General Mathematics Units 1 & 2

Chapter 10 — Applications of trigonometry

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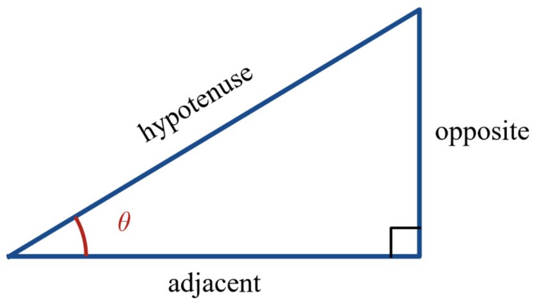
Contents

Theory

A **right-angled triangle** contains one angle of exactly 90°. A remarkable number of practical measurement problems — the height of a tower, the distance across a river, the slope of a ramp — can be solved by drawing a right-angled triangle and using the two tools of this section: Pythagoras' theorem and the trigonometric ratios. First the sides must be named. The side opposite the right angle is the **hypotenuse**; it is always the longest side. Relative to one of the other two (acute) angles, called θ in the diagram below, the side directly across the triangle from θ is the **opposite** side, and the remaining side — the one that, together with the hypotenuse, forms the angle θ — is the **adjacent** side.



Pythagoras' theorem. In every right-angled triangle the square of the hypotenuse equals the sum of the squares of the other two sides:

$$c^2 = a^2 + b^2,$$

where c is the length of the hypotenuse and a and b are the lengths of the two shorter sides. To find the hypotenuse, compute $c = \sqrt{a^2 + b^2}$; to find a shorter side, rearrange before taking the square root, giving $a = \sqrt{c^2 - b^2}$. A quick check is always available: the hypotenuse must come out **longer** than each of the other two sides, and a shorter side must come out shorter than the hypotenuse. Pythagoras' theorem links the three **sides** only — it says nothing about the angles.

The trigonometric ratios. When an angle is involved, we use the three **trigonometric ratios**. For an acute angle θ in a right-angled triangle,

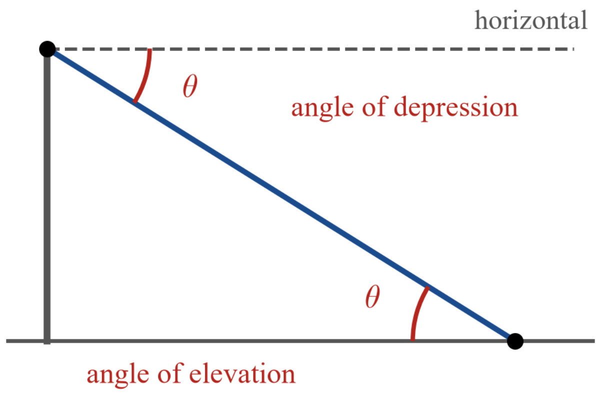
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

The memory aid **SOH CAH TOA** records the three definitions: **S**ine is **O**pposite over **H**ypotenuse, **C**osine is **A**djacent over **H**ypotenuse, **T**angent is **O**pposite over **A**djacent. In this course all angles are measured in **degrees**.

Finding an unknown side. Mark the known angle, then label the three sides *opposite*, *adjacent* and *hypotenuse* relative to it. Exactly one of the three ratios links the side you know with the side you want — choose it, write the equation, and solve. If the unknown is the numerator, multiply: from $\sin\theta = \frac{x}{c}$ we get $x = c\sin\theta$. If the unknown is the denominator, divide: from $\sin\theta = \frac{a}{x}$ we get $x = \frac{a}{\sin\theta}$.

Finding an unknown angle. If two sides are known, the value of one of the ratios is known, and the angle is recovered with an **inverse** trigonometric function: $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ on your CAS. For example, if $\tan\theta = 0.75$, then $\theta = \tan^{-1}(0.75) \approx 36.9°$ — read $\tan^{-1}(0.75)$ as "the angle whose tangent is 0.75".

Angles of elevation and depression. Both angles are measured **from the horizontal**. When an observer looks **up** at an object, the angle between the horizontal and the line of sight is the **angle of elevation**; when an observer looks **down**, it is the **angle of depression**. Because the two horizontal lines in the diagram below are parallel, the angle of depression from a point A down to a point B equals the angle of elevation from B up to A (alternate angles). This lets every “looking down” problem be redrawn as a right-angled triangle at ground level.



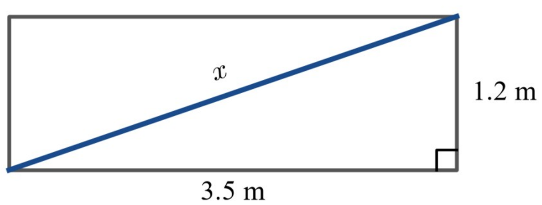
Sensible accuracy. The lengths and angles in these problems are measurements, so they are only approximate, and an answer should not pretend to be more precise than the data. Round only at the **final** step — keep the full calculator value throughout the working — and then give lengths to one or two decimal places (or to the nearest metre for large distances) and angles to one decimal place (or to the nearest degree). An answer like 23.517461 m claims far more precision than a tape-measured triangle can justify.

Using CAS. Check the calculator is in **Degree** mode before doing anything else — in Radian mode every trigonometric answer in this chapter would be wrong. The sin, cos and tan keys evaluate the ratios directly (for example $\sin 34°$), and the inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ recover an angle from a ratio. When the unknown is a denominator, bracket the division carefully, and store or re-use the full-precision value rather than retyping a rounded one. The by-hand skill in this section is choosing the right ratio and writing the correct equation; the CAS only evaluates what you set up.

Worked examples

Example 1 — scaffolded

A farm gate on a cattle property near Shepparton is 3.5 m wide and 1.2 m high. A diagonal steel brace runs from the bottom hinge corner to the opposite top corner to stop the gate sagging. Find the length x of the brace.



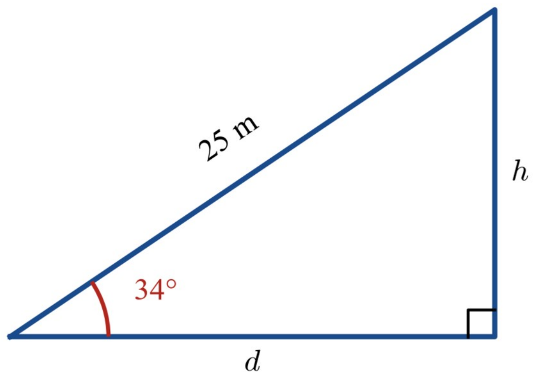
Working	Comment
The brace is the hypotenuse of a right-angled triangle whose shorter sides are 3.5 m and 1.2 m.	The corner of the gate is a right angle, and the brace is the side opposite it.
$x^2 = 3.5^2 + 1.2^2$	Pythagoras’ theorem $c^2 = a^2 + b^2$, with the hypotenuse alone on one side.
$x^2 = 12.25 + 1.44 = 13.69$	Square each shorter side first, then add.
$x = \sqrt{13.69} = 3.7$	Take the square root of the total (not of each term separately).
The brace is 3.7 m long.	Check: 3.7 m is longer than both 3.5 m and 1.2 m, as a

Working	Comment
	hypotenuse must be.

So the diagonal brace is exactly 3.7 m long.

Example 2 — scaffolded

The flying fox at a school camp in the Grampians has a straight cable 25 m long that makes an angle of 34° with the horizontal. Find, correct to two decimal places: (a) the vertical drop h from the top platform to the bottom of the cable; (b) the horizontal distance d covered by the ride.

