.00$ each year, so $V_0=32\,000$, $V_{n+1}=V_n-4000$. Iterating, $V_1=28\,000$, $V_2=24\,000$, $V_3=20\,000$, $V_4=\$16\,000.00$. After four years the car is worth $16,000.00.

Q9. Using 8,000 g at $0.05 per gram loses $D=0.05\times 8000=\$400.00$ of value each year, so $V_0=7200$, $V_{n+1}=V_n-400$. Iterating, $V_1=6800$, $V_2=6400$, $V_3=\$6\,000.00$. After three years the printer is worth $6,000.00.

Q10. Each year's simple interest is $\frac{5.2}{100}\times P=156$, so $P=\frac{156}{0.052}=\$3\,000.00$. The recurrence is $V_0=3000$, $V_{n+1}=V_n+156$, and after four years $V_4=3000+4\times 156=3000+624=\$3\,624.00$.

Q11. The yearly interest is $\frac{7}{100}\times 8000=\$560.00$. Over five years the total interest is $5\times 560=\$2\,800.00$, so the balance is $8000+2800=\$10\,800.00$.

Q12. The value falls $6,000 over four years along a straight line, so the annual depreciation is $D=\frac{15\,000-9000}{4}=\$1\,500.00$. The recurrence is $V_0=15\,000$, $V_{n+1}=V_n-1500$. Continuing past year 4: $V_5=9000-1500=7500$ and $V_6=7500-1500=\$6\,000.00$, so after six years the machine is worth $6,000.00.

Q13. Running 500 hours at $1.20 per hour loses $D=1.20\times 500=\$600.00$ of value each year, so $V_0=8500$, $V_{n+1}=V_n-600$. After five years, $V_5=8500-5\times 600=8500-3000=\$5\,500.00$.

Q14. The monthly rate is $\frac{6.2}{12}\%$, so the monthly interest is $D=\frac{6.2}{12}\times\frac{4500}{100}=\$23.25$. The recurrence, with n in months, is $V_0=4500$, $V_{n+1}=V_n+23.25$. After eight months, $V_8=4500+8\times 23.25=4500+186=\$4\,686.00$.

Q15. Machine A: $D=\$1\,000.00$, so $V_0=12\,000$, $V_{n+1}=V_n-1000$ and after four years $V_4=12\,000-4000=\$8\,000.00$. Machine B: driving 3,000 km at $0.50 per km loses $D=0.50\times 3000=\$1\,500.00$ each year, so $V_0=12\,000$, $V_{n+1}=V_n-1500$ and after four years $V_4=12\,000-6000=\$6\,000.00$. Machine A is worth $8000-6000=\$2\,000.00$ more after four years.

Q16. Over five years the asset loses $5\times 850=\$4\,250.00$. Since it is worth $5,750.00 after that loss, the purchase price was $5750+4250=\$10\,000.00$.

Q17. The total interest is $7440-6000=\$1\,440.00$ over four years, so the yearly interest is $\frac{1440}{4}=\$360.00$. As a percentage of the $6,000 principal the rate is $\frac{360}{6000}\times 100=6\,\%$ per annum, and the recurrence is $V_0=6000$, $V_{n+1}=V_n+360$.

Q18. The van must lose $40\,000-25\,000=\$15\,000.00$ of value. At $0.30 per kilometre this takes $\frac{15\,000}{0.30}=50\,000$ km. (Modelling per kilometre, $V_{n+1}=V_n-0.30$ with n counting kilometres, the value reaches $25,000.00 at $n=50\,000$.)

Q19. With $D=\$3\,500.00$ each year, $V_0=21\,000$, $V_{n+1}=V_n-3500$. After four years $V_4=21\,000-4\times 3500=21\,000-14\,000=\$7\,000.00$. The value reaches $0 when the total depreciation equals the purchase price: $\frac{21\,000}{3500}=6$ years.

Q20. (a) The starting term is the purchase price, \$13,000.00. (b) Each year \$1,625.00 is subtracted, so the annual depreciation is \$1,625.00. (c) As a percentage of the purchase price this is $\frac{1625}{13000} \times 100 = 12.5\%$. (d) Iterating, $V_1 = 11375$, $V_2 = 9750$, $V_3 = 8125$, $V_4 = \$6500.00$. (e) The value falls by exactly \$1,625.00 every year, a constant first difference, so the plotted points are equally spaced and lie on a straight line. If the value were instead *multiplied* by a constant each year, the amount of change would be different every year (a constant *ratio* rather than a constant difference), and the graph would be a curve.

Theory

In an earlier section you met the first-order linear recurrence relation

$$V_0 = a, V_{n+1} = V_n + D,$$

in which each term is found from the one before it by adding a fixed amount D, the **common difference**. When $D > 0$ the sequence rises by equal steps (**linear growth**); when $D < 0$ it falls by equal steps (**linear decay**). Generating a distant term by iterating one step at a time is slow — to reach V_{50} you would apply the rule fifty times. This section develops an **explicit** (closed-form) rule that produces *any* term directly.

Building the explicit rule.

Start at V_0 and add D repeatedly:

$$V_1 = V_0 + D, V_2 = V_1 + D = V_0 + 2D, V_3 = V_0 + 3D, \ldots$$

Each extra step adds one more D, so after n steps exactly n lots of D have been added to the starting value. This gives the **explicit rule** for a linear model:

$$V_n = V_0 + nD.$$

Here V_0 is the starting value and D is the constant change per period. Because V_n is a degree-one (linear) expression in n, the points (n, V_n) lie on a **straight line** of gradient D and vertical intercept V_0 — which is why these are called **linear** (or straight-line) models.

Finding any term. Substitute the period number straight into the rule; for example $V_{25} = V_0 + 25D$, with no iterating required.

Solving for the period n. To find when the value reaches a target T, put $V_n = T$ and solve

$$T = V_0 + nD \Rightarrow n = \frac{T - V_0}{D}.$$

The period counter n takes whole-number values. If the solution is not a whole number, interpret it: a growing value first *reaches or passes* a target, and a falling value first *drops below* a target, at the next whole period up.

Simple interest. Interest is called **simple** when it is calculated only on the original principal P, so the same amount is added each period. At a rate of $r\%$ per period the interest per period is the constant

$$D = \frac{r}{100} \times P,$$

so simple interest is a linear-growth model with $V_0 = P$:

$$V_n = P + \frac{r}{100} Pn = P\left(1 + \frac{rn}{100}\right).$$

The **total interest** after n periods is $V_n - P = \frac{r}{100} Pn$.

Flat-rate (straight-line) depreciation. The value of an asset is written down by the same amount each period, most often a fixed percentage $r\%$ of the **original purchase price** V_0. The depreciation per period is the constant

$$d = \frac{r}{100} \times V_0,$$

so the book value is a linear-decay model:

$$V_n = V_0 - d\,n.$$

The asset reaches a stated scrap (written-down) value when V_n equals it; solve for n as above.

Unit-cost depreciation. Here the value falls by a fixed amount for each **unit of use** — each kilometre driven, each item produced — rather than with time. If the asset loses c dollars per unit and has been used for m units,

$$V = V_0 - c\,m,$$

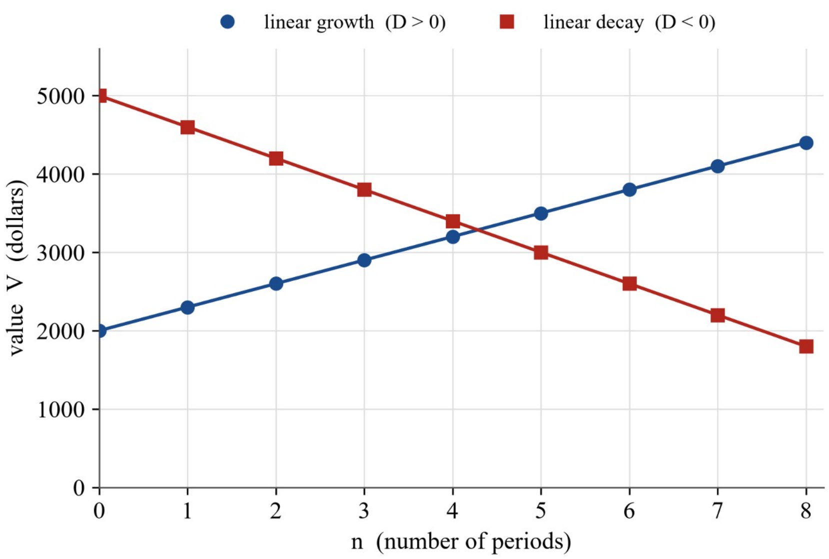
which is the same linear rule with $D = -c$ and the number of units m in place of n.

Both flat-rate and unit-cost depreciation are linear models: the book value falls along a straight line. (Reducing-balance depreciation, met later in this chapter, instead *multiplies* by a fixed factor each period and is **not** linear.)

Model	Constant change D	Explicit rule
Simple interest	$+\frac{r}{100}P$	$V_n = P + \frac{r}{100}P\,n$
Flat-rate depreciation	$-\frac{r}{100}V_0$	$V_n = V_0 - \frac{r}{100}V_0\,n$
Unit-cost depreciation	$-c$ per unit used	$V = V_0 - c\,m$

Using CAS. General Mathematics is technology-active. You may generate the sequence from the recurrence, or simply evaluate the explicit rule for the required n; to find the period for a target value, solve the linear equation $V_0 + n\,D = T$ with the equation solver. Work with exact values and round only the final money answer to the nearest cent.

The graph below shows the two shapes: the terms of a linear-growth model and a linear-decay model each sit exactly on a straight line, rising or falling by the same step every period.



Worked examples

Example 1 — scaffolded

A savings plan begins with $500.00, and $120.00 is added at the end of each month; the account pays no interest. Let V_n be the balance after n months. Find an explicit rule for V_n, the balance after 18 months, and the number of months until the balance reaches $3,380.00.

Working	Comment
$V_0 = 500$; each month adds $120.00, so $D = 120$.	The balance rises by the same amount each month — linear growth.
$V_{n+1} = V_n + 120$.	The recurrence: add the fixed amount to the previous balance.
$V_n = V_0 + nD = 500 + 120n$.	The explicit rule: n lots of $120.00 added to the $500.00 start.
$V_{18} = 500 + 120 \times 18 = 500 + 2160 = 2660$.	Substitute $n = 18$; the balance is $2,660.00.
$3380 = 500 + 120n \Rightarrow 120n = 2880 \Rightarrow n = 24$.	Solve for n: the balance reaches $3,380.00 after 24 months.

Example 2 — scaffolded

$4,000.00 is invested at 5% per annum simple interest, with interest paid at the end of each year. Let V_n be the value after n years. Find the explicit rule, the value after 6 years, the total interest earned in that time, and when the investment is first worth $6,000.00.

Working	Comment
$D = \frac{5}{100} \times 4000 = 200$.	Simple interest: 5% of the original $4,000.00 every year.
$V_{n+1} = V_n + 200$, so $V_n = 4000 + 200n$.	Linear growth from the principal $4,000.00.
$V_6 = 4000 + 200 \times 6 = 5200$.	Value after 6 years: $5,200.00.
Interest $= V_6 - 4000 = 1200$.	Total interest $= 6 \times \$200.00 = \$1\,200.00$.
$6000 = 4000 + 200n \Rightarrow 200n = 2000 \Rightarrow n = 10$.	The investment first reaches $6,000.00 after 10 years.

Example 3 — unscaffolded

A commercial oven is bought for $18,000.00 and depreciated by the flat-rate method at 12% of its purchase price each year. Find its value after 5 years, and the first year in which its book value drops below $3,600.00.

The annual depreciation is a fixed percentage of the purchase price,

$$d = \frac{12}{100} \times 18000 = 2160,$$

so $2,160.00 is written off every year. With $V_0 = 18000$ the book value follows the linear-decay rule

$$V_n = 18000 - 2160n.$$

After 5 years, $V_5 = 18000 - 2160 \times 5 = 18000 - 10800 = 7200$, that is $7,200.00.

To find when the value drops below $3,600.00, solve

$$18000 - 2160n = 3600 \Rightarrow 2160n = 14400 \Rightarrow n = \frac{14400}{2160} = 6.67\,(2\text{ d.p.}).$$

Depreciation is applied in whole years, so the book value is still $5,040.00 after 6 years ($18000 - 2160 \times 6$) and $2,880.00 after 7 years. It therefore first drops below $3,600.00 during the 7th year.

∴ the oven is worth $7,200.00 after 5 years, and its book value first falls below $3,600.00 after 7 years.

Example 4 — unscaffolded

A delivery van is bought for $45,000.00 and depreciated by the unit-cost method at 30 cents per kilometre travelled. (a) Find its value after it has travelled 60,000 km. (b) The van is written off when its book value falls to $9,000.00; how far will it have travelled by then?

Each kilometre reduces the value by $0.30, so after m kilometres the book value is

$$V = 45000 - 0.30\,m.$$

(a) After $m = 60\,000$ km,

$$V = 45000 - 0.30 \times 60000 = 45000 - 18000 = 27000,$$

that is $27,000.00.