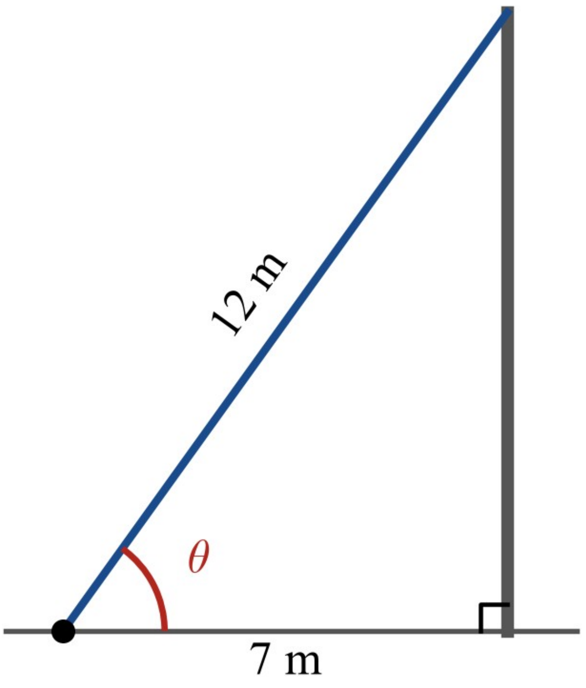


Working	Comment
Relative to the 34° angle: the drop h is the **opposite** side, the distance d is the **adjacent** side, and the 25 m cable is the **hypotenuse**.	Always label the sides relative to the known angle before choosing a ratio.
(a) $\sin 34° = \frac{h}{25}$, so $h = 25 \sin 34°$.	Opposite and hypotenuse are involved, so use SOH; the unknown is the numerator, so multiply.
$h = 25 \times 0.5592 \approx 13.98$	With the CAS in Degree mode, $\sin 34° = 0.5592$ (to four decimal places).
(b) $\cos 34° = \frac{d}{25}$, so $d = 25 \cos 34° = 20.7259\ldots \approx 20.73$.	Adjacent and hypotenuse are involved, so use CAH.
Check: $13.98^2 + 20.73^2 = 625.17 \approx 625 = 25^2$.	Pythagoras' theorem confirms the two answers are consistent with the 25 m cable.

So the ride drops about 13.98 m vertically and covers about 20.73 m horizontally.

Example 3 — unscaffolded

A 12 m guy wire steadies a communications mast on a farm near Colac. The wire runs from the top of the mast to an anchor point on level ground 7 m from the base of the mast. Find the angle θ that the wire makes with the ground, correct to one decimal place.

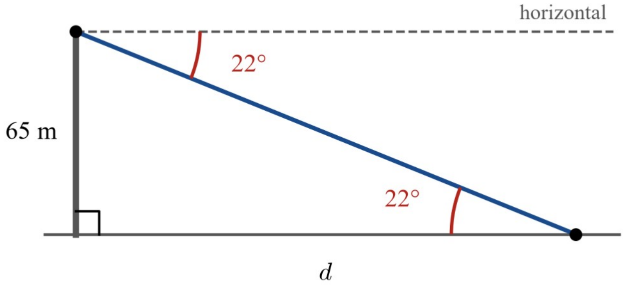


Relative to the angle θ at the anchor point, the 7 m along the ground is the **adjacent** side and the 12 m wire is the **hypotenuse**, so the cosine ratio links the two known sides: $\cos\theta = \frac{7}{12} = 0.5833\ldots$ Applying the inverse cosine, $\theta = \cos^{-1}\left(\frac{7}{12}\right) = 54.3146\ldots° \approx 54.3°$. As a check, the answer is sensible: the wire is much longer than the 7 m base, so it must rise steeply, at noticeably more than 45°.

∴ the guy wire makes an angle of about 54.3° with the ground.

Example 4 — unscaffolded

From the top of a vertical 65 m cliff at Cape Woolamai, a walker sees a kayaker on the sea below at an angle of depression of 22°. How far is the kayaker from the base of the cliff, to the nearest metre?



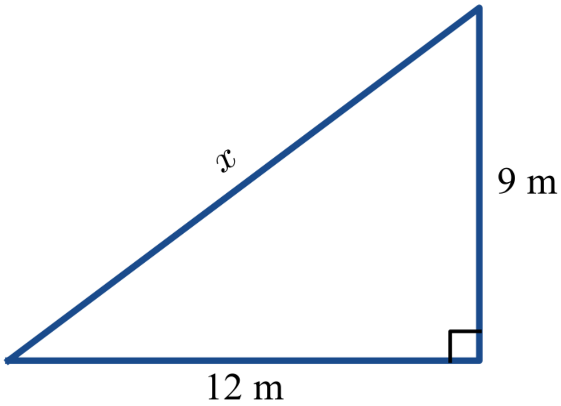
The angle of depression is measured from the horizontal at the top of the cliff down to the line of sight. Since that horizontal is parallel to the sea surface, the angle of elevation of the walker from the kayak is also 22° (alternate angles). In the right-angled triangle formed by the cliff and the sea, the 65 m cliff is the side **opposite** the 22° angle and the distance d is the **adjacent** side, so $\tan 22° = \frac{65}{d}$. The unknown is the denominator, so $d = \frac{65}{\tan 22°} = \frac{65}{0.4040\ldots} = 160.88\ldots \approx 161$ m.

∴ the kayaker is about 161 m from the base of the cliff.

Practice questions

Scaffolded

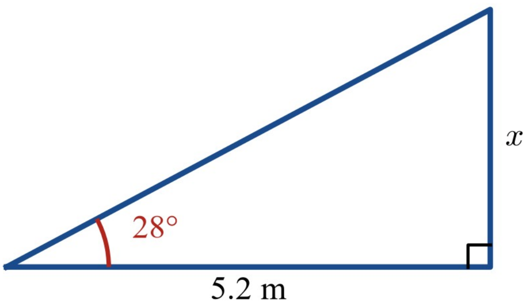
Q1. At a community garden in Bendigo, a triangular garden bed has two straight edges 12 m and 9 m long that meet at a right angle. The third edge, of length x, is to be lined with timber.



(a) Explain how you know the side labelled x is the hypotenuse.
(b) Write down an equation for x using Pythagoras' theorem.
(c) Solve the equation to find x.
(d) Check that your answer is reasonable by comparing it with the other two sides.

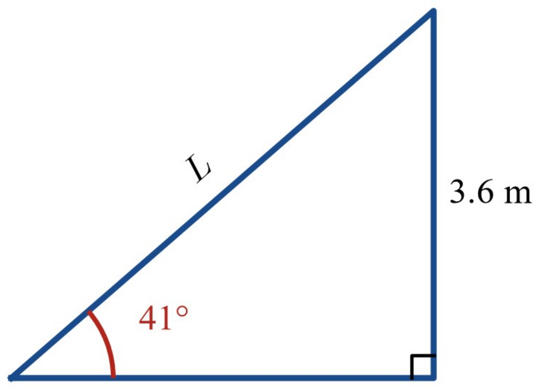
Q2. A 6.5 m ladder leans against the wall of a house in Hobart. The foot of the ladder rests on level ground 2.5 m from the base of the wall. (a) Which of the three lengths in this right-angled triangle is the hypotenuse? (b) Write down an equation relating the three lengths, using h for the height reached on the wall. (c) Rearrange the equation and solve it to find h. (d) Write your answer as a sentence.

Q3. A loading ramp at a Dandenong warehouse makes an angle of 28° with the horizontal, and the horizontal distance from the foot of the ramp to the loading dock is 5.2 m. Let x be the height of the dock.



(a) Relative to the 28° angle, state which side of the triangle is the opposite side and which is the adjacent side.
(b) Which trigonometric ratio links the known side and the unknown side? Explain your choice.
(c) Write an equation and solve it to find x, correct to two decimal places.

Q4. The straight slide at a Geelong aquatic centre makes an angle of 41° with the ground, and its top platform is 3.6 m above the ground. Let L be the length of the slide.



(a) Relative to the 41° angle, state which side is the opposite side and which is the hypotenuse.
(b) Write an equation linking 41°, 3.6 and L.
(c) Solve the equation to find L, correct to two decimal places. (Take care: the unknown is the denominator.)

Q5. A 4.5 m flagpole at a Mildura school casts a shadow 7.2 m long on level ground. Let θ be the angle of elevation of the sun. (a) In the right-angled triangle formed by the pole, its shadow and the sun's ray, state which side is opposite θ and which is adjacent to θ. (b) Show that $\tan\theta = 0.625$. (c) Find θ, correct to one decimal place. (d) State θ to the nearest degree.

Q6. From a point on level ground 40 m from the base of a mobile phone tower, the angle of elevation of the top of the tower is 31°. (a) Which trigonometric ratio should be used to find the height of the tower? Explain your choice. (b) Write an equation for the height h of the tower. (c) Solve it to find h, correct to one decimal place. (d) Interpret your answer in context, and state one assumption the calculation relies on.

Unscaffolded

Q7. Two straight roads meet at right angles at the corner of a nature reserve near Ballarat. Walking from the reserve's entrance to the picnic ground along the roads means 3.0 km along one road, then 1.6 km along the other. A proposed walking track would run straight across the reserve from the entrance to the picnic ground. Find the length of the proposed track, and how much shorter the track would be than the route along the roads.

Q8. At a country show, a 10 m rope runs from the top of a marquee pole to a peg on level ground 6 m from the base of the pole. Find the height of the pole.

Q9. On the beach at Torquay, Priya flies a kite on a straight 45 m string held at ground level. The string makes an angle of 52° with the horizontal. Find the height of the kite above the ground, correct to one decimal place.

Q10. A worksite safety rule requires the foot of any ladder to be placed so the ladder makes an angle of 68° with level ground. Find how far from the wall the foot of a 5.8 m ladder should be placed, correct to two decimal places.

Q11. A 7.5 m grain auger loads a silo on a farm near Horsham. The intake end of the auger sits on the ground 6.9 m horizontally from the point directly below its top end. Find the angle the auger makes with the horizontal, correct to one decimal place.

Q12. An escalator at Southern Cross Station rises a vertical height of 4.8 m and is inclined at 27° to the horizontal. Find the length of the escalator, correct to one decimal place.

Q13. From a point on level ground 65 m from the base of a Melbourne office tower, the angle of elevation of the top of the tower is 58°. Find the height of the tower, to the nearest metre.

Q14. The lamp of a lighthouse stands 48 m above sea level. From the lamp, the angle of depression of a fishing boat is 25°. Find the horizontal distance from the boat to the base of the lighthouse, to the nearest metre.

Q15. A right-angled triangular sail on a yacht at Sandringham has its two shorter sides, 5.6 m and 3.3 m long, at right angles to each other. (a) Find the length of the third side of the sail. (b) Find the angle between the 5.6 m side and the third side, correct to one decimal place.

Q16. A surveyor wants the width of the Murray River at a spot where the banks are parallel. From a point A on the south bank directly opposite a red gum T on the north bank, she walks 12.5 m along her bank to a point B. From B, the angle between the bank and the line of sight to the tree ($\angle ABT$) is 44°. (a) Which trigonometric ratio should she use to find the width AT, and why can none of the other ratios (or Pythagoras' theorem) be used directly here? (b) Find the width of the river, correct to one decimal place.

Q17. A rescue helicopter hovers 90 m above the sea. The winch operator sights a swimmer at an angle of depression of 19°. (a) Find the horizontal distance from the point directly below the helicopter to the swimmer, to the nearest metre. (b) Find the direct straight-line distance from the helicopter to the swimmer, to the nearest metre. (c) Explain, using the triangle, why the answer to part (b) must be larger than both 90 m and your answer to part (a).

Q18. A student stands on level ground 38 m from the base of a communications tower in Wodonga. Using a clinometer held at eye height, 1.6 m above the ground, she measures the angle of elevation of the top of the tower as 52°. Find the height of the tower, correct to one decimal place. (Remember that the 38 m triangle sits on top of her eye level, not on the ground.)

Answers

Q1. (a) It is the side opposite the right angle. (b) $x^2=12^2+9^2$. (c) $x=15$ m. (d) $15>12$ and $15>9$, as a hypotenuse must be. **Q2.** (a) The 6.5 m ladder. (b) $6.5^2=h^2+2.5^2$. (c) $h=6$ m. (d) The ladder reaches exactly 6 m up the wall. **Q3.** (a) Opposite $=x$; adjacent $=5.2$ m. (b) tan (it links opposite and adjacent). (c) $x=5.2\tan 28°\approx 2.76$ m. **Q4.** (a) Opposite $=3.6$ m; hypotenuse $=L$. (b) $\sin 41°=\dfrac{3.6}{L}$. (c) $L=\dfrac{3.6}{\sin 41°}\approx 5.49$ m. **Q5.** (a) Opposite = pole (4.5 m); adjacent = shadow (7.2 m). (b) $\tan\theta=\dfrac{4.5}{7.2}=0.625$. (c) $\theta\approx 32.0°$. (d) $32°$. **Q6.** (a) tan — the known 40 m is adjacent and the unknown height is opposite. (b) $\tan 31°=\dfrac{h}{40}$. (c) $h\approx 24.0$ m. (d) The tower is about 24 m tall; assumes the ground is level (and the tower vertical). **Q7.** Track $=3.4$ km; it is 1.2 km shorter than the 4.6 km road route. **Q8.** 8 m. **Q9.** $45\sin 52°\approx 35.5$ m. **Q10.** $5.8\cos 68°\approx 2.17$ m. **Q11.** $\cos^{-1}(0.92)\approx 23.1°$. **Q12.** $\dfrac{4.8}{\sin 27°}\approx 10.6$ m. **Q13.** $65\tan 58°\approx 104$ m. **Q14.** $\dfrac{48}{\tan 25°}\approx 103$ m. **Q15.** (a) 6.5 m. (b) $\tan^{-1}\left(\dfrac{3.3}{5.6}\right)\approx 30.5°$. **Q16.** (a) tan — only the opposite (AT) and adjacent (12.5 m) are involved; sin and cos need the hypotenuse, and Pythagoras needs two known sides. (b) $12.5\tan 44°\approx 12.1$ m. **Q17.** (a) $\dfrac{90}{\tan 19°}\approx 261$ m. (b) $\dfrac{90}{\sin 19°}\approx 276$ m. (c) It is the hypotenuse, which is the longest side of the triangle. **Q18.** $38\tan 52°+1.6\approx 50.2$ m.

Fully-worked solutions

Q1. (a) The 12 m and 9 m edges meet at the right angle, so the side x is opposite the right angle — it is the hypotenuse. (b) By Pythagoras' theorem, $x^2=12^2+9^2$. (c) $x^2=144+81=225$, so $x=\sqrt{225}=15$ m. (d) The answer is reasonable: 15 m is longer than both 12 m and 9 m, which the hypotenuse must be.

Q2. (a) The ladder itself (6.5 m) is opposite the right angle between the wall and the ground, so it is the hypotenuse. (b) $6.5^2=h^2+2.5^2$. (c) Rearranging, $h^2=6.5^2-2.5^2=42.25-6.25=36$, so $h=\sqrt{36}=6$ m. (d) The ladder reaches exactly 6 m up the wall.

Q3. (a) The height x of the dock is across the triangle from the 28° angle, so it is the opposite side; the 5.2 m along the ground runs from the angle to the right angle, so it is the adjacent side. (b) The tangent ratio, because tan links exactly the opposite and adjacent sides — the hypotenuse (the ramp surface) is neither known nor wanted. (c) $\tan 28°=\dfrac{x}{5.2}$, so $x=5.2\tan 28°=5.2\times 0.5317\ldots=2.7648\ldots\approx 2.76$ m.