(b) Set $V = 9000$ and solve for m:

$$45000 - 0.30\,m = 9000 \Rightarrow 0.30\,m = 36000 \Rightarrow m = \frac{36000}{0.30} = 120000.$$

∴ the van is worth $27,000.00 after 60,000 km, and it will have travelled 120,000 km when it is written off at $9,000.00.

Practice questions

Scaffolded

Q1. A sequence has $V_0 = 300$ and $V_{n+1} = V_n + 45$. (a) State the common difference D. (b) Write the explicit rule for V_n. (c) Find V_{12}. (d) Find the value of n for which $V_n = 1200$.

Q2. $2,500.00 is invested at 4% per annum simple interest. (a) Find the interest added each year, D. (b) Write the recurrence relation. (c) Write the explicit rule for the value V_n after n years. (d) Find the value after 8 years. (e) Find the total interest earned in 8 years.

Q3. A laptop bought for $3,200.00 is depreciated by the flat-rate method at 15% of its purchase price each year. (a) Find the annual depreciation d. (b) Write the explicit rule for the book value V_n after n years. (c) Find the value after 4 years. (d) Find the value after 6 years.

Q4. A photocopier bought for $12,000.00 is depreciated by the unit-cost method at 5 cents per copy made. (a) Write the rule for the value V after m copies. (b) Find the value after 40,000 copies. (c) Find the number of copies made when the value has fallen to $4,000.00.

Q5. $5,000.00 is invested at 6% per annum simple interest. (a) Write the explicit rule for the value V_n after n years. (b) Find the value after 5 years. (c) After how many whole years does the investment first exceed $7,000.00?

Q6. A van bought for $28,000.00 is depreciated by the flat-rate method at 12% of its purchase price each year. (a) Find the annual depreciation d. (b) Write the explicit rule for the book value V_n after n years. (c) Find the value after 5 years. (d) In which year does the book value first drop below $7,000.00?

Unscaffolded

Q7. A sequence has $V_0 = 850$ and $V_{n+1} = V_n - 60$. Write the explicit rule, find V_9, and state whether the model is linear growth or linear decay.

Q8. $6,000.00 is invested at 3.5% per annum simple interest. Find the value after 10 years and the total interest earned.

Q9. A machine bought for $24,000.00 is depreciated by the flat-rate method at 8% of its purchase price each year. Write the explicit rule, find the book value after 7 years, and find the first year in which the value drops below $5,000.00.

Q10. A car bought for $32,000.00 is depreciated by the unit-cost method at 22 cents per kilometre. Find its value after 50,000 km, and the distance travelled when its value has fallen to $10,000.00.

Q11. $1,500.00 is invested at 8% per annum simple interest. After how many whole years is the investment first worth more than $3,000.00?

Q12. Investment A of $8,000.00 earns 5% per annum simple interest; investment B of $10,000.00 earns 3% per annum simple interest. After how many years are the two investments equal in value, and what is that value?

Q13. An asset is depreciated by the flat-rate method. Its book value is $9,400.00 after 3 years and $6,600.00 after 7 years. Find the annual depreciation D, the purchase price V_0, and the explicit rule.

Q14. A taxi bought for $38,000.00 is depreciated by the unit-cost method at 15 cents per kilometre, and it travels 40,000 km each year. (a) Write a rule for its book value V_t after t years. (b) Find its value after 4 years. (c) In which year does its value first drop below $9,000.00?

Q15. An investment earns simple interest. Its value after 6 years is $4,540.00, and the total interest earned in that time is $540.00. Find the annual interest D, the principal invested, and the annual interest rate.

Q16. An asset's value falls by the same number of dollars each year. Explain why the explicit rule $V_n = V_0 + nD$ applies, why the graph of value against time is a straight line, and state the sign of D.

Q17. A machine costing $50,000.00 has a scrap value of $5,000.00 at the end of a useful life of 9 years, and is depreciated by the flat-rate (straight-line) method to that scrap value. Find the annual depreciation, the explicit rule, and the book value after 4 years.

Q18. An investment of $12,000.00 grows to $15,600.00 in 5 years under simple interest. Find the annual interest, the annual interest rate, the explicit rule, and the value after 12 years.

Q19. A truck bought for $60,000.00 is written off at $12,000.00 after travelling 300,000 km, using unit-cost depreciation. Find the depreciation per kilometre (in cents), the explicit rule, and the book value after 175,000 km.

Q20. A delivery vehicle costing $40,000.00 may be depreciated either by the flat-rate method at 10% of its purchase price each year, or by the unit-cost method at 20 cents per kilometre with an expected 32,000 km travelled each year. (a) Write a per-year rule for the book value under each method. (b) Find the book value under each method after 6 years. (c) State which method gives the lower book value after 6 years.

Answers

Q1. (a) $D=45$; (b) $V_n=300+45n$; (c) $V_{12}=840$; (d) $n=20$. **Q2.** (a) $D=\$100.00$; (b) $V_{n+1}=V_n+100$; (c) $V_n=2500+100n$; (d) \$3,300.00; (e) \$800.00. **Q3.** (a) $d=\$480.00$; (b) $V_n=3200-480n$; (c) \$1,280.00; (d) \$320.00. **Q4.** (a) $V=12000-0.05m$; (b) \$10,000.00; (c) 160,000 copies. **Q5.** (a) $V_n=5000+300n$; (b) \$6,500.00; (c) 7 years (value \$7,100.00). **Q6.** (a) $d=\$3360.00$; (b) $V_n=28000-3360n$; (c) \$11,200.00; (d) year 7 (value \$4,480.00). **Q7.** $V_n=850-60n$; $V_9=\$310.00$; linear decay. **Q8.** \$8,100.00; total interest \$2,100.00. **Q9.** $V_n=24000-1920n$; after 7 years \$10,560.00; first below \$5,000.00 after 10 years (\$4,800.00). **Q10.** $V=32000-0.22k$; after 50,000 km \$21,000.00; value \$10,000.00 at 100,000 km. **Q11.** $V_n=1500+120n$; first exceeds \$3,000.00 after 13 years (value \$3,060.00). **Q12.** Equal after 20 years, both worth \$16,000.00. **Q13.** $D=-\$700.00$; $V_0=\$11500.00$; $V_n=11500-700n$. **Q14.** (a) $V_t=38000-6000t$; (b) \$14,000.00; (c) year 5 (value \$8,000.00). **Q15.** $D=\$90.00$; principal \$4,000.00; rate 2.25% per annum. **Q16.** The value changes by the same amount each year, so D is a common difference and n lots of D are added to V_0, giving $V_n=V_0+nD$; being linear in n the points lie on a straight line of gradient D; here $D<0$ (decay). **Q17.** $d=\$5000.00$; $V_n=50000-5000n$; after 4 years \$30,000.00. **Q18.** $D=\$720.00$; rate 6% per annum; $V_n=12000+720n$; after 12 years \$20,640.00. **Q19.** 16 cents per km; $V=60000-0.16k$; after 175,000 km \$32,000.00. **Q20.** (a) Flat-rate $V=40000-4000t$, unit-cost $V=40000-6400t$; (b) flat-rate \$16,000.00, unit-cost \$1,600.00; (c) the unit-cost method gives the lower book value.

Fully-worked solutions

Q1. The common difference is $D=45$. Adding n lots of 45 to $V_0=300$ gives the explicit rule $V_n=300+45n$. Then $V_{12}=300+45\times12=300+540=840$. Solving $1200=300+45n$ gives $45n=900$, so $n=20$.

Q2. Simple interest adds the same amount each year: $D=\frac{4}{100}\times2500=100$, so the recurrence is $V_{n+1}=V_n+100$ and the explicit rule is $V_n=2500+100n$. After 8 years, $V_8=2500+100\times8=3300$, i.e. \$3,300.00. The total interest is $V_8-2500=800$, i.e. \$800.00 (which is $8\times\$100.00$).

Q3. Flat-rate depreciation writes off a fixed percentage of the purchase price each year: $d=\frac{15}{100}\times3200=480$, so $V_n=3200-480n$. After 4 years, $V_4=3200-480\times4=3200-1920=1280$, i.e. \$1,280.00. After 6 years, $V_6=3200-480\times6=3200-2880=320$, i.e. \$320.00.

Q4. Unit-cost depreciation removes a fixed amount per copy: $\$0.05$ each, so $V=12000-0.05m$. After 40,000 copies, $V=12000-0.05\times40000=12000-2000=10000$, i.e. \$10,000.00. Setting $V=4000$: $4000=12000-0.05m$, so $0.05m=8000$ and $m=\frac{8000}{0.05}=160000$ copies.

Q5. The annual interest is $D=\frac{6}{100}\times5000=300$, so $V_n=5000+300n$. After 5 years, $V_5=5000+300\times5=6500$, i.e. \$6,500.00. Solving $7000=5000+300n$ gives $n=\frac{2000}{300}=6.67$ (2 d.p.); since n must be a whole number, check: $V_6=6800$ (not yet above \$7,000.00) and $V_7=5000+2100=7100$. So the investment first exceeds \$7,000.00 after 7 years.

Q6. The annual depreciation is $d=\frac{12}{100}\times28000=3360$, so $V_n=28000-3360n$. After 5 years, $V_5=28000-3360\times5=28000-16800=11200$, i.e. \$11,200.00. Solving $7000=28000-3360n$ gives $3360n=21000$, so $n=6.25$; since depreciation applies in whole years, $V_6=28000-20160=7840$ (still above) and $V_7=28000-23520=4480$. The value first drops below \$7,000.00 in year 7.

Q7. The common difference is $D=-60$, so $V_n=850-60n$. Then $V_9=850-60\times9=850-540=310$, i.e. \$310.00. Because the value falls by a constant amount each step ($D<0$), the model is **linear decay**.

Q8. The annual interest is $D=\frac{3.5}{100}\times6000=210$, so $V_n=6000+210n$. After 10 years, $V_{10}=6000+210\times10=6000+2100=8100$, i.e. \$8,100.00, and the total interest is $V_{10}-6000=2100$, i.e. \$2,100.00.

Q9. The annual depreciation is $d=\frac{8}{100}\times24000=1920$, so $V_n=24000-1920n$. After 7 years, $V_7=24000-1920\times7=24000-13440=10560$, i.e. \$10,560.00. Solving $5000=24000-1920n$ gives $1920n=19000$, so $n=9.90$ (2 d.p.); checking whole years, $V_9=24000-17280=6720$ (still above) and $V_{10}=24000-19200=4800$. The value first drops below \$5,000.00 after 10 years.

Q10. With \$0.22 removed per kilometre, $V=32000-0.22k$. After 50,000 km, $V=32000-0.22\times50000=32000-11000=21000$, i.e. \$21,000.00. Setting $V=10000$: $10000=32000-0.22k$, so $0.22k=22000$ and $k=\frac{22000}{0.22}=100000$ km.

Q11. The annual interest is $D=\frac{8}{100}\times1500=120$, so $V_n=1500+120n$. Solving $3000=1500+120n$ gives $n=\frac{1500}{120}=12.5$; since n is a whole number, check: $V_{12}=1500+1440=2940$ (not yet above \$3,000.00) and $V_{13}=1500+1560=3060$. The investment is first worth more than \$3,000.00 after 13 years.

Q12. The two values are $V_A=8000+400n$ (since $\frac{5}{100}\times8000=400$) and $V_B=10000+300n$ (since $\frac{3}{100}\times10000=300$). They are equal when $8000+400n=10000+300n$, i.e. $100n=2000$, so $n=20$. Then $V_A=8000+400\times20=16000$ and $V_B=10000+300\times20=16000$. The investments are equal after 20 years, both worth \$16,000.00.

Q13. With the linear rule $V_n=V_0+nD$, the two readings give $V_0+3D=9400$ and $V_0+7D=6600$. Subtracting the first from the second removes V_0: $4D=6600-9400=-2800$, so $D=-700$ (a depreciation of \$700.00 each year). Then $V_0=9400-3D=9400-3\times(-700)=9400+2100=11500$, i.e. \$11,500.00, and the rule is $V_n=11500-700n$.

Q14. At 15 cents per kilometre over 40,000 km, the value falls by $0.15\times40000=6000$ each year, so the per-year rule is $V_t=38000-6000t$. After 4 years, $V_4=38000-6000\times4=38000-24000=14000$, i.e. \$14,000.00. Solving $9000=38000-6000t$ gives $6000t=29000$, so $t=4.83$ (2 d.p.); checking whole years, $V_4=14000$ (above) and $V_5=38000-30000=8000$. The value first drops below \$9,000.00 in year 5.

Q15. Under simple interest the total interest is nD: here $6D=540$, so $D=90$ (\$90.00 per year). The value after 6 years is $P+6D=4540$, so $P=4540-540=4000$, i.e. a principal of \$4,000.00. The annual rate is $r=\frac{D}{P}\times100=\frac{90}{4000}\times100=2.25\%$.