Q4. (a) The 3.6 m height is opposite the 41° angle; the slide itself, L, is opposite the right angle, so it is the hypotenuse. (b) Opposite and hypotenuse give the sine ratio: $\sin 41°=\dfrac{3.6}{L}$. (c) The unknown is the denominator, so divide: $L=\dfrac{3.6}{\sin 41°}=\dfrac{3.6}{0.6560\ldots}=5.4873\ldots\approx 5.49$ m.

Q5. (a) The sun's ray is the hypotenuse; the 4.5 m pole stands across the triangle from θ, so it is the opposite side, and the 7.2 m shadow, running from θ to the base of the pole, is the adjacent side. (b) $\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}=\dfrac{4.5}{7.2}=0.625$. (c) $\theta=\tan^{-1}(0.625)=32.0053\ldots°\approx 32.0°$. (d) To the nearest degree, $\theta=32°$.

Q6. (a) The tangent ratio: relative to the 31° angle, the known 40 m along the ground is the adjacent side and the unknown height is the opposite side, and tan is the ratio that links those two. (b) $\tan 31°=\dfrac{h}{40}$. (c) $h=40\tan 31°=40\times 0.6008\ldots=24.0344\ldots\approx 24.0$ m. (d) The top of the tower is about 24 m above the ground at the observer's position. The calculation assumes the ground between the point and the tower is level (and that the tower is vertical); on sloping ground the 40 m and the height would not meet at a right angle.

Q7. The two roads meet at right angles, so the track is the hypotenuse of a right-angled triangle with shorter sides 3.0 km and 1.6 km. Its length is $\sqrt{3.0^2+1.6^2}=\sqrt{9+2.56}=\sqrt{11.56}=3.4$ km. The road route is $3.0+1.6=4.6$ km, so the track saves $4.6-3.4=1.2$ km.

Q8. The rope is the hypotenuse. With h the height of the pole, $h^2=10^2-6^2=100-36=64$, so $h=\sqrt{64}=8$ m.

Q9. The height of the kite is the side opposite the 52° angle and the 45 m string is the hypotenuse, so $\sin 52° = \frac{h}{45}$ and $h=45\sin 52° = 45\times 0.7880\ldots = 35.4604\ldots \approx 35.5$ m.

Q10. The distance d from the wall to the foot of the ladder is adjacent to the 68° angle and the 5.8 m ladder is the hypotenuse, so $\cos 68° = \frac{d}{5.8}$ and $d=5.8\cos 68° = 5.8\times 0.3746\ldots = 2.1727\ldots \approx 2.17$ m.

Q11. The 6.9 m horizontal distance is adjacent to the required angle θ and the 7.5 m auger is the hypotenuse, so $\cos\theta = \frac{6.9}{7.5}=0.92$. Hence $\theta = \cos^{-1}(0.92)=23.0739\ldots° \approx 23.1°$.

Q12. The 4.8 m rise is opposite the 27° angle and the escalator's length L is the hypotenuse, so $\sin 27° = \frac{4.8}{L}$. The unknown is the denominator, so $L=\frac{4.8}{\sin 27°}=\frac{4.8}{0.4539\ldots}=10.5729\ldots \approx 10.6$ m.

Q13. The height h is opposite the 58° angle and the 65 m along the ground is adjacent, so $\tan 58° = \frac{h}{65}$ and $h=65\tan 58° = 65\times 1.6003\ldots = 104.0217\ldots$ To the nearest metre the tower is about 104 m tall.

Q14. By alternate angles, the angle of elevation of the lamp from the boat is also 25°. The 48 m height is opposite this angle and the horizontal distance d is adjacent, so $\tan 25° = \frac{48}{d}$ and $d=\frac{48}{\tan 25°}=\frac{48}{0.4663\ldots}=102.9363\ldots$ To the nearest metre, the boat is about 103 m from the base of the lighthouse.

Q15. (a) The third side is the hypotenuse: $\sqrt{5.6^2+3.3^2}=\sqrt{31.36+10.89}=\sqrt{42.25}=6.5$ m. (b) At the vertex where the 5.6 m side meets the third side, the 3.3 m side is opposite and the 5.6 m side is adjacent, so $\tan\theta = \frac{3.3}{5.6}=0.5892\ldots$ and $\theta = \tan^{-1}\left(\frac{3.3}{5.6}\right)=30.5102\ldots° \approx 30.5°$.

Q16. (a) In the triangle ABT, the right angle is at A (the bank and the width AT are perpendicular because A is directly opposite T and the banks are parallel). Relative to the 44° angle at B, the width AT is the opposite side and the 12.5 m walked is the adjacent side, so the **tangent** ratio must be used. Sine and cosine each involve the hypotenuse BT, which is neither known nor wanted, and Pythagoras' theorem needs two known sides, but only one (12.5 m) is known. (b) $\tan 44° = \frac{AT}{12.5}$, so $AT=12.5\tan 44° = 12.5\times 0.9656\ldots = 12.0711\ldots \approx 12.1$ m.

Q17. (a) By alternate angles, the angle of elevation of the helicopter from the swimmer is 19°. The 90 m height is opposite this angle and the horizontal distance d is adjacent, so $\tan 19° = \frac{90}{d}$ and $d=\frac{90}{\tan 19°}=\frac{90}{0.3443\ldots}=261.3789\ldots \approx 261$ m. (b) The direct distance s is the hypotenuse, and the 90 m height is opposite the 19° angle, so $\sin 19° = \frac{90}{s}$ and $s=\frac{90}{\sin 19°}=\frac{90}{0.3255\ldots}=276.4398\ldots \approx 276$ m. (c) The direct line of sight is the hypotenuse of the right-angled triangle, and the hypotenuse is always the longest side — longer than the 90 m vertical side and longer than the horizontal side found in part (a).

Q18. From eye level, the top of the tower is $38\tan 52°$ above the horizontal, because the 38 m is adjacent to the 52° angle and the rise is opposite it: $38\tan 52° = 38 \times 1.2799\ldots = 48.6377\ldots \approx 48.64$ m. The clinometer sits 1.6 m above the ground, so the height of the tower is $48.6377\ldots + 1.6 = 50.2377\ldots \approx 50.2$ m.

Theory

The trigonometric ratios of the previous section apply only to **right-angled** triangles, yet the triangles that arise in surveying, navigation and design usually contain no right angle. The **sine rule** extends trigonometry to every triangle.

Labelling a triangle. In triangle ABC the capital letters A, B and C name the vertices and also the angles at those vertices. The lower-case letters name the sides, and each side takes the letter of the angle **opposite** it: side a (joining B and C) is opposite angle A, side b is opposite angle B, and side c is opposite angle C. The same pattern is used whatever the letters: in triangle PQR, side p is opposite angle P. A side together with its opposite angle is called a **matching pair**.

The sine rule. In any triangle ABC,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

In words: every side, divided by the sine of its opposite angle, gives the same number. The rule holds in acute, obtuse and right-angled triangles alike. To use it you must know at least one complete matching pair, and you always work with just **two** of the three fractions at a time.

Finding a side. When **two angles and one side** are known, the angle sum $A+B+C=180°$ supplies the third angle, so a complete matching pair is always available. Choose the fraction containing the unknown side, put it equal to the known pair, and write the unknown on top before rearranging:

$$\frac{b}{\sin B}=\frac{a}{\sin A} \text{ so } b=\frac{a\sin B}{\sin A}.$$

Finding an angle. When **two sides and a non-included angle** are known (an angle opposite one of the known sides), flip every fraction so that the sines are on top:

$$\frac{\sin B}{b}=\frac{\sin A}{a} \text{ so } \sin B=\frac{b\sin A}{a},$$

and then apply inverse sine, $B=\sin^{-1}\left(\frac{b\sin A}{a}\right)$.