Q16. Because the value changes by the same number of dollars every year, that fixed change is a common difference D. Starting from V_0 and applying the change n times adds exactly n lots of D, giving $V_n=V_0+nD$. This rule is linear (degree one) in n, so plotting value against time produces a straight line whose gradient is D and whose vertical intercept is V_0. Since the value is falling, $D<0$.

Q17. Straight-line depreciation spreads the total fall in value evenly over the useful life. The total depreciation is $50000-5000=45000$ over 9 years, so the annual depreciation is $d=\frac{45000}{9}=5000$ ($5,000.00 per year), and $V_n=50000-5000n$. After 4 years, $V_4=50000-5000\times 4=50000-20000=30000$, i.e. $30,000.00. (As a check, after the 9-year life $V_9=50000-45000=5000$, the scrap value.)

Q18. The total simple interest over 5 years is $15600-12000=3600$, so the annual interest is $D=\frac{3600}{5}=720$ ($720.00 per year). The annual rate is $r=\frac{720}{12000}\times 100=6\%$, and the explicit rule is $V_n=12000+720n$. After 12 years, $V_{12}=12000+720\times 12=12000+8640=20640$, i.e. $20,640.00.

Q19. The total depreciation is $60000-12000=48000$ over 300,000 km, so the cost per kilometre is $\frac{48000}{300000}=0.16$ dollars, i.e. 16 cents per km, and $V=60000-0.16k$. After 175,000 km, $V=60000-0.16\times 175000=60000-28000=32000$, i.e. $32,000.00.

Q20. (a) Flat-rate: the annual depreciation is $\frac{10}{100}\times 40000=4000$, so $V=40000-4000t$. Unit-cost: the annual depreciation is $0.20\times 32000=6400$, so $V=40000-6400t$. (b) After 6 years the flat-rate value is $40000-4000\times 6=40000-24000=16000$ ($16,000.00) and the unit-cost value is $40000-6400\times 6=40000-38400=1600$ ($1,600.00). (c) The unit-cost method gives the lower book value after 6 years ($1,600.00 compared with $16,000.00), because its yearly depreciation of $6,400.00 exceeds the flat-rate $4,000.00.

Chapter 6 Modelling growth and decay using recursion — 6D Modelling geometric growth and decay with recursion

Theory

Many financial quantities change by the same **percentage** each period rather than by the same dollar **amount**. When a quantity is repeatedly **multiplied** by a fixed number, its growth or decay is **geometric**, and it is modelled by a first-order recurrence relation of the form

$$V_{n+1}=RV_n, V_0=a,$$

where V_n is the value after n periods, a is the starting value, and the constant R is the **growth or decay multiplier**. This is the multiplying partner of the *adding* rule $V_{n+1}=V_n+D$ met in an earlier section: there a fixed amount was added each period; here a fixed *factor* is applied.

Growth or decay. Every term is R times the one before it, so the size of R decides the behaviour of the sequence.

- If $R>1$, each term is larger than the last and the value **grows** (for example, a compound-interest investment).
- If $0<R<1$, each term is a fraction of the last and the value **decays** (for example, a depreciating asset).
- If $R=1$, the value never changes.

The multiplier from a percentage rate. If a quantity increases by $r\,\%$ each period, each new value is the old value plus $\frac{r}{100}$ of it:

$$V_{n+1}=V_n+\frac{r}{100}V_n=\left(1+\frac{r}{100}\right)V_n,$$

so $R=1+\frac{r}{100}$. If instead it decreases by $r\,\%$ each period, the same argument gives $R=1-\frac{r}{100}$. In short,

$$R=1\pm\frac{r}{100}\left(+\text{for growth}, -\text{for decay}\right).$$

An increase of $6\,\%$ gives $R=1.06$; a decrease of $15\,\%$ gives $R=0.85$.

Compound interest is a geometric model. With **compound interest** the interest for each period is calculated on the *whole current balance* and added to it, so the balance is multiplied by $R=1+\frac{r}{100}$ each period. This is exactly the geometric rule above. (Contrast **simple interest**, met in an earlier section, where the same dollar amount is added each period — an *arithmetic* model.) When a rate is quoted as a **nominal annual rate** compounded k times a year, the **per-period rate** is

$$r=\frac{\text{annual rate}}{k}.$$

For example, $4.8\,\%$ per annum compounding monthly gives $r=\frac{4.8}{12}=0.4\,\%$ per month, so $R=1.004$; the same $4.8\,\%$ compounding quarterly gives $r=\frac{4.8}{4}=1.2\,\%$ per quarter, so $R=1.012$.

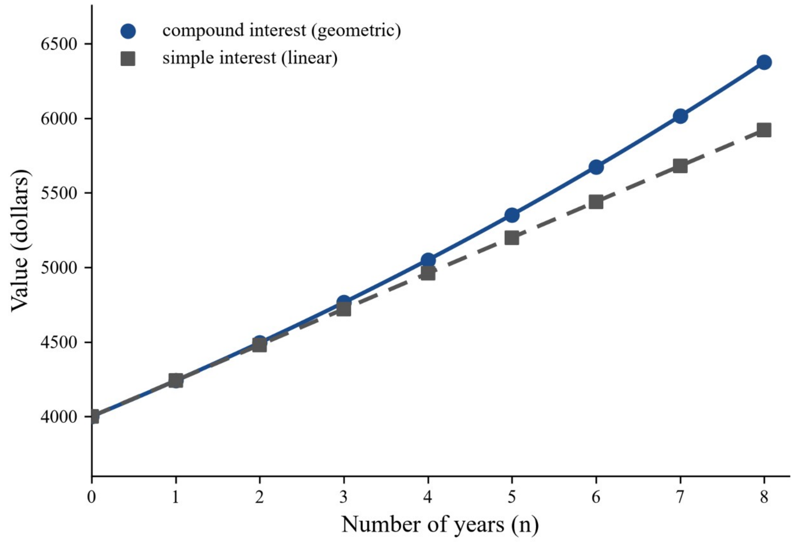
Reducing-balance depreciation is a geometric model. Under **reducing-balance** depreciation an asset loses a fixed *percentage* of its *current* value each period. Losing $r\,\%$ each period multiplies the value by $R=1-\frac{r}{100}$, with $0<R<1$ — geometric decay. (Contrast **flat-rate** depreciation from an earlier section, where the *same dollar amount* is subtracted each year — again an arithmetic model.)

Generating the value sequence. To find later values you **iterate**: apply the rule term by term, $V_1 = R V_0$, then $V_2 = R V_1$, and so on. Because these are amounts of money, compute with full accuracy and **round to the nearest cent only when you write a value down**; rounding part-way through and then continuing lets small errors build up.

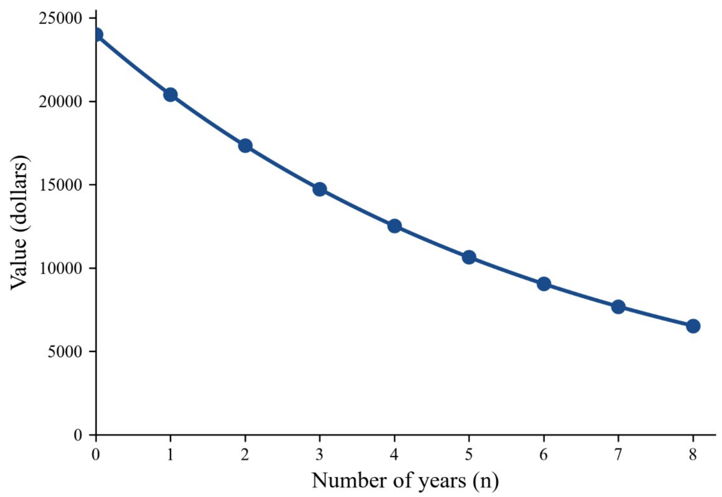
The constant-ratio signature. Because every term is the previous term multiplied by the *same* number, the ratio of consecutive terms is constant:

$$\frac{V_{n+1}}{V_n} = R \text{ for every } n.$$

Testing whether consecutive values share a **common ratio** is how you *recognise* a geometric model — an arithmetic model has a common *difference* instead. Plotted against n, the terms of a geometric sequence lie on a **curve**, not a straight line: for $R > 1$ the points climb ever more steeply, while for $0 < R < 1$ they fall towards zero, flattening as they go. The graph below compares a compound-interest balance (geometric, curved) with a simple-interest balance on the same principal and rate (arithmetic, straight).



The next graph shows reducing-balance depreciation: the value drops by less each year (the same percentage of a smaller amount), so the curve flattens towards the horizontal axis.



A rule that gives V_n directly from n, without stepping through every earlier term, is developed later in this chapter.

Using CAS. General Mathematics is a technology-active, open-book course. Enter the starting value and the rule $V_{n+1} = R V_n$ into your calculator's recursion or sequence facility and generate the terms; this is convenient for reading off a value after many periods or for finding the first period at which a balance passes a target. Learn the by-hand iteration below so you can set the model up correctly and check that the technology output is reasonable.

Worked examples

Example 1 — scaffolded

An investment of $\$4\,000$ earns compound interest at 6% per annum, compounding annually. (a) State the multiplier R and write a recurrence relation for the value V_n after n years. (b) Find the value after each of the first three years. (c) State whether the model shows growth or decay.

Working	Comment
$R = 1 + \frac{6}{100} = 1.06$.	An increase of 6%: $R = 1 + r/100$.
$V_0 = 4000, V_{n+1} = 1.06V_n$.	Starting value and recurrence relation.
$V_1 = 1.06 \times 4000 = 4240$.	Value after 1 year: $\$4\,240.00$.
$V_2 = 1.06 \times 4240 = 4494.40$.	After 2 years: $\$4\,494.40$.
$V_3 = 1.06 \times 4494.40 = 4764.064$.	After 3 years: $\$4\,764.06$ (nearest cent).
$R = 1.06 > 1$, so the value grows.	Each term is larger than the last.

Example 2 — scaffolded

A commercial oven bought for $\$24\,000$ depreciates by 15% of its value each year (reducing-balance). (a) Write a recurrence relation for its value V_n after n years. (b) Find its value after three years. (c) Check that the ratio of consecutive values is constant.

Working	Comment
$R = 1 - \frac{15}{100} = 0.85$.	A decrease of 15%: $R = 1 - r/100$, and $0 < R < 1$.
$V_0 = 24000, V_{n+1} = 0.85V_n$.	Reducing-balance model.
$V_1 = 0.85 \times 24000 = 20400$.	After 1 year: $\$20\,400.00$.
$V_2 = 0.85 \times 20400 = 17340$.	After 2 years: $\$17\,340.00$.
$V_3 = 0.85 \times 17340 = 14739$.	After 3 years: $\$14\,739.00$.
$\frac{20400}{24000} = \frac{17340}{20400} = \frac{14739}{17340} = 0.85$.	Constant ratio — the geometric signature.

Example 3 — unscaffolded

A savings account pays interest at a nominal 7.2% per annum, compounding monthly. An amount of $\$8\,000$ is deposited and left untouched. Write a recurrence relation for the balance after n months and find the balance after three months.

The nominal annual rate is shared equally over the 12 monthly periods, so the per-period rate is $r = \frac{7.2}{12} = 0.6\%$ per month, giving a multiplier

$$R = 1 + \frac{0.6}{100} = 1.006.$$

With $V_0 = 8000$ the recurrence relation is $V_{n+1} = 1.006V_n$. Iterating,

$$V_1 = 1.006 \times 8000 = 8048,$$

$$V_2 = 1.006 \times 8048 = 8096.288,$$

$$V_3 = 1.006 \times 8096.288 = 8144.865728.$$

$\therefore$ the recurrence relation is $V_{n+1}=1.006V_n$, $V_0=8000$, and after three months the balance is $\$8\,144.87$ (to the nearest cent).

Example 4 — unscaffolded

A car bought for $\$36\,000$ has its value modelled by $V_{n+1}=0.88V_n$, $V_0=36000$, where V_n is its value after n years. State the annual depreciation rate, find the car's value after five years, and explain why a graph of V_n against n is curved rather than straight.

Here $R=0.88$. Writing $R=1-\frac{r}{100}$ gives $\frac{r}{100}=0.12$, so $r=12$: the car loses 12% of its value each year, and since $0<R<1$ the model is geometric decay. Iterating from $V_0=36000$,

$$V_1=31680, V_2=27878.40, V_3=24532.992, V_4=21589.03296, V_5=18998.3490048.$$

So after five years the value is $\$18\,998.35$. Each value is the previous one multiplied by the *same* factor 0.88, so the amount lost each year is 12% of a smaller and smaller value: the drops get smaller every year. The points therefore fall by less and less and lie on a curve that flattens towards the axis, not on a straight line (which would require an equal drop each year).

$\therefore$ the car depreciates at 12% per year, is worth $\$18\,998.35$ after five years, and its value–time graph is a decreasing curve.

Practice questions

Scaffolded

Q1. For each annual change, state whether it represents growth or decay and give the multiplier R. (a) an increase of 8% (b) a decrease of 8% (c) an increase of 3.5% (d) a decrease of 12.5%

Q2. $\$5\,000$ is invested at 6% per annum, compounding annually. (a) State the multiplier R. (b) Write a recurrence relation for the value V_n after n years. (c) Find the value after 1, 2 and 3 years.