The ambiguous case. Inverse sine only ever returns an acute angle, but for any value k with $0<k<1$ there are **two** angles between $0°$ and $180°$ whose sine is k: the acute angle $\theta=\sin^{-1}(k)$ and the obtuse angle $180°-\theta$. Both are possible sizes for an angle of a triangle, so two sides and a non-included angle can describe **two different triangles** — the known side opposite the given angle can “swing” to meet the base in two places, giving an acute triangle and an obtuse one. Whenever you find an angle with the sine rule, test the obtuse candidate against the angle sum: if the given angle plus $180°-\theta$ is still less than $180°$, a second triangle exists and **both** answers must be reported. Two triangles can only occur when the side opposite the given angle is **shorter** than the other given side (a longer side must sit opposite a larger angle). And if the calculation produces $\sin\theta>1$, no triangle with the given measurements exists at all.

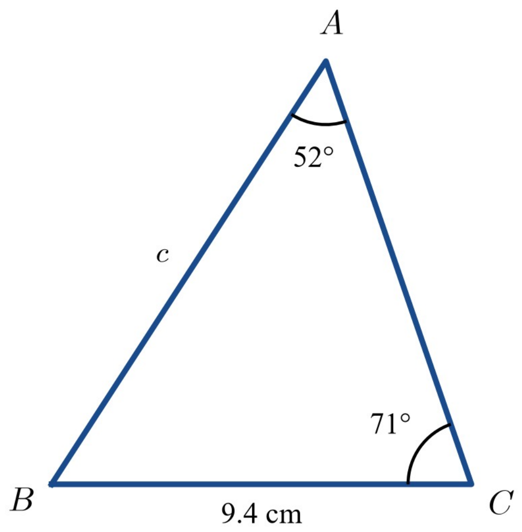
Applications. The sine rule is the surveyor’s workhorse: measure a **baseline** between two accessible points, measure the angle from each end of the baseline to an inaccessible object — a hut across a creek, a windmill in a locked paddock — and the sine rule delivers the distances that cannot be taped directly. Measure what you can reach; calculate what you cannot.

Using CAS. Work in **degree mode** — check the setting before you start. The sine rule rearrangements can be evaluated directly, or you can enter the equation and use solve. Remember that $\sin^{-1}$ on a CAS returns only the acute angle for a positive input; the calculator will never volunteer the obtuse solution, so in the ambiguous case you must form $180°-\theta$ yourself and test it with the angle sum. Keep unrounded values on the calculator (store them or use the answer memory) and round only the final answer, so that rounding errors do not build up through a multi-step calculation.

Worked examples

Example 1 — scaffolded

In triangle ABC, $A=52°$, $C=71°$ and $a=9.4$ cm. Find c, correct to one decimal place.

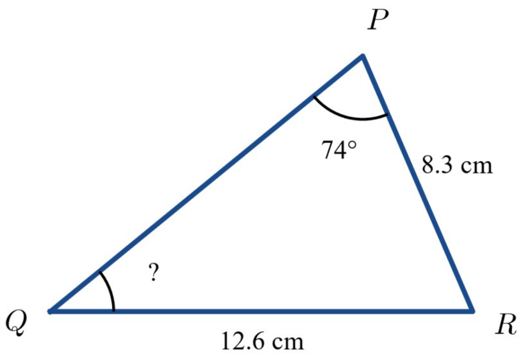


Working	Comment
The known matching pair is $a=9.4$ cm opposite $A=52°$; the unknown side c is opposite the known angle $C=71°$.	The sine rule needs a complete side–angle pair plus the pair containing the unknown.
$\dfrac{c}{\sin C}=\dfrac{a}{\sin A}$, so $\dfrac{c}{\sin 71°}=\dfrac{9.4}{\sin 52°}$.	Use only the two fractions involved, with the unknown side on top.
$c=\dfrac{9.4\sin 71°}{\sin 52°}$	Multiply both sides by $\sin 71°$.
$c=11.278\ldots\approx 11.3$ cm.	Evaluate with CAS in degree mode; round only at the end.

So the side c is approximately 11.3 cm. (The third angle, $B=180°-52°-71°=57°$, was not needed here, but it is always available from the angle sum.)

Example 2 — scaffolded

In triangle PQR, $p=12.6$ cm, $q=8.3$ cm and $P=74°$. Find Q and R, each correct to one decimal place.

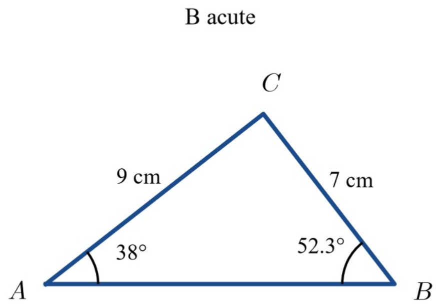


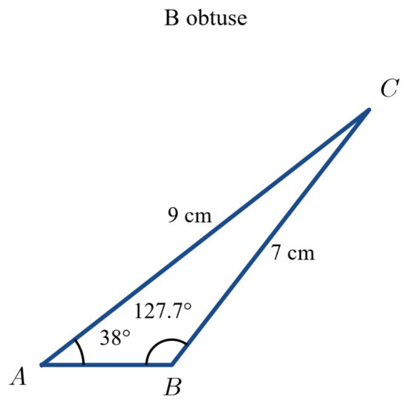
Working	Comment
The known matching pair is $p=12.6$ cm opposite $P=74°$; the side $q=8.3$ cm is opposite the unknown angle Q.	Two sides and a non-included angle: use the sine rule to find an angle.
$\frac{\sin Q}{q}=\frac{\sin P}{p}$, so $\sin Q=\frac{8.3\sin 74°}{12.6}=0.6332$ (4 d.p.).	Flip the fractions so the sines are on top, with the unknown angle's sine uppermost.
$Q=\sin^{-1}(0.6332\ldots)\approx 39.3°$.	Inverse sine returns the acute solution.
Obtuse candidate: $180°-39.3°=140.7°$, but $74°+140.7°=214.7°>180°$, which is impossible.	Always test the second solution against the angle sum. Here $q<p$, so Q must be smaller than P: only the acute answer survives.
$R=180°-74°-39.3°=66.7°$.	The angle sum gives the third angle.

So $Q\approx 39.3°$ and $R\approx 66.7°$.

Example 3 — unscaffolded

In triangle ABC, $A=38°$, $a=7$ cm and $b=9$ cm. Show that two triangles are possible, and find both possible sizes of angle B and the corresponding sizes of angle C, correct to one decimal place.



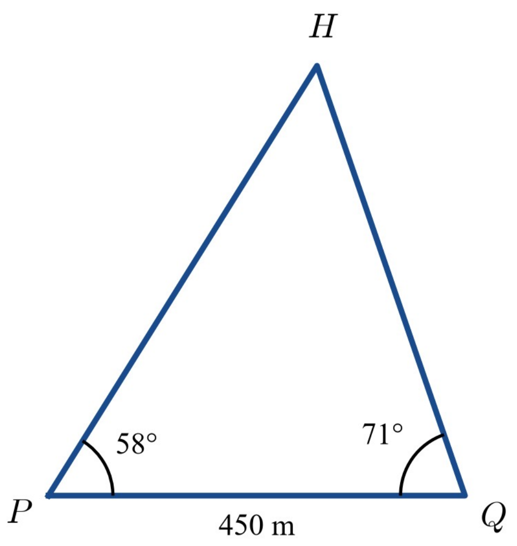


The known matching pair is $a=7$ opposite $A=38°$, and $b=9$ is opposite the unknown B. The sine rule gives $\sin B=\frac{b\sin A}{a}=\frac{9\sin 38°}{7}=0.7916$ (4 d.p.). The acute solution is $B=\sin^{-1}(0.7916\ldots)\approx 52.3°$, and the obtuse candidate is $180°-52.3°=127.7°$. Checking the angle sum: $38°+127.7°=165.7°<180°$, so the obtuse angle also fits in a triangle with $A=38°$ — two triangles are possible, as expected, because the side opposite the given angle (7 cm) is shorter than the other given side (9 cm). The angle sum now gives the two corresponding values of C: if $B\approx 52.3°$ then $C=180°-38°-52.3°=89.7°$, and if $B\approx 127.7°$ then $C=180°-38°-127.7°=14.3°$.