Q3. A laptop bought for $\$2\,000$ depreciates by 20% each year (reducing-balance). (a) State the multiplier R. (b) Write a recurrence relation for its value V_n after n years. (c) Find its value after 1, 2 and 3 years.

Q4. $\$12\,000$ is invested at a nominal 6% per annum, compounding quarterly. (a) Find the interest rate per quarter. (b) State the multiplier R. (c) Write a recurrence relation and find the value after 1, 2 and 3 quarters.

Q5. The value of a photocopier after n years is modelled by $V_{n+1}=0.9V_n$, $V_0=3000$. (a) State the annual depreciation rate. (b) State whether the model shows growth or decay. (c) Find its value after 1, 2 and 3 years.

Q6. The value (in dollars) of an investment at the end of each of four consecutive years is $800, 840, 882, 926.10$. (a) Show that the ratio of consecutive values is constant. (b) State the multiplier R and the annual percentage growth rate. (c) Write a recurrence relation for the value V_n after n years.

Unscaffolded

Q7. $\$7\,000$ is invested at 5% per annum, compounding annually. Write a recurrence relation for the balance after n years, and find the balance after 4 years.

Q8. A tractor bought for $\$85\,000$ depreciates by 9% per annum (reducing-balance). Write a recurrence relation for its value, and find its value after 3 years.

Q9. $\$15\,000$ is invested at a nominal 7.2% per annum, compounding monthly. State the monthly interest rate and the multiplier R, and find the balance after 6 months.

Q10. An investment is modelled by $V_{n+1}=1.025V_n$, $V_0=20000$, where each period is a quarter. State the quarterly growth rate, state whether the model shows growth or decay, and find the value after 4 quarters.

Q11. A printing press bought for $\$9\,000$ is depreciated at 4% of its value each quarter (reducing-balance). State the multiplier R, and find its value after 1 year (4 quarters).

Q12. $\$2\,500$ is invested at a nominal 8% per annum, compounding half-yearly. State the interest rate per half-year and the multiplier R, and find the balance after 2 years.

Q13. Under a compound-interest model the balance is $\$5\,136$ after 1 year and $\$5\,495.52$ after 2 years. (a) Find the multiplier R. (b) State the annual interest rate. (c) Find the amount originally invested.

Q14. A delivery van worth $\$28\,000$ when new is worth $\$23\,800$ after one year under a reducing-balance model. (a) Find the multiplier R and the annual depreciation rate. (b) Predict its value after 3 years.

Q15. $\$1\,000$ is invested at 12% per annum, compounding annually. Find its value after 10 years.

Q16. $\$6\,000$ is invested at 5% per annum, compounding annually. By iterating the recurrence relation, find the first year at the end of which the balance exceeds $\$7\,000$.

Q17. A quantity is modelled by $V_{n+1}=RV_n$. (a) Explain why the ratio $\frac{V_{n+1}}{V_n}$ is the same for every n. (b) Describe the shape of the graph of V_n against n when $R>1$, and when $0<R<1$.

Q18. One investment grows by $\$50$ each year; another grows by 5% each year. State which one is modelled by a geometric recurrence relation, and explain how the shape of its value–time graph differs from that of the other.

Q19. A business buys equipment for $\$50\,000$ that depreciates at 25% per annum (reducing-balance). (a) Write a recurrence relation for its value V_n after n years. (b) Find its value after 4 years. (c) The equipment is scrapped once its value first falls below $\$15\,000$. In which year does this happen?

Q20. $\$3\,000$ is invested at a nominal 9.6% per annum, compounding quarterly. State the interest rate per quarter and the multiplier R, and find the balance after 1 year and after 2 years.

Answers

Q1. (a) growth, $R = 1.08$; (b) decay, $R = 0.92$; (c) growth, $R = 1.035$; (d) decay, $R = 0.875$. **Q2.** $R = 1.06$; $V_{n+1} = 1.06V_n$, $V_0 = 5000$; $\$5\,300.00$, $\$5\,618.00$, $\$5\,955.08$. **Q3.** $R = 0.80$; $V_{n+1} = 0.80V_n$, $V_0 = 2000$; $\$1\,600.00$, $\$1\,280.00$, $\$1\,024.00$. **Q4.** (a) 1.5% per quarter; (b) $R = 1.015$; (c) $V_{n+1} = 1.015V_n$, $V_0 = 12000$; $\$12\,180.00$, $\$12\,362.70$, $\$12\,548.14$. **Q5.** (a) 10%; (b) decay; (c) $\$2\,700.00$, $\$2\,430.00$, $\$2\,187.00$. **Q6.** (a) $840 \div 800 = 882 \div 840 = 926.10 \div 882 = 1.05$; (b) $R = 1.05$, growth of 5%; (c) $V_{n+1} = 1.05V_n$, $V_0 = 800$. **Q7.** $V_{n+1} = 1.05V_n$, $V_0 = 7000$; $\$8\,508.54$. **Q8.** $V_{n+1} = 0.91V_n$, $V_0 = 85000$; $\$64\,053.54$. **Q9.** 0.6% per month, $R = 1.006$; $\$15\,548.17$. **Q10.** 2.5% per quarter; growth; $\$22\,076.26$. **Q11.** $R = 0.96$; $\$7\,644.12$. **Q12.** 4% per half-year, $R = 1.04$; $\$2\,924.65$. **Q13.** (a) $R = 1.07$; (b) 7%; (c) $\$4\,800.00$. **Q14.** (a) $R = 0.85$, depreciation of 15%; (b) $\$17\,195.50$. **Q15.** $\$3\,105.85$. **Q16.** Year 4 (the balance is $\$7\,293.04$, having been $\$6\,945.75$ after year 3). **Q17.** (a) each term is the previous one multiplied by R, so dividing consecutive terms always gives R; (b) $R > 1$: an increasing curve that gets steeper; $0 < R < 1$: a decreasing curve that flattens towards zero. **Q18.** The 5% investment (constant ratio 1.05) is geometric; its value–time graph is a curve, whereas the $\$50$ investment (constant difference) gives a straight line. **Q19.** (a) $V_{n+1} = 0.75V_n$, $V_0 = 50000$; (b) $\$15\,820.31$; (c) year 5. **Q20.** 2.4% per quarter, $R = 1.024$; $\$3\,298.53$ after 1 year, $\$3\,626.78$ after 2 years.

Fully-worked solutions

Q1. Using $R = 1 \pm \frac{r}{100}$: (a) $R = 1 + 0.08 = 1.08$ (growth); (b) $R = 1 - 0.08 = 0.92$ (decay); (c) $R = 1 + 0.035 = 1.035$ (growth); (d) $R = 1 - 0.125 = 0.875$ (decay).

Q2. An increase of 6% gives $R = 1 + \frac{6}{100} = 1.06$, so $V_{n+1} = 1.06V_n$ with $V_0 = 5000$. Iterating: $V_1 = 1.06 \times 5000 = 5300$; $V_2 = 1.06 \times 5300 = 5618$; $V_3 = 1.06 \times 5618 = 5955.08$. The values are $\$5\,300.00$, $\$5\,618.00$ and $\$5\,955.08$.

Q3. A decrease of 20% gives $R = 1 - \frac{20}{100} = 0.80$, so $V_{n+1} = 0.80V_n$ with $V_0 = 2000$. Iterating: $V_1 = 0.80 \times 2000 = 1600$; $V_2 = 0.80 \times 1600 = 1280$; $V_3 = 0.80 \times 1280 = 1024$. The values are $\$1\,600.00$, $\$1\,280.00$ and $\$1\,024.00$.

Q4. (a) The nominal rate is spread over 4 quarters: $r = \frac{6}{4} = 1.5\%$ per quarter. (b) $R = 1 + \frac{1.5}{100} = 1.015$. (c) $V_{n+1} = 1.015V_n$ with $V_0 = 12000$: $V_1 = 1.015 \times 12000 = 12180$; $V_2 = 1.015 \times 12180 = 12362.70$; $V_3 = 1.015 \times 12362.70 = 12548.1405$. The values are $\$12\,180.00$, $\$12\,362.70$ and $\$12\,548.14$.

Q5. (a) $R = 0.9 = 1 - \frac{r}{100}$ gives $\frac{r}{100} = 0.1$, so $r = 10$: a depreciation rate of 10% per year. (b) Since $0 < R < 1$ the model shows decay. (c) $V_1 = 0.9 \times 3000 = 2700$; $V_2 = 0.9 \times 2700 = 2430$; $V_3 = 0.9 \times 2430 = 2187$. The values are $\$2\,700.00$, $\$2\,430.00$ and $\$2\,187.00$.

Q6. (a) $\frac{840}{800} = 1.05$, $\frac{882}{840} = 1.05$ and $\frac{926.10}{882} = 1.05$: the ratio of consecutive values is constant, so the growth is geometric. (b) The common ratio is $R = 1.05$, which is $1 + \frac{5}{100}$, an annual growth rate of 5%. (c) $V_{n+1} = 1.05V_n$ with $V_0 = 800$.

Q7. An increase of 5% gives $R = 1.05$, so $V_{n+1} = 1.05V_n$ with $V_0 = 7000$. Iterating: $V_1 = 7350$, $V_2 = 7717.50$, $V_3 = 8103.375$, $V_4 = 8508.54375$. After 4 years the balance is $\$8\,508.54$.

Q8. A decrease of 9% gives $R=1-0.09=0.91$, so $V_{n+1}=0.91V_n$ with $V_0=85000$. Iterating: $V_1=77350$, $V_2=70388.50$, $V_3=64053.535$. After 3 years the value is $\$64\,053.54$.

Q9. The monthly rate is $r=\frac{7.2}{12}=0.6\%$, so $R=1+\frac{0.6}{100}=1.006$ and $V_{n+1}=1.006V_n$ with $V_0=15000$. Iterating six times gives $V_1=15090$, $V_2=15180.54$, $V_3=15271.62324$, $V_4=15363.25297\ldots$, $V_5=15455.43249\ldots$, $V_6=15548.16509\ldots$. After 6 months the balance is $\$15\,548.17$.

Q10. Here $R=1.025=1+\frac{2.5}{100}$, a growth rate of 2.5% per quarter; since $R>1$ the model shows growth. Iterating $V_{n+1}=1.025V_n$ from $V_0=20000$: $V_1=20500$, $V_2=21012.50$, $V_3=21537.8125$, $V_4=22076.2578125$. After 4 quarters the value is $\$22\,076.26$.

Q11. A decrease of 4% per quarter gives $R=1-\frac{4}{100}=0.96$, so $V_{n+1}=0.96V_n$ with $V_0=9000$. One year is 4 quarters: $V_1=8640$, $V_2=8294.40$, $V_3=7962.624$, $V_4=7644.11904$. After 1 year the value is $\$7\,644.12$.

Q12. Compounding half-yearly means 2 periods a year, so $r=\frac{8}{2}=4\%$ per half-year and $R=1.04$. With $V_0=2500$ and 2 years $=4$ half-years: $V_1=2600$, $V_2=2704$, $V_3=2812.16$, $V_4=2924.6464$. After 2 years the balance is $\$2\,924.65$.

Q13. (a) The balance is multiplied by the same R each year, so $R=\frac{5495.52}{5136}=1.07$. (b) $R=1+\frac{r}{100}=1.07$ gives $r=7$: an interest rate of 7% per annum. (c) The amount after 1 year is R times the original, so $V_0=\frac{5136}{1.07}=4800$; the original investment was $\$4\,800.00$.

Q14. (a) $R=\frac{23800}{28000}=0.85$, and $R=1-\frac{r}{100}=0.85$ gives $r=15$: a depreciation rate of 15% per year. (b) $V_{n+1}=0.85V_n$ with $V_0=28000$: $V_2=0.85\times23800=20230$ and $V_3=0.85\times20230=17195.50$. After 3 years the value is $\$17\,195.50$.

Q15. With $R=1.12$ and $V_0=1000$, iterate $V_{n+1}=1.12V_n$ ten times:
$1120, 1254.40, 1404.928, 1573.51936, 1762.3416832, 1973.822685\ldots, 2210.681407\ldots, 2475.963176\ldots, 2773.07875$
. After 10 years the value is $\$3\,105.85$.

Q16. With $R=1.05$ and $V_0=6000$: $V_1=6300$, $V_2=6615$, $V_3=6945.75$, $V_4=7293.0375$. The balance first exceeds $\$7\,000$ at the end of **year 4**, when it reaches $\$7\,293.04$ (after year 3 it was only $\$6\,945.75$).

Q17. (a) In $V_{n+1}=RV_n$ every term is the previous term multiplied by the same constant R, so $\frac{V_{n+1}}{V_n}=R$ for every n — a constant ratio. (b) When $R>1$ each term is larger than the last by a growing amount, so the points form an increasing curve that gets steeper. When $0<R<1$ each term is a fraction of the last, so the points form a decreasing curve that flattens as it approaches zero.

Q18. Growing by a fixed $\$50$ each year adds a constant *amount*: this is $V_{n+1}=V_n+50$, an arithmetic (linear) model whose graph is a straight line. Growing by 5% each year multiplies by a constant *factor* $R=1.05$: this is $V_{n+1}=1.05V_n$, a geometric model. So the 5% investment is the geometric one, and its value–time graph is an upward curve that steepens over time, rather than the straight line of the $\$50$ investment.

Q19. (a) A decrease of 25% gives $R=1-\frac{25}{100}=0.75$, so $V_{n+1}=0.75V_n$ with $V_0=50000$. (b) Iterating: $V_1=37500$, $V_2=28125$, $V_3=21093.75$, $V_4=15820.3125$, so after 4 years the value is $\$15\,820.31$. (c) Continuing,

$V_5 = 0.75 \times 15820.3125 = 11865.234375$. Since $V_4 = \$15\,820.31$ is still above $\$15\,000$ but $V_5 = \$11\,865.23$ is below it, the value first falls below $\$15\,000$ during **year 5**.

Q20. The quarterly rate is $r = \frac{9.6}{4} = 2.4\,\%$, so $R = 1 + \frac{2.4}{100} = 1.024$ and $V_{n+1} = 1.024\,V_n$ with $V_0 = 3000$. One year is 4 quarters: $V_4 = 3000 \times 1.024^4$; iterating gives $V_1 = 3072$, $V_2 = 3145.728$, $V_3 = 3221.225472$, $V_4 = 3298.534883\ldots$, so the balance after 1 year is $\$3\,298.53$. Continuing to 8 quarters (2 years): $V_5 = 3377.699720\ldots$, $V_6 = 3458.764513\ldots$, $V_7 = 3541.774862\ldots$, $V_8 = 3626.777458\ldots$, so the balance after 2 years is $\$3\,626.78$.

Theory

An earlier section modelled geometric growth and decay with the recurrence relation

$$V_{n+1} = R V_n,$$

with a stated starting value V_0. Each term is the previous term multiplied by the constant **growth multiplier** R. To reach a term far along the sequence this way you must generate every term before it. This section develops an **explicit rule** (a closed-form rule) that gives V_n directly from n.

Deriving the explicit rule. Apply the recurrence repeatedly, beginning at V_0:

$$V_1 = R V_0, V_2 = R V_1 = R^2 V_0, V_3 = R V_2 = R^3 V_0,$$

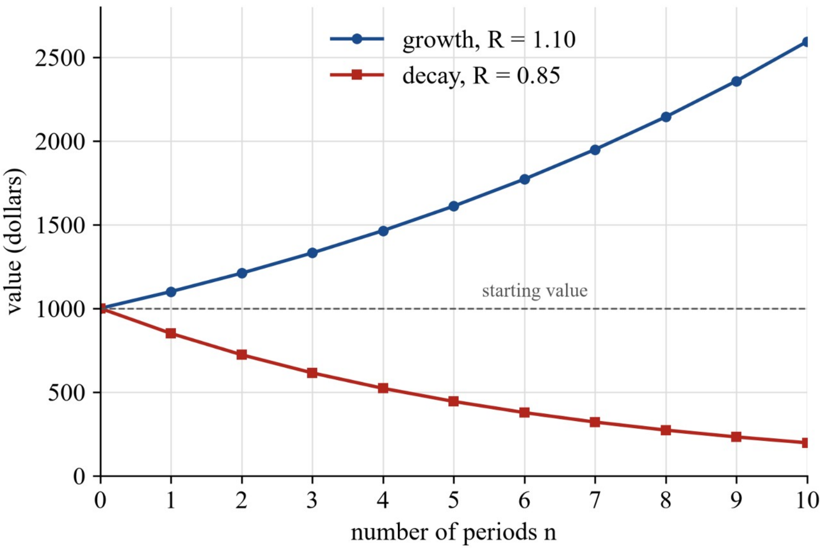
and each further step contributes one more factor of R. After n periods there are n factors, so

$$V_n = R^n V_0.$$

This is the **explicit rule** for a geometric model: the starting value V_0 is multiplied by R once for each of the n periods. Because the rule involves R^n, the terms change by a constant *ratio* rather than a constant amount, so a graph of V_n against n is a **curve** — rising ever more steeply when $R > 1$, and falling towards zero when $0 < R < 1$.

- If $R > 1$ the model **grows**: each term is larger than the last.
- If $0 < R < 1$ the model **decays**: each term is smaller than the last.

The explicit rule gives exactly the same terms as iterating the recurrence — it is a shortcut, and any value it produces can be checked by generating a few terms directly.



Compound interest. When a principal P is invested at a fixed rate per period, the interest earned each period is added to the balance, so the next period's interest is calculated on a larger amount. This is **compound interest**. If the rate is r% per compounding period, each balance is multiplied by

$$R = 1 + \frac{r}{100}$$

to obtain the next, so with $V_0 = P$ the explicit rule $V_n = R^n V_0$ becomes the **compound-interest formula**

$$A = P\left(1 + \frac{r}{100}\right)^n,$$

where A is the **future value** (the amount after n periods), P is the **principal** (the starting amount), r is the interest rate **per compounding period** and n is the **number of compounding periods**.

Interest is usually quoted as a *nominal annual rate* but compounded several times a year. If the nominal annual rate is compounded k times per year, then

$$r = \frac{\text{annual rate}}{k}, n = k \times (\text{number of years}).$$

For example, 6% per annum compounding monthly gives $r = 6/12 = 0.5\%$ per month, so $R = 1.005$, and 3 years is $n = 36$ months.

Reducing-balance depreciation. Many assets lose a fixed *percentage* of their value each period. This is **reducing-balance depreciation**: if the value falls by $r\%$ per period, each value is multiplied by

$$R = 1 - \frac{r}{100} (0 < R < 1),$$

so the explicit rule is

$$V_n = \left(1 - \frac{r}{100}\right)^n V_0,$$

where V_0 is the purchase price and V_n is the value after n periods. Because $0 < R < 1$ the value decays towards zero but never reaches it.

Solving for the number of periods. Sometimes the target value is known and the unknown is n — for example, *how many years until an investment first exceeds a given amount*, or *until an asset first drops below a given value*. Writing the target value as T and dividing $R^n V_0 = T$ by the starting value gives

$$R^n = \frac{T}{V_0},$$

an equation with the unknown in the exponent. General Mathematics is technology-active, so solve it with the CAS `solve` command (or read n from a finance solver). Alternatively — and this is often quicker — evaluate V_n for successive whole values of n, by hand or with the calculator's sequence facility, and take the first one that meets the condition. That systematic search always works here because $R^n V_0$ moves steadily in one direction: always up when $R > 1$, always down when $0 < R < 1$.

Because interest is only added (and depreciation only applied) at whole-number periods, the terms exist only for whole-number n. The solver returns the exact value at which the target is reached, and it is rarely a whole number. If that value is, say, $n = 6.64$, the target is first met at the **next whole period**, $n = 7$: round *up* — for growth and for decay alike — then confirm by evaluating V_6 and V_7.

Beyond this course. The equation $R^n = T / V_0$ can also be solved algebraically, by taking logarithms of both sides. That method belongs to Mathematical Methods and is *not* required in General Mathematics: solving with technology, then confirming with the two neighbouring whole periods, is the expected working here.

Using CAS. The compound-interest formula and the depreciation rule can each be evaluated directly for a given n, and the sequence facility will list the terms of $V_{n+1} = R V_n$ if you would rather read a value off the list. Set the rule up by hand first, so that you enter it correctly and can judge whether the technology's answer is reasonable.

Worked examples

Example 1 — scaffolded

\$5,000 is invested at 6% per annum, compounding annually. (a) Write down the growth multiplier R and the explicit rule for the value after n years. (b) Use the rule to find the value after 4 years, to the nearest cent.

Working	Comment
$R=1+\frac{6}{100}=1.06$.	A 6% rise multiplies each balance by 1.06.
$V_n=R^n V_0=1.06^n\times 5\,000$.	Explicit rule with principal $V_0=5\,000$.
$V_4=1.06^4\times 5\,000=1.26247696\times 5\,000$.	Substitute $n=4$ to find the term directly.
$=6\,312.3848$, so $V_4\approx \$6\,312.38$.	Round the future value to the nearest cent.

Example 2 — scaffolded

A machine is bought for \$24,000 and depreciates by 15% each year by the reducing-balance method. (a) Write the explicit rule for its value after n years. (b) Find its value after 3 years. (c) Find the first year at the end of which its value is below \$10,000.

Working	Comment
$R=1-\frac{15}{100}=0.85$.	Reducing balance: the value falls to 85% each year.
$V_n=0.85^n\times 24\,000$.	Explicit rule with $V_0=24\,000$.
$V_3=0.85^3\times 24\,000=0.614125\times 24\,000=\$14\,739.00$.	Value after 3 years.
Need $0.85^n\times 24\,000<10\,000$.	Set up the condition for part (c).
$0.85^n<\frac{10\,000}{24\,000}=\frac{5}{12}$.	Divide both sides by $24\,000$.
Solving $0.85^n=\frac{5}{12}$ with CAS gives $n\approx 5.39$.	The value passes \$10,000 part-way through year 6.
So $n=6$.	The value only changes at whole years, and it is falling, so round up.
Check: $V_5=\$10\,648.93$ ($>10\,000$), $V_6=\$9\,051.59$ ($<10\,000$).	Confirms year 6 is the first below \$10,000.

Example 3 — unscaffolded

\$8,000 is invested at a nominal 4.8% per annum, compounding monthly. Find (a) the value after 2 years, and (b) the number of whole months for the investment to first exceed \$10,000.

The monthly rate is $r=4.8/12=0.4\%$, so $R=1+0.4/100=1.004$ and the explicit rule is $V_n=1.004^n\times 8\,000$, with n measured in months.

(a) Two years is $n=24$ months, so

$$V_{24}=1.004^{24}\times 8\,000=1.1005483\ldots\times 8\,000\approx \$8\,804.39.$$

(b) Require $1.004^n\times 8\,000>10\,000$, that is $1.004^n>1.25$. Solving this equation with the CAS `solve` command,

$$1.004^n=1.25 \text{ gives } n\approx 55.90.$$

Since n counts whole months and the balance is rising, round up to $n = 56$. Checking, $V_{55} \approx \$9964.24$ and $V_{56} \approx \$10004.10$, so the balance first exceeds \$10,000 after 56 months.

∴ the value after 2 years is \$8,804.39, and the investment first exceeds \$10,000 after 56 months (4 years 8 months).

Example 4 — unscaffolded

A car is bought for \$36,000 and depreciates by 12% per year by the reducing-balance method. (a) Write the explicit rule and find the car's value after 5 years. (b) Find the number of whole years for the car to first be worth less than half its purchase price.

Each year the value falls to 88% of the previous year, so $R = 1 - 12/100 = 0.88$ and

$$V_n = 0.88^n \times 36000.$$

(a) After 5 years,

$$V_5 = 0.88^5 \times 36000 = 0.5277319\ldots \times 36000 \approx \$18998.35.$$

(b) Half the purchase price is \$18,000, so require $0.88^n \times 36000 < 18000$, that is $0.88^n < 0.5$. Solving with CAS,

$$0.88^n = 0.5 \text{ gives } n \approx 5.42,$$

and the value is falling, so it stays below half from the next whole year on: $n = 6$. Indeed $V_5 \approx \$18998.35$ (just above half) while $V_6 \approx \$16718.55$ (below half).

∴ the value after 5 years is \$18,998.35, and the car is first worth less than half its purchase price after 6 years.

Practice questions

Scaffolded

Q1. A geometric model has $V_{n+1} = 1.05V_n$ with $V_0 = 2000$. (a) Write down the growth multiplier R. (b) Write the explicit rule for V_n. (c) Use the rule to find V_1, V_2 and V_3. (d) Use the rule to find V_{10}, to the nearest cent.

Q2. \$4,000 is invested at 7% per annum, compounding annually. (a) State the growth multiplier R. (b) Write the compound-interest rule $A = P(1 + r/100)^n$ for this investment. (c) Find the value after 5 years. (d) Find the value after 10 years.

Q3. A laptop bought for \$1,800 depreciates by 20% per year (reducing balance). (a) State the multiplier R. (b) Write the explicit rule for its value after n years. (c) Find its value after 1, 2 and 3 years. (d) Find its value after 4 years.

Q4. \$3,000 is invested at 8% per annum, compounding annually. Find the number of whole years for the value to first exceed \$5,000. (a) Write the inequality $3000 \times 1.08^n > 5000$. (b) Divide both sides by 3000 to show $1.08^n > 5/3$. (c) Solve $1.08^n = 5/3$ with CAS, giving n correct to two decimal places. (d) Round up to find the number of whole years, and check by evaluating V_6 and V_7.

Q5. \$1,500 is invested at a nominal 6% per annum, compounding monthly. (a) Find the monthly rate and the multiplier R. (b) State the number of compounding periods in 3 years. (c) Find the value after 3 years, to the nearest cent.