$\therefore$ two triangles are possible: $B\approx 52.3°$ with $C\approx 89.7°$, or $B\approx 127.7°$ with $C\approx 14.3°$.

Example 4 — unscaffolded

Two park rangers are at checkpoints P and Q, 450 m apart on a straight walking track. Across a flooded creek they can both see a hut H. Using a compass they measure $\angle HPQ = 58°$ and $\angle HQP = 71°$. Find the distance from each checkpoint to the hut, to the nearest metre.



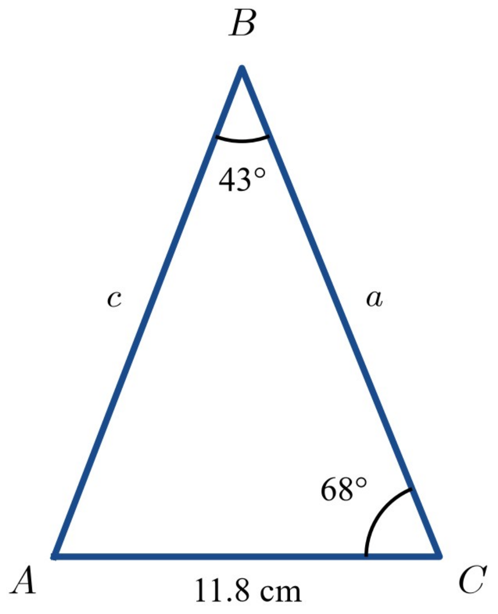
The angle sum gives the angle at the hut: $\angle PHQ = 180° - 58° - 71° = 51°$, and this angle is opposite the measured baseline $PQ = 450$ m, so the matching pair is complete. The side QH is opposite the angle at P, so $QH = \dfrac{450 \sin 58°}{\sin 51°} = 491.055\ldots \approx 491$ m. The side PH is opposite the angle at Q, so $PH = \dfrac{450 \sin 71°}{\sin 51°} = 547.494\ldots \approx 547$ m. As a check, the largest angle ($71°$) is opposite the longest side (PH), and the smallest angle ($51°$) is opposite the shortest side (the 450 m baseline), as must be the case.

$\therefore$ the hut is approximately 491 m from checkpoint Q and 547 m from checkpoint P.

Practice questions

Scaffolded

Q1. In triangle ABC, $B = 43°$, $C = 68°$ and $b = 11.8$ cm.



(a) Find A.

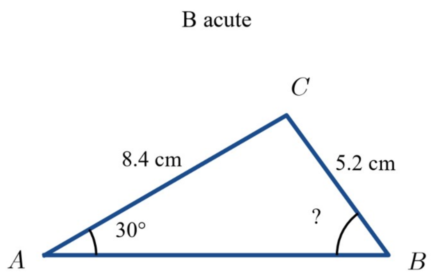
(b) Find c, correct to one decimal place.

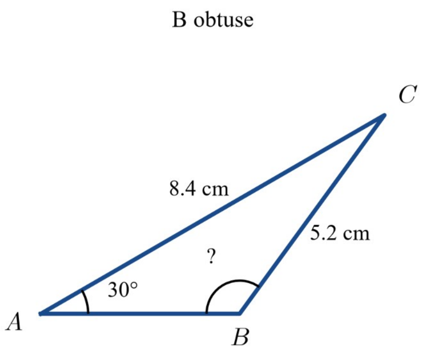
(c) Find a, correct to one decimal place.

Q2. In triangle ABC, $a = 14.2$ cm, $b = 9.6$ cm and $A = 81°$. (a) Use the sine rule to show that $\sin B \approx 0.6677$. (b) Hence find B, correct to one decimal place. (c) Explain why the obtuse angle with the same sine, $180° - B$, cannot be an angle of this triangle. (d) Find C, correct to one decimal place.

Q3. A triangular garden bed has corners A, B and C. The side AC is 24 m long, and the angles at A and C are $36°$ and $59°$ respectively. (a) Find the angle at B. (b) Find the length of side BC, correct to one decimal place. (c) Find the length of side AB, correct to one decimal place. (d) The council will edge all three sides with timber. Find the total length of edging required, to the nearest metre.

Q4. In triangle ABC, $A = 30°$, $a = 5.2$ cm and $b = 8.4$ cm.





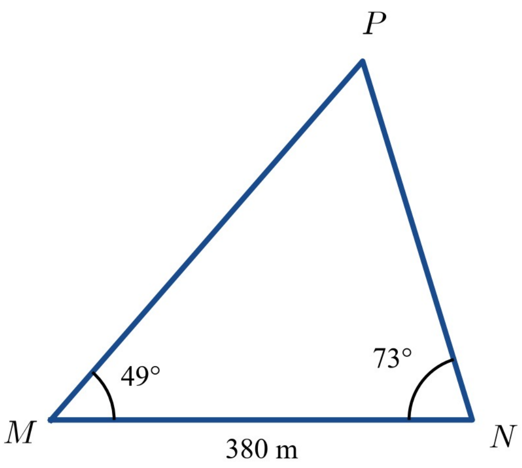
(a) Show that $\sin B = \dfrac{21}{26} \approx 0.8077$.

(b) Find the acute solution for B, correct to one decimal place.

(c) Find the obtuse solution for B, and use the angle sum to explain why it is also a valid answer.

(d) Find the two possible sizes of C, correct to one decimal place.

Q5. Mia and Noah stand at points M and N, 380 m apart on a straight beach. A pontoon P is anchored offshore, with $\angle PMN = 49°$ and $\angle PNM = 73°$.



(a) Find $\angle MPN$.
(b) Find the distance from N to the pontoon, to the nearest metre.
(c) Find the distance from M to the pontoon, to the nearest metre.
(d) Who is closer to the pontoon? Interpret your answers in the context of the question.

Q6. For each triangle described below, state whether the **sine rule** can be used to find the requested unknown, and give a brief reason. (Do not carry out any calculation.) (a) $A=40°$, $B=76°$ and $c=12$ cm; find a. (b) $a=8$ cm, $b=11$ cm and the included angle $C=52°$; find c. (c) $A=33°$, $a=6$ cm and $b=9$ cm; find B. (d) $a=5$ cm, $b=7$ cm and $c=9$ cm; find A.

Unscaffolded

Q7. In triangle ABC, $A=44°$, $B=66°$ and $a=15.3$ cm. Find b, correct to one decimal place.

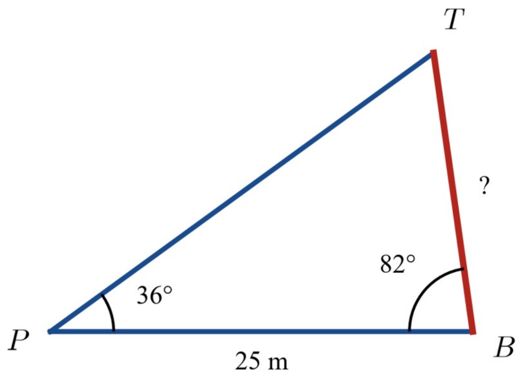
Q8. In triangle KLM, $k=18.4$ cm, $l=11.7$ cm and $K=98°$. Find L, correct to one decimal place.