Q6. A machine bought for \$50,000 depreciates by 10% per year (reducing balance). (a) Write the explicit rule for its value after n years. (b) Use the rule to find its value after 3 years. (c) Check the answer to (b) by iterating the recurrence $V_{n+1} = 0.9V_n$ from $V_0 = 50000$.

Unscaffolded

Q7. \$6,000 is invested at 5.5% per annum, compounding annually. Find its value after 8 years.

Q8. A painting bought for $12,000 appreciates in value by 8% each year. Find its value after 6 years and after 12 years.

Q9. A delivery van bought for $45,000 depreciates by 18% per year (reducing balance). Find its value after 4 years and after 7 years.

Q10. $10,000 is invested at a nominal 3.6% per annum, compounding monthly. Find its value after 5 years.

Q11. $2,500 is invested at 9% per annum, compounding annually. Find the number of whole years for the investment to double in value.

Q12. A forklift bought for $60,000 depreciates by 22% per year (reducing balance). After how many whole years does its value first fall below $20,000?

Q13. $5,000 is invested at a nominal 4% per annum, compounding quarterly. Find its value after 3 years and after 5 years.

Q14. Two investments each start with $8,000 and run for 10 years. Investment A earns 6% per annum compounding annually; investment B earns a nominal 5.9% per annum compounding monthly. Find the value of each after 10 years and state which is worth more.

Q15. A tractor bought for $80,000 depreciates by 16% per year (reducing balance). (a) Find its value after 5 years. (b) The owner will sell it once its value first drops below $30,000; after how many years is that?

Q16. $20,000 is invested at a nominal 3% per annum, compounding monthly. Find the number of whole months for the balance to first exceed $25,000, and express this in years and months.

Q17. A share portfolio worth $15,000 grows by 11% each year. (a) Write the explicit rule and find its value after 5 years. (b) Show, by iterating the recurrence for the first three years, that the rule gives the same values.

Q18. $9,000 is invested at a nominal 8% per annum, compounding quarterly. Find the number of whole quarters for the balance to first exceed $12,000, and express this in years.

Q19. Office equipment bought for $28,000 depreciates by 25% per year (reducing balance). (a) Find its value after 2 years and after 4 years. (b) What percentage of its original value remains after 4 years?

Q20. $7,000 is invested at 6% per annum, compounding annually. (a) Find its value after 12 years. (b) Find the number of whole years for the investment to double in value. (c) Comment on how your answers to (a) and (b) are consistent.

Answers

Q1. $R=1.05$; $V_n=1.05^n\times 2\,000$; $V_1=\$2\,100.00$, $V_2=\$2\,205.00$, $V_3=\$2\,315.25$; $V_{10}\approx\$3\,257.79$. **Q2.** $R=1.07$; $A=4\,000\times 1.07^n$; after 5 years \$5,610.21; after 10 years \$7,868.61. **Q3.** $R=0.8$; $V_n=1\,800\times 0.8^n$; \$1,440.00, \$1,152.00, \$921.60; after 4 years \$737.28. **Q4.** $1.08^n>5/3$; solving $1.08^n=5/3$ gives $n\approx 6.64$; 7 years. **Q5.** $r=0.5\,\%$, $R=1.005$; $n=36$; \$1,795.02. **Q6.** $V_n=50\,000\times 0.9^n$; after 3 years \$36,450.00; iterating gives \$45,000.00, \$40,500.00, \$36,450.00 — the same. **Q7.** \$9,208.12. **Q8.** After 6 years \$19,042.49; after 12 years \$30,218.04. **Q9.** After 4 years \$20,345.48; after 7 years \$11,217.85. **Q10.** \$11,968.95. **Q11.** $1.09^n>2$; solving $1.09^n=2$ gives $n\approx 8.04$; 9 years. **Q12.** $0.78^n<1/3$; solving $0.78^n=1/3$ gives $n\approx 4.42$; 5 years. **Q13.** $R=1.01$ per quarter; after 3 years \$5,634.13; after 5 years \$6,100.95. **Q14.** A \$14,326.78; B \$14,411.06; investment B is worth more. **Q15.** (a) \$33,456.96; (b) 6 years. **Q16.** $1.0025^n>1.25$; solving $1.0025^n=1.25$ gives $n\approx 89.37$; 90 months = 7 years 6 months. **Q17.** (a) $V_n=15\,000\times 1.11^n$, after 5 years \$25,275.87; (b) iterating gives \$16,650.00, \$18,481.50, \$20,514.47, matching the rule. **Q18.** $1.02^n>4/3$; solving $1.02^n=4/3$ gives $n\approx 14.53$; 15 quarters = 3.75 years (3 years 9 months). **Q19.** (a) \$15,750.00 and \$8,859.38; (b) $0.75^4=31.64\,\%$. **Q20.** (a) \$14,085.38; (b) $1.06^n>2$, solving $1.06^n=2$ gives $n\approx 11.90$, 12 years; (c) after 12 years the value has just passed double, agreeing with (b).

Fully-worked solutions

Q1. The multiplier is $R=1.05$, so $V_n=1.05^n\times 2\,000$. Then $V_1=1.05\times 2\,000=\$2\,100.00$, $V_2=1.05^2\times 2\,000=1.1025\times 2\,000=\$2\,205.00$ and $V_3=1.05^3\times 2\,000=1.157625\times 2\,000=\$2\,315.25$. For the tenth term, $V_{10}=1.05^{10}\times 2\,000=1.628894\ldots\times 2\,000\approx\$3\,257.79$.

Q2. A $7\,\%$ annual rise gives $R=1.07$, so $A=4\,000\times 1.07^n$. After 5 years, $A=1.07^5\times 4\,000=1.402551\ldots\times 4\,000\approx\$5\,610.21$. After 10 years, $A=1.07^{10}\times 4\,000=1.967151\ldots\times 4\,000\approx\$7\,868.61$.

Q3. A $20\,\%$ fall gives $R=1-0.20=0.8$, so $V_n=1\,800\times 0.8^n$. Then $V_1=0.8\times 1\,800=\$1\,440.00$, $V_2=0.64\times 1\,800=\$1\,152.00$ and $V_3=0.512\times 1\,800=\$921.60$. After 4 years, $V_4=0.8^4\times 1\,800=0.4096\times 1\,800=\$737.28$.

Q4. (a) The value after n years is $3\,000\times 1.08^n$, so the condition is $3\,000\times 1.08^n>5\,000$. (b) Dividing by $3\,000$ gives $1.08^n>5/3$. (c) Solving the equation $1.08^n=5/3$ with the CAS `solve` command gives $n\approx 6.64$. (d) Interest is added only at whole years and the balance is rising, so the value first exceeds \$5,000 at $n=7$ years. (Check: $V_6=1.08^6\times 3\,000\approx\$4\,760.62<5\,000$ and $V_7=1.08^7\times 3\,000\approx\$5\,141.47>5\,000$.)

Q5. (a) The monthly rate is $r=6/12=0.5\,\%$, so $R=1+0.5/100=1.005$. (b) Three years is $n=12\times 3=36$ months. (c) Then $V_{36}=1.005^{36}\times 1\,500=1.196680\ldots\times 1\,500\approx\$1\,795.02$.

Q6. (a) A $10\,\%$ fall gives $R=0.9$, so $V_n=50\,000\times 0.9^n$. (b) After 3 years, $V_3=0.9^3\times 50\,000=0.729\times 50\,000=\$36\,450.00$. (c) Iterating the recurrence from $V_0=50\,000$: $V_1=0.9\times 50\,000=\$45\,000.00$, $V_2=0.9\times 45\,000=\$40\,500.00$, $V_3=0.9\times 40\,500=\$36\,450.00$ — the same value the rule gives.

Q7. With $R=1.055$, $V_8=1.055^8\times 6\,000=1.534687\ldots\times 6\,000\approx\$9\,208.12$.

Q8. An $8\,\%$ annual rise gives $R=1.08$. After 6 years, $V_6=1.08^6\times 12\,000=1.586874\ldots\times 12\,000\approx\$19\,042.49$; after 12 years, $V_{12}=1.08^{12}\times 12\,000=2.518170\ldots\times 12\,000\approx\$30\,218.04$.

Q9. An $18\,\%$ fall gives $R=0.82$. After 4 years, $V_4=0.82^4\times 45\,000=0.452122\ldots\times 45\,000\approx\$20\,345.48$; after 7 years, $V_7=0.82^7\times 45\,000=0.249285\ldots\times 45\,000\approx\$11\,217.85$.

Q10. The monthly rate is $r = 3.6/12 = 0.3\,\%$, so $R = 1.003$ and $n = 12 \times 5 = 60$. Then $V_{60} = 1.003^{60} \times 10\,000 = 1.196895\ldots \times 10\,000 \approx \$\,11\,968.95$.

Q11. With $R = 1.09$, "doubling" means $1.09^n \times 2500 > 5\,000$, that is $1.09^n > 2$. Solving $1.09^n = 2$ with CAS gives $n \approx 8.04$, and the balance is rising, so round up to $n = 9$ years. (Check: $V_8 = 1.09^8 \times 2500 \approx \$\,4\,981.41 < 5\,000$ and $V_9 = 1.09^9 \times 2500 \approx \$\,5\,429.73 > 5\,000$.)

Q12. A $22\,\%$ fall gives $R = 0.78$. Require $0.78^n \times 60\,000 < 20\,000$, that is $0.78^n < 1/3$. Solving $0.78^n = 1/3$ with CAS gives $n \approx 4.42$, and the value is falling, so it is first below \$20,000 at $n = 5$ years. (Check: $V_4 \approx \$\,22\,209.03 > 20\,000$ and $V_5 \approx \$\,17\,323.05 < 20\,000$.)

Q13. The quarterly rate is $r = 4/4 = 1\,\%$, so $R = 1.01$. Three years is $n = 12$ quarters: $V_{12} = 1.01^{12} \times 5\,000 = 1.126825\ldots \times 5\,000 \approx \$\,5\,634.13$. Five years is $n = 20$ quarters: $V_{20} = 1.01^{20} \times 5\,000 = 1.220190\ldots \times 5\,000 \approx \$\,6\,100.95$.

Q14. Investment A: $R = 1.06$ and $n = 10$, so $V_{10} = 1.06^{10} \times 8\,000 = 1.790848\ldots \times 8\,000 \approx \$\,14\,326.78$. Investment B: the monthly rate is $5.9/12\,\%$, so $R = 1 + 5.9/1200 = 1.004916\overline{6}$ and $n = 120$ months, giving $V_{120} = R^{120} \times 8\,000 = 1.801382\ldots \times 8\,000 \approx \$\,14\,411.06$. Since $\$\,14\,411.06 > \$\,14\,326.78$, investment B is worth more after 10 years.

Q15. (a) A $16\,\%$ fall gives $R = 0.84$, so $V_5 = 0.84^5 \times 80\,000 = 0.418212\ldots \times 80\,000 \approx \$\,33\,456.96$. (b) Require $0.84^n \times 80\,000 < 30\,000$, that is $0.84^n < 0.375$. Solving $0.84^n = 0.375$ with CAS gives $n \approx 5.63$, and the value is falling, so round up to $n = 6$ years. (Check: $V_5 \approx \$\,33\,456.96 > 30\,000$ and $V_6 \approx \$\,28\,103.84 < 30\,000$.)

Q16. The monthly rate is $r = 3/12 = 0.25\,\%$, so $R = 1.0025$. Require $1.0025^n \times 20\,000 > 25\,000$, that is $1.0025^n > 1.25$. Solving $1.0025^n = 1.25$ with CAS gives $n \approx 89.37$, and the balance is rising, so round up to $n = 90$ months. (Check: $V_{89} \approx \$\,24\,976.98 < 25\,000$ and $V_{90} \approx \$\,25\,039.42 > 25\,000$.) In years, $90 \div 12 = 7$ remainder 6, so 7 years 6 months.

Q17. (a) With $R = 1.11$, $V_n = 15\,000 \times 1.11^n$, so $V_5 = 1.11^5 \times 15\,000 = 1.685058\ldots \times 15\,000 \approx \$\,25\,275.87$. (b) Iterating from $V_0 = 15\,000$: $V_1 = 1.11 \times 15\,000 = \$\,16\,650.00$, $V_2 = 1.11 \times 16\,650 = \$\,18\,481.50$, $V_3 = 1.11 \times 18\,481.50 = \$\,20\,514.47$. The rule gives the same first three terms ($1.11^1, 1.11^2, 1.11^3$ times $15\,000$), confirming they agree.

Q18. The quarterly rate is $r = 8/4 = 2\,\%$, so $R = 1.02$. Require $1.02^n \times 9\,000 > 12\,000$, that is $1.02^n > 4/3$. Solving $1.02^n = 4/3$ with CAS gives $n \approx 14.53$, and the balance is rising, so round up to $n = 15$ quarters. (Check: $V_{14} \approx \$\,11\,875.31 < 12\,000$ and $V_{15} \approx \$\,12\,112.82 > 12\,000$.) In years, $15 \div 4 = 3.75$ years (3 years 9 months).