Q9. In triangle ABC, $B=51°$, $C=62°$ and $c=14.8$ m. Find A, and find a and b correct to one decimal place.

Q10. In triangle ABC, $A=42°$, $a=8.1$ cm and $b=10.4$ cm. Show that two triangles are possible, and find both possible sizes of B, correct to one decimal place.

Q11. Points X and Y are 260 m apart on a straight country road. A windmill W stands in a locked paddock, with $\angle WXY=67°$ and $\angle WYX=54°$. Find the distance from X to the windmill, to the nearest metre.

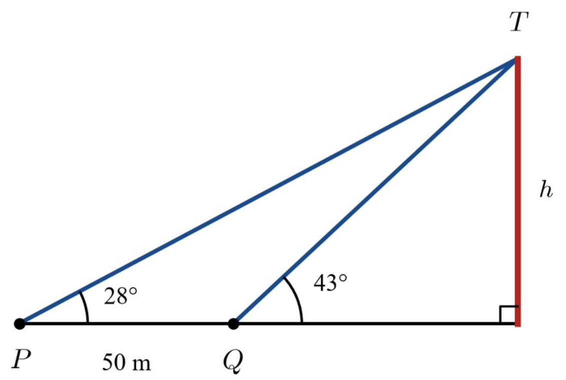
Q12. A river red gum leans towards Priya, its trunk making an angle of $82°$ with the level ground on the side facing her. Priya stands 25 m from the base of the trunk, and from where she stands the angle of elevation of the top of the tree is $36°$. Find the length of the leaning trunk, correct to one decimal place.



Q13. In triangle ABC, $B = 25°$, $b = 6.4$ cm and $c = 12.8$ cm. Two triangles are possible. For each triangle, find C and A correct to one decimal place, and the side a correct to one decimal place.

Q14. In an orienteering event, competitors leave the start S along a straight path that makes an angle of $34°$ with the straight road through S. A shed H stands beside the road, 6.0 km from S. Dana radios in that she is resting at a drink station on the path exactly 4.2 km from the shed. Explain why this information does not fix Dana's position on the path uniquely, and find both possible distances from S to Dana, correct to one decimal place.

Q15. From a point P on level ground, the angle of elevation of the top T of a transmission tower is $28°$. After walking 50 m on level ground directly towards the tower, to a point Q in line with P and the base of the tower, the angle of elevation is $43°$.



Use the sine rule in triangle PQT to find QT, correct to one decimal place, and hence find the height of the tower, to the nearest metre.

Q16. A triangular racing sail has corner angles of $54°$ and $63°$ at the two ends of its lower edge, which is 5.6 m long. Find the size of the third angle and the lengths of the other two sides, and hence show that the perimeter of the sail is approximately 16.3 m.

Q17. Ari claims to have drawn a triangle ABC with $A = 40°$, $a = 5$ cm and $b = 10$ cm. Use the sine rule to explain why this is impossible.

Q18. Fire-spotting towers stand at points A and B, 600 m apart on a straight firebreak. Both towers sight a column of smoke at F, with $\angle FAB = 47°$ and $\angle FBA = 58°$. (a) Find the distances AF and BF, to the nearest metre. (b) Hence find the shortest distance from the fire to the firebreak, to the nearest metre.

Answers

Q1. (a) $A=69°$. (b) $c\approx16.0$ cm. (c) $a\approx16.2$ cm. **Q2.** (a) $\sin B=\frac{9.6\sin 81°}{14.2}\approx0.6677$. (b) $B\approx41.9°$. (c) $180°-41.9°=138.1°$, and $81°+138.1°=219.1°>180°$, so it cannot fit in the triangle. (d) $C\approx57.1°$. **Q3.** (a) $B=85°$. (b) $BC\approx14.2$ m. (c) $AB\approx20.7$ m. (d) 59 m. **Q4.** (a) $\sin B=\frac{8.4\sin 30°}{5.2}=\frac{4.2}{5.2}=\frac{21}{26}\approx0.8077$. (b) $B\approx53.9°$. (c) $B\approx126.1°$; valid because $30°+126.1°=156.1°<180°$. (d) $C\approx96.1°$ or $C\approx23.9°$. **Q5.** (a) $\angle MPN=58°$. (b) $NP\approx338$ m. (c) $MP\approx429$ m. (d) Noah — his distance (338 m) is less than Mia's (429 m). **Q6.** (a) Yes — the angle sum gives $C=64°$, so the matching pair c and C is known. (b) No — the only known angle lies between the two known sides, so no matching pair is known (a different rule, met in the next section, is needed). (c) Yes — the pair a and A is known; because the side opposite the given angle is the shorter of the two given sides, this is the ambiguous case and there may be two answers. (d) No — no angle is known, so every fraction in the sine rule contains an unknown. **Q7.** $b\approx20.1$ cm. **Q8.** $L\approx39.0°$. **Q9.** $A=67°$, $a\approx15.4$ m, $b\approx13.0$ m. **Q10.** $\sin B\approx0.8591$; $B\approx59.2°$ or $B\approx120.8°$ (both valid, since $42°+120.8°<180°$). **Q11.** $XW\approx245$ m. **Q12.** Approximately 16.6 m. **Q13.** Triangle 1: $C\approx57.7°$, $A\approx97.3°$, $a\approx15.0$ cm. Triangle 2: $C\approx122.3°$, $A\approx32.7°$, $a\approx8.2$ cm. **Q14.** $\sin D\approx0.7988$ gives two valid angles at the drink station ($53.0°$ or $127.0°$), so two positions on the path fit the report: $SD\approx7.5$ km or $SD\approx2.4$ km. **Q15.** $QT\approx90.7$ m; height ≈62 m. **Q16.** Third angle $=63°$; the sides are 5.6 m (opposite the second $63°$ angle) and ≈5.1 m; perimeter ≈16.3 m. **Q17.** $\sin B=2\sin 40°\approx1.286>1$, which is impossible — no such triangle exists. **Q18.** (a) $AF\approx527$ m, $BF\approx454$ m. (b) ≈385 m.

Fully-worked solutions

Q1. (a) By the angle sum, $A=180°-43°-68°=69°$. (b) The known matching pair is $b=11.8$ opposite $B=43°$, and c is opposite $C=68°$:

$$\frac{c}{\sin 68°}=\frac{11.8}{\sin 43°}\text{ so }c=\frac{11.8\sin 68°}{\sin 43°}=16.042\ldots\approx16.0\,\text{cm}.$$

(c) Similarly a is opposite $A=69°$: $a=\frac{11.8\sin 69°}{\sin 43°}=16.152\ldots\approx16.2$ cm.

Q2. (a) Flipping the sine rule so the sines are on top, $\frac{\sin B}{b}=\frac{\sin A}{a}$, so $\sin B=\frac{9.6\sin 81°}{14.2}=0.66773\ldots\approx0.6677$. (b) $B=\sin^{-1}(0.6677\ldots)=41.892\ldots\approx41.9°$. (c) The obtuse angle with the same sine is $180°-41.9°=138.1°$. But $81°+138.1°=219.1°>180°$, so the two angles alone would already exceed the angle sum of a triangle. Only the acute answer is possible (as expected: $b<a$, so B must be smaller than A). (d) $C=180°-81°-41.9°=57.1°$.

Q3. (a) The angle at B is $180°-36°-59°=85°$. (b) Side BC is opposite the $36°$ angle at A, and the known pair is $AC=24$ m opposite $B=85°$: $BC=\frac{24\sin 36°}{\sin 85°}=14.160\ldots\approx14.2$ m. (c) Side AB is opposite the $59°$ angle at C: $AB=\frac{24\sin 59°}{\sin 85°}=20.650\ldots\approx20.7$ m. (d) Using the unrounded values, the perimeter is $24+14.160\ldots+20.650\ldots=58.81\ldots\approx59$ m of edging.