Q19. A $25\,\%$ fall gives $R = 0.75$, so $V_n = 28\,000 \times 0.75^n$. (a) $V_2 = 0.75^2 \times 28\,000 = 0.5625 \times 28\,000 = \$\,15\,750.00$ and $V_4 = 0.75^4 \times 28\,000 = 0.316406\ldots \times 28\,000 \approx \$\,8\,859.38$. (b) The fraction of the original value remaining after 4 years is $0.75^4 = 0.31640625$, that is $31.64\,\%$.

Q20. (a) With $R = 1.06$, $V_{12} = 1.06^{12} \times 7\,000 = 2.012196\ldots \times 7\,000 \approx \$\,14\,085.38$. (b) Doubling means $1.06^n > 2$, and solving $1.06^n = 2$ with CAS gives $n \approx 11.90$, so rounding up to the next whole year gives $n = 12$ years. (Check: $V_{11} \approx \$\,13\,288.09 < 14\,000$ and $V_{12} \approx \$\,14\,085.38 > 14\,000$.) (c) The two results agree: the value after 12 years, \$14,085.38, has just passed double the \$7,000 principal (\$14,000), which is exactly why 12 years is the first whole year the investment more than doubles.

Theory

Interest on an investment or a loan is almost always advertised as a **nominal annual interest rate** — a headline rate quoted *per annum* (p.a.), such as “6% p.a.” A nominal rate is only a starting point, because interest is usually **compounded** (added to the balance) more than once a year: monthly, quarterly, weekly or daily. This section shows how to split a nominal rate into the rate that actually applies each compounding period, how to count the periods, and how to build a single **effective annual rate** that lets you compare accounts on equal terms.

The rate per compounding period.

If interest is compounded k times per year, the nominal annual rate is shared equally between the periods. The **rate per compounding period** is

$$r = \frac{\text{nominal annual rate}}{k},$$

so a nominal rate of $i\%$ p.a. gives $r = \frac{i}{k}\%$ each period. The usual values of k are set out below.

Compounded	Periods per year k	Rate per period r
annually	1	$i\%$
semi-annually (half-yearly)	2	$\frac{i}{2}\%$
quarterly	4	$\frac{i}{4}\%$
monthly	12	$\frac{i}{12}\%$
weekly	52	$\frac{i}{52}\%$
daily	365	$\frac{i}{365}\%$

For example, 6% p.a. compounded monthly means $r = \frac{6}{12} = 0.5\%$ **per month**, not 6% per month. The matching **growth multiplier** for each period is

$$R = 1 + \frac{r}{100}.$$

Counting the periods. Because the recurrence relation of this chapter, $V_{n+1} = RV_n$ with V_0 the starting amount, advances **one compounding period at a time**, the number of periods must match the compounding frequency. Over a term of T years,

$$n = k \times T.$$

So 4 years compounded monthly is $n = 12 \times 4 = 48$ months, and the value after 4 years is V_{48}. Always convert the years to periods *before* iterating or substituting.

The effective annual interest rate.

Two accounts can advertise the same nominal rate yet return different amounts, because compounding more often lets earlier interest start earning interest sooner. To compare fairly we ask: *what single rate, compounded just once a year,* *would produce the same growth?* That rate is the **effective annual interest rate**. If the rate per period is $r\,\%$ and there are k periods in a year, then one year multiplies the balance by $R^k=\left(1+\dfrac{r}{100}\right)^k$, so

$$r_{\text{eff}}=\left[\left(1+\frac{r}{100}\right)^k-1\right]\times 100\,\%.$$

Written directly from a nominal rate of $i\,\%$ p.a. (so that $r=\dfrac{i}{k}$), this is

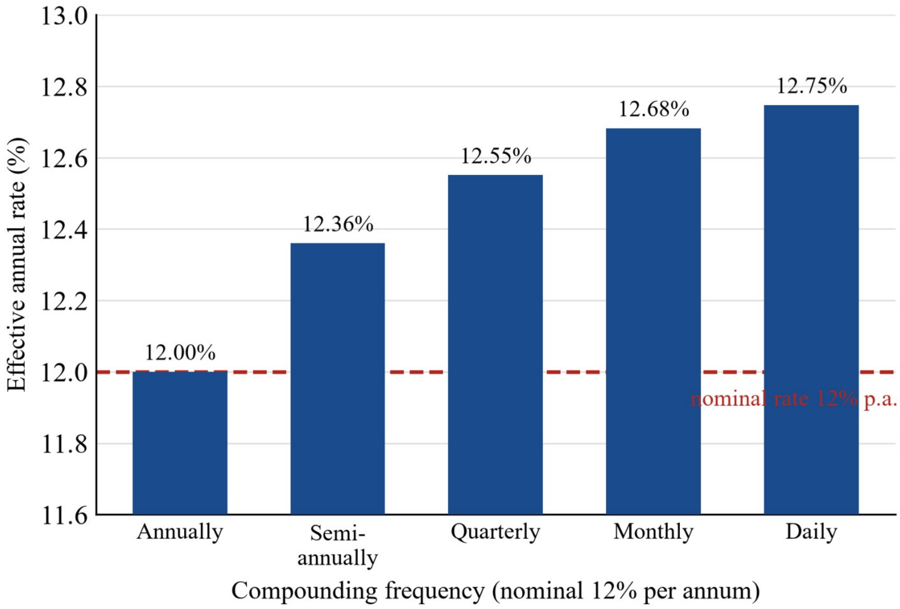
$$r_{\text{eff}}=\left[\left(1+\frac{i}{100\,k}\right)^k-1\right]\times 100\,\%.$$

Using effective rates to compare. The effective rate puts every account on the same footing — one year, compounded once — so the rates can be compared directly, whatever their compounding frequencies:

- for an **investment**, the account with the **higher** effective rate returns more;
- for a **loan**, the offer with the **lower** effective rate costs less.

Nominal rates cannot be compared this way unless they are compounded equally often.

The effect of more frequent compounding. For a *fixed* nominal rate, the effective rate **increases** as the compounding becomes more frequent, but by smaller and smaller amounts. When interest is compounded annually ($k=1$) there is nothing to convert, so the effective rate **equals** the nominal rate. The bar chart below shows the effective rate of a nominal 12% p.a. rate climbing from 12.00% (annually) towards about 12.75% (daily).



Rounding convention. Interest calculations are done on a CAS calculator, and per-period rates are often non-terminating (for instance $8\div 12=0.6\overline{6}\,\%$). Keep the **full** value in the calculator through every step and round **only the final answer** — effective rates are reported here correct to **two decimal places**. When two effective rates are very close, keep extra decimals so that the comparison is not decided by rounding.

Worked examples

Example 1 — scaffolded

A nominal interest rate of 7.2% p.a. is applied to an investment. Find the rate per compounding period, the period growth multiplier R, and the number of compounding periods n when the interest is compounded: (a) monthly for 4 years; (b) quarterly for 4 years; (c) daily for 3 years.

Working	Comment
(a) $k=12$, so $r=\frac{7.2}{12}=0.6\%$ per month.	Divide the nominal rate by the number of periods per year.
$R=1+\frac{0.6}{100}=1.006$.	Period growth multiplier $R=1+\frac{r}{100}$.
$n=12\times 4=48$ months.	$n=k\times T$; the value after 4 years is V_{48}.
(b) $k=4$, so $r=\frac{7.2}{4}=1.8\%$ per quarter.	Four quarters in a year.
$R=1+\frac{1.8}{100}=1.018$; $n=4\times 4=16$ quarters.	
(c) $k=365$, so $r=\frac{7.2}{365}\approx 0.019726\%$ per day.	Keep $7.2\div 365$ in the calculator; do not round yet.
$R=1+\frac{7.2}{365\times 100}\approx 1.00019726$; $n=365\times 3=1095$ days.	

Example 2 — scaffolded

Interest of 8% p.a. is compounded quarterly. Find the effective annual interest rate, correct to two decimal places.

Working	Comment
$k=4$, so $r=\frac{8}{4}=2\%$ per quarter.	Rate per compounding period.
$R=1+\frac{2}{100}=1.02$.	Growth multiplier per quarter.
$R^4=1.02^4=1.08243216$.	One year is four quarters, so multiply by R four times.
$r_{\text{eff}}=(1.08243216-1)\times 100\%=8.243216\%$.	Subtract 1 and convert to a percentage.
$r_{\text{eff}}\approx 8.24\%$.	Round only at the end.

An effective rate of 8.24% means that 8% p.a. compounded quarterly grows an investment by the same amount in a year as 8.24% p.a. compounded annually.

Example 3 — unscaffolded

Two banks advertise savings accounts. Account A pays 6.0% p.a. compounded quarterly; Account B pays 5.95% p.a. compounded monthly. Which account gives the better return?

For Account A, the rate per quarter is $\frac{6.0}{4}=1.5\%$, so $R=1.015$ and

$$r_{\text{eff}}=\left(1.015^4-1\right)\times 100\%=6.136355\ldots\%\approx 6.14\%.$$

For Account B, the rate per month is $\frac{5.95}{12}\%$, so $R=1+\frac{5.95}{1200}$ and

$$r_{\text{eff}}=\left[\left(1+\frac{5.95}{1200}\right)^{12}-1\right]\times 100\%=6.114973\ldots\%\approx 6.11\%.$$

Account A has the higher effective rate (6.14% against 6.11%), so it returns more over a year. Even though Account B compounds more often, Account A’s higher nominal rate outweighs that advantage.

∴ Account A gives the better return.

Example 4 — unscaffolded

A nominal interest rate of 12% p.a. is offered. Find the effective annual rate, correct to two decimal places, when the interest is compounded annually, semi-annually, quarterly, monthly and daily, and describe the pattern.

Divide 12% by k, raise the multiplier to the power k, subtract 1 and convert to a percentage in each case.

Compounded	k	Rate per period	$r_{\text{eff}} = (R^k - 1) \times 100\,\%$
annually	1	12%	12.00%
semi-annually	2	6%	$(1.06^2 - 1) \times 100\,\% = 12.3$
quarterly	4	3%	$(1.03^4 - 1) \times 100\,\% \approx 12.5$
monthly	12	1%	$(1.01^{12} - 1) \times 100\,\% \approx 12.6$
daily	365	$\dfrac{12}{365}\,\%$	$\left[\left(1 + \dfrac{12}{36500}\right)^{365} - 1\right] \times 100$

The effective rate rises as the interest is compounded more frequently, because interest begins to earn interest sooner. The increases get smaller each time (0.36, then 0.19, then 0.13, then 0.07 of a per cent), so the effective rate approaches a ceiling rather than growing without limit. Compounded annually, the effective rate equals the nominal 12%.

∴ the effective rates are 12.00%, 12.36%, 12.55%, 12.68% and 12.75%, increasing with the compounding frequency by ever-smaller steps.

Practice questions

Scaffolded

Q1. Find the interest rate per compounding period for each nominal annual rate. (a) 9% p.a. compounded monthly. (b) 10% p.a. compounded quarterly. (c) 7.3% p.a. compounded daily. (d) 6% p.a. compounded semi-annually.

Q2. Find the number of compounding periods n for each term. (a) 5 years, compounded monthly. (b) 3 years, compounded quarterly. (c) 2 years, compounded daily. (d) 4 years, compounded semi-annually. (e) 10 years, compounded weekly.

Q3. Interest of 8% p.a. is compounded monthly. Find the effective annual rate by completing the steps below. (a) State the number of compounding periods per year, k. (b) Find the rate per month r (keep the exact value $8 \div 12$). (c) Write down the growth multiplier $R = 1 + \dfrac{r}{100}$. (d) Evaluate R^{12}. (e) Find $r_{\text{eff}} = (R^{12} - 1) \times 100\,\%$, correct to two decimal places.

Q4. Interest of 6% p.a. is compounded daily ($k = 365$). Find the effective annual rate by completing the steps below. (a) Find the rate per day r. (b) Write down the growth multiplier $R = 1 + \dfrac{r}{100}$. (c) Evaluate R^{365}. (d) Find r_{eff}, correct to two decimal places.

Q5. Account A pays 7.2% p.a. compounded monthly; Account B pays 7.35% p.a. compounded quarterly. (a) Find the effective annual rate of Account A. (b) Find the effective annual rate of Account B. (c) State which account gives the better return.

Q6. A nominal rate of 10% p.a. is offered. (a) Find the effective annual rate if it is compounded quarterly. (b) Find the effective annual rate if it is compounded monthly. (c) State which is larger, and explain why.

Unscaffolded

Q7. A personal loan is charged interest at 18% p.a. compounded monthly. Find the interest rate per month and the number of compounding periods in 3 years.

Q8. An investment earns 4.8% p.a. compounded monthly. Find the effective annual interest rate, correct to two decimal places.

Q9. Find the effective annual interest rate for 9% p.a. compounded quarterly, correct to two decimal places.

Q10. Find the effective annual interest rate for 15% p.a. compounded daily, correct to two decimal places.

Q11. An investor compares two term deposits: A at 8.4% p.a. compounded monthly and B at 8.5% p.a. compounded quarterly. Find the effective annual rate of each and state which is the better investment.

Q12. Two personal loans are offered for the same amount: Loan A at 11.4% p.a. compounded monthly and Loan B at 11.5% p.a. compounded quarterly. Find the effective annual rate of each, correct to four decimal places, and state which loan is cheaper.

Q13. A nominal rate of 12% p.a. is available. Show, by calculating both effective rates, that compounding monthly gives a higher effective annual rate than compounding annually.

Q14. $5000 is invested at 6% p.a. compounded monthly for 2 years. (a) Find the effective annual interest rate, correct to two decimal places. (b) Find the value of the investment after 2 years, to the nearest cent, and confirm that using the effective rate over 2 whole years gives the same value as iterating the monthly recurrence for 24 months.

Q15. (a) Explain why two savings accounts advertising the same nominal interest rate, but compounding at different frequencies, do not give the same return over a year. (b) State the one compounding frequency for which the effective annual rate equals the nominal rate.

Q16. A credit card charges interest at 1.5% per month. (a) State the nominal annual interest rate. (b) Find the effective annual interest rate, correct to two decimal places.

Q17. Three savings accounts are advertised: Account A: 5% p.a. compounded semi-annually. Account B: 4.9% p.a. compounded monthly. Account C: 5.1% p.a. compounded quarterly. Find the effective annual rate of each and rank the accounts from best to worst for an investor.

Q18. A term deposit pays 7% p.a. compounded weekly ($k=52$). Find the effective annual interest rate, correct to two decimal places.

Q19. A store offers finance at 24% p.a. Find the effective annual interest rate if the interest is charged (a) monthly and (b) daily, each correct to two decimal places, and state which charging arrangement costs the customer more.

Q20. A borrower compares two car loans of equal size. Loan P is charged 9.6% p.a. compounded monthly; Loan Q is charged 9.7% p.a. compounded quarterly. (a) Find the effective annual rate of each loan, correct to two decimal places. (b) State which loan is cheaper. (c) Explain how the loan with the lower nominal rate turns out to be the cheaper one here, even though it compounds more frequently.

Answers

Q1. (a) 0.75% per month; (b) 2.5% per quarter; (c) 0.02% per day; (d) 3% per half-year. **Q2.** (a) 60; (b) 12; (c) 730; (d) 8; (e) 520. **Q3.** (a) $k=12$; (b) $r=\frac{8}{12}\approx 0.6667\%$; (c) $R=1.00\overline{6}$; (d) $R^{12}\approx 1.083000$; (e) $r_{\text{eff}}\approx 8.30\%$. **Q4.** (a) $r=\frac{6}{365}\approx 0.016438\%$; (b) $R\approx 1.00016438$; (c) $R^{365}\approx 1.061831$; (d) $r_{\text{eff}}\approx 6.18\%$. **Q5.** (a) 7.44%; (b) 7.56%; (c) Account B. **Q6.** (a) 10.38%; (b) 10.47%; (c) monthly is larger — more frequent compounding earns interest on interest sooner. **Q7.** 1.5% per month; $n=36$. **Q8.** 4.91%. **Q9.** 9.31%. **Q10.** 16.18%. **Q11.** A: 8.73%; B: 8.77%; Account B is the better investment. **Q12.** A: 12.0149%; B: 12.0055%; Loan B is cheaper. **Q13.** Annually 12.00%; monthly 12.68%; monthly is higher. **Q14.** (a) 6.17%; (b) \$5635.80 by both methods. **Q15.** (a) More frequent compounding adds interest sooner, so it earns further interest within the year; (b) compounded annually. **Q16.** (a) 18% p.a.; (b) 19.56%. **Q17.** A: 5.06%; B: 5.01%; C: 5.20%; best to worst C, A, B. **Q18.** 7.25%. **Q19.** (a) 26.82%; (b) 27.11%; charging daily costs more. **Q20.** (a) P: 10.03%; Q: 10.06%; (b) Loan P; (c) P's lower nominal rate outweighs its more frequent compounding, so its effective rate is lower.

Fully-worked solutions

Q1. Divide each nominal rate by the number of compounding periods per year. (a) $\frac{9}{12}=0.75\%$ per month. (b) $\frac{10}{4}=2.5\%$ per quarter. (c) $\frac{7.3}{365}=0.02\%$ per day. (d) $\frac{6}{2}=3\%$ per half-year.

Q2. Multiply the number of years by the periods per year, $n=k\times T$. (a) $12\times 5=60$. (b) $4\times 3=12$. (c) $365\times 2=730$. (d) $2\times 4=8$. (e) $52\times 10=520$.

Q3. (a) Monthly compounding gives $k=12$. (b) The rate per month is $r=\frac{8}{12}\approx 0.6667\%$; keep the exact value $8\div 12$ in the calculator. (c) $R=1+\frac{8/12}{100}=1.00\overline{6}$. (d) $R^{12}=1.00\overline{6}^{12}\approx 1.083000$. (e) $r_{\text{eff}}=(R^{12}-1)\times 100\%=8.2999\ldots\%\approx 8.30\%$.

Q4. (a) The rate per day is $r=\frac{6}{365}\approx 0.016438\%$. (b) $R=1+\frac{6}{365\times 100}\approx 1.00016438$. (c) $R^{365}\approx 1.061831$. (d) $r_{\text{eff}}=(R^{365}-1)\times 100\%=6.1831\ldots\%\approx 6.18\%$.

Q5. Account A: the rate per month is $\frac{7.2}{12}=0.6\%$, so $R=1.006$ and $r_{\text{eff}}=(1.006^{12}-1)\times 100\%=7.4424\ldots\%\approx 7.44\%$. Account B: the rate per quarter is $\frac{7.35}{4}=1.8375\%$, so $R=1.018375$ and $r_{\text{eff}}=(1.018375^{4}-1)\times 100\%=7.5550\ldots\%\approx 7.56\%$. Since $7.56\%>7.44\%$, Account B gives the better return.

Q6. (a) Quarterly: $r=\frac{10}{4}=2.5\%$, $R=1.025$ and $r_{\text{eff}}=(1.025^{4}-1)\times 100\%=10.3812\ldots\%\approx 10.38\%$. (b) Monthly: $r=\frac{10}{12}\approx 0.8333\%$, $R\approx 1.008333$ and $r_{\text{eff}}=\left[\left(1+\frac{10}{1200}\right)^{12}-1\right]\times 100\%=10.4713\ldots\%\approx 10.47\%$. (c) The monthly effective rate (10.47%) is larger, because compounding more often means interest is added to the balance sooner and then itself earns interest within the year.

Q7. Monthly compounding gives $r=\frac{18}{12}=1.5\%$ per month. In 3 years there are $n=12\times 3=36$ compounding periods.

Q8. The rate per month is $\frac{4.8}{12}=0.4\,\%$, so $R=1.004$ and

$$r_{\text{eff}}=\left(1.004^{12}-1\right)\times 100\,\%=4.9070\ldots\%\approx 4.91\,\%.$$

Q9. The rate per quarter is $\frac{9}{4}=2.25\,\%$, so $R=1.0225$ and

$$r_{\text{eff}}=\left(1.0225^{4}-1\right)\times 100\,\%=9.3083\ldots\%\approx 9.31\,\%.$$

Q10. The rate per day is $\frac{15}{365}$ %, so $R=1+\frac{15}{36500}$ and

$$r_{\text{eff}}=\left[\left(1+\frac{15}{36500}\right)^{365}-1\right]\times 100\,\%=16.1798\ldots\%\approx 16.18\,\%.$$

Q11. Deposit A: $r=\frac{8.4}{12}=0.7\,\%$, $R=1.007$ and $r_{\text{eff}}=\left(1.007^{12}-1\right)\times 100\,\%=8.7310\ldots\%\approx 8.73\,\%$. Deposit B: $r=\frac{8.5}{4}=2.125\,\%$, $R=1.02125$ and $r_{\text{eff}}=\left(1.02125^{4}-1\right)\times 100\,\%=8.7747\ldots\%\approx 8.77\,\%$. Since $8.77\,\%>8.73\,\%$, Deposit B is the better investment.

Q12. Loan A: $r=\frac{11.4}{12}=0.95\,\%$, $R=1.0095$ and $r_{\text{eff}}=\left(1.0095^{12}-1\right)\times 100\,\%=12.0149\,\%$ (to 4 d.p.). Loan B: $r=\frac{11.5}{4}=2.875\,\%$, $R=1.02875$ and $r_{\text{eff}}=\left(1.02875^{4}-1\right)\times 100\,\%=12.0055\,\%$ (to 4 d.p.). The two rates both round to 12.01% at two decimal places, so the extra decimals are needed: $12.0055\,\%<12.0149\,\%$, so Loan B is (marginally) cheaper.

Q13. Compounded annually, $k=1$ and the effective rate equals the nominal rate, 12.00%. Compounded monthly, $r=\frac{12}{12}=1\,\%$, $R=1.01$ and $r_{\text{eff}}=\left(1.01^{12}-1\right)\times 100\,\%=12.6825\ldots\%\approx 12.68\,\%$. Since $12.68\,\%>12.00\,\%$, monthly compounding gives the higher effective rate.

Q14. (a) The rate per month is $\frac{6}{12}=0.5\,\%$, so $R=1.005$ and $r_{\text{eff}}=\left(1.005^{12}-1\right)\times 100\,\%=6.1677\ldots\%\approx 6.17\,\%$.
(b) Iterating the monthly recurrence for $n=12\times 2=24$ months,

$$V_{24}=5000\times 1.005^{24}\approx \$\,5635.80.$$

Using the effective annual rate over 2 whole years,

$$5000\times\left(1+\frac{6.1677\ldots}{100}\right)^{2}=5000\times 1.005^{24}\approx \$\,5635.80,$$

since one year at the effective rate is exactly 1.005^{12}. Both methods give \$5635.80.

Q15. (a) With more frequent compounding, interest is added to the balance earlier in the year, and that interest then earns further interest before the year ends; the more often this happens, the more the balance grows, so equal nominal rates give different yearly returns. (b) The effective rate equals the nominal rate only when interest is **compounded annually** ($k=1$), because then there is a single period and no interest-on-interest within the year.

Q16. (a) A monthly rate of 1.5% corresponds to a nominal annual rate of $1.5\times 12=18\,\%$ p.a. (b) With $R=1.015$,

$$r_{\text{eff}}=\left(1.015^{12}-1\right)\times 100\,\%=19.5618\ldots\%\approx 19.56\,\%.$$

Q17. Account A: $r=\frac{5}{2}=2.5\,\%$, $R=1.025$ and $r_{\text{eff}}=\left(1.025^2-1\right)\times 100\,\%=5.0625\,\%\approx 5.06\,\%$. Account B: $r=\frac{4.9}{12}\,\%$, so $r_{\text{eff}}=\left[\left(1+\frac{4.9}{1200}\right)^{12}-1\right]\times 100\,\%=5.0115\ldots\%\approx 5.01\,\%$. Account C: $r=\frac{5.1}{4}=1.275\,\%$, $R=1.01275$ and $r_{\text{eff}}=\left(1.01275^4-1\right)\times 100\,\%=5.1983\ldots\%\approx 5.20\,\%$. Ranking the effective rates $5.20\,\%>5.06\,\%>5.01\,\%$, the order from best to worst is C, A, B.

Q18. The rate per week is $\frac{7}{52}\,\%$, so $R=1+\frac{7}{5200}$ and

$$r_{\text{eff}}=\left[\left(1+\frac{7}{5200}\right)^{52}-1\right]\times 100\,\%=7.2457\ldots\%\approx 7.25\,\%.$$

Q19. (a) Monthly: $r=\frac{24}{12}=2\,\%$, $R=1.02$ and $r_{\text{eff}}=\left(1.02^{12}-1\right)\times 100\,\%=26.8241\ldots\%\approx 26.82\,\%$. (b) Daily: $r=\frac{24}{365}\,\%$, so $r_{\text{eff}}=\left[\left(1+\frac{24}{36500}\right)^{365}-1\right]\times 100\,\%=27.1148\ldots\%\approx 27.11\,\%$. Because $27.11\,\%>26.82\,\%$, charging interest daily costs the customer more.

Q20. (a) Loan P: $r=\frac{9.6}{12}=0.8\,\%$, $R=1.008$ and $r_{\text{eff}}=\left(1.008^{12}-1\right)\times 100\,\%=10.0338\ldots\%\approx 10.03\,\%$. Loan Q: $r=\frac{9.7}{4}=2.425\,\%$, $R=1.02425$ and $r_{\text{eff}}=\left(1.02425^4-1\right)\times 100\,\%=10.0585\ldots\%\approx 10.06\,\%$. (b) Loan P has the lower effective rate (10.03% against 10.06%), so Loan P is cheaper. (c) More frequent compounding does raise Loan P's effective rate above its 9.6% nominal, but only to 10.03%; Loan Q starts from a higher nominal rate (9.7%), and that head start outweighs Loan P's more frequent compounding, so Loan P still ends up with the lower effective rate.