Q4. (a) $\sin B=\frac{b\sin A}{a}=\frac{8.4\sin 30°}{5.2}=\frac{8.4\times0.5}{5.2}=\frac{4.2}{5.2}=\frac{21}{26}=0.80769\ldots\approx0.8077$. (b) $B=\sin^{-1}\left(\frac{21}{26}\right)=53.871\ldots\approx53.9°$. (c) The obtuse angle with the same sine is $180°-53.9°=126.1°$. Checking the angle sum: $30°+126.1°=156.1°<180°$, so a triangle containing both $A=30°$ and $B=126.1°$ exists. This is the ambiguous case — the given side $a=5.2$ cm, opposite the given angle, is shorter than $b=8.4$ cm — so both answers are valid. (d) If $B\approx53.9°$ then $C=180°-30°-53.9°=96.1°$; if $B\approx126.1°$ then $C=180°-30°-126.1°=23.9°$.

Q5. (a) By the angle sum, $\angle MPN = 180° - 49° - 73° = 58°$. (b) NP is opposite the $49°$ angle at M, and the known pair is $MN = 380$ m opposite the $58°$ angle at P: $NP = \dfrac{380\sin 49°}{\sin 58°} = 338.176\ldots \approx 338$ m. (c) MP is opposite the $73°$ angle at N: $MP = \dfrac{380\sin 73°}{\sin 58°} = 428.508\ldots \approx 429$ m. (d) Noah is closer: he is about 338 m from the pontoon while Mia is about 429 m from it. This matches the geometry — the larger angle of sight is at Noah's end, so the pontoon lies closer to his end of the beach.

Q6. (a) **Yes.** The angle sum gives $C = 180° - 40° - 76° = 64°$, so the side $c = 12$ cm and its opposite angle form a known matching pair, and a (opposite the known $A = 40°$) follows from the sine rule. (b) **No.** The known angle is *included* between the two known sides, so it is not opposite either of them: no matching pair is known, and every fraction of the sine rule contains an unknown. A different rule (met in the next section) handles this situation. (c) **Yes.** The pair $a = 6$ cm opposite $A = 33°$ is known, and $b = 9$ cm is opposite the unknown B. Because the side opposite the given angle (6 cm) is the shorter of the two given sides, this is the ambiguous case: the obtuse candidate must be tested, and there may be two valid triangles. (d) **No.** Three sides but no angle are known, so no matching pair is available and every fraction of the sine rule contains an unknown sine.

Q7. The known pair is $a = 15.3$ cm opposite $A = 44°$, and b is opposite $B = 66°$:

$$b = \frac{15.3\sin 66°}{\sin 44°} = 20.121\ldots \approx 20.1\,\text{cm}.$$

Q8. The known pair is $k = 18.4$ cm opposite $K = 98°$, and $l = 11.7$ cm is opposite the unknown L: $\sin L = \dfrac{11.7\sin 98°}{18.4} = 0.62968\ldots \approx 0.6297$, so $L = \sin^{-1}(0.6297\ldots) = 39.026\ldots \approx 39.0°$. The obtuse candidate $180° - 39.0° = 141.0°$ fails the angle-sum test, since $98° + 141.0° > 180°$ (the triangle already contains the obtuse angle K), so $L \approx 39.0°$ is the only answer.

Q9. By the angle sum, $A = 180° - 51° - 62° = 67°$. The known pair is $c = 14.8$ m opposite $C = 62°$. Then $a = \dfrac{14.8\sin 67°}{\sin 62°} = 15.429\ldots \approx 15.4$ m and $b = \dfrac{14.8\sin 51°}{\sin 62°} = 13.026\ldots \approx 13.0$ m.

Q10. $\sin B = \dfrac{b\sin A}{a} = \dfrac{10.4\sin 42°}{8.1} = 0.85913\ldots \approx 0.8591$. The acute solution is $B = \sin^{-1}(0.8591\ldots) = 59.219\ldots \approx 59.2°$; the obtuse candidate is $180° - 59.2° = 120.8°$, and since $42° + 120.8° = 162.8° < 180°$ it also fits with $A = 42°$. Both triangles exist (the side opposite the given angle, 8.1 cm, is shorter than 10.4 cm), so $B \approx 59.2°$ or $B \approx 120.8°$.

Q11. The angle at the windmill is $180° - 67° - 54° = 59°$, opposite the measured baseline $XY = 260$ m — a complete matching pair. The distance XW is opposite the $54°$ angle at Y:

$$XW = \frac{260\sin 54°}{\sin 59°} = 245.394\ldots \approx 245\,\text{m}.$$

Q12. Let B be the base of the trunk and T the top of the tree, and let P be where Priya stands. In triangle PBT the angle at P is $36°$ and the angle at B (between the trunk and the ground on Priya's side) is $82°$, so the angle at T is $180° - 36° - 82° = 62°$. The known pair is $PB = 25$ m opposite $T = 62°$, and the trunk BT is opposite the $36°$ angle at P:

$$BT = \frac{25\sin 36°}{\sin 62°} = 16.642\ldots \approx 16.6\,\text{m}.$$

Q13. The known pair is $b = 6.4$ cm opposite $B = 25°$, and $c = 12.8$ cm is opposite the unknown C: $\sin C = \dfrac{12.8\sin 25°}{6.4} = 2\sin 25° = 0.84523\ldots \approx 0.8452$. The acute solution is $C = \sin^{-1}(0.8452\ldots) = 57.697\ldots \approx 57.7°$; the obtuse candidate is $180° - 57.7° = 122.3°$, valid because

$25° + 122.3° = 147.3° < 180°$. Triangle 1: $C \approx 57.7°$, so $A = 180° - 25° - 57.7° = 97.3°$ and $a = \frac{6.4 \sin 97.3°}{\sin 25°} = 15.020\ldots \approx 15.0$ cm (using the unrounded angle). Triangle 2: $C \approx 122.3°$, so $A = 180° - 25° - 122.3° = 32.7°$ and $a = \frac{6.4 \sin 32.7°}{\sin 25°} = 8.180\ldots \approx 8.2$ cm.

Q14. In the triangle formed by the start S, the shed H and the drink station D, the angle at S is $34°$, the side $SH = 6.0$ km is opposite the unknown angle at D, and the side $DH = 4.2$ km is opposite the $34°$ angle. Then $\sin D = \frac{6.0 \sin 34°}{4.2} = 0.79884\ldots \approx 0.7988$, giving $D = 53.020\ldots \approx 53.0°$ or $D = 180° - 53.0° = 127.0°$. Both are valid, because $34° + 127.0° = 161.0° < 180°$ — the reported distance 4.2 km is opposite the given angle and is shorter than $SH = 6.0$ km, so the circle of radius 4.2 km around the shed cuts the path at **two** points. That is why Dana's report does not fix her position. The angle at H is $180° - 34° - 53.0° = 93.0°$ or $180° - 34° - 127.0° = 19.0°$, so the distance from the start is $SD = \frac{4.2 \sin 93.0°}{\sin 34°} = 7.500\ldots \approx 7.5$ km, or $SD = \frac{4.2 \sin 19.0°}{\sin 34°} = 2.447\ldots \approx 2.4$ km (unrounded angles used throughout).

Q15. In triangle PQT, the angle at P is $28°$. The angle $\angle PQT$ is the straight angle at Q less the elevation on the tower side: $180° - 43° = 137°$. The angle sum then gives $\angle PTQ = 180° - 28° - 137° = 15°$. The known pair is $PQ = 50$ m opposite the $15°$ angle at T, and QT is opposite the $28°$ angle at P: $QT = \frac{50 \sin 28°}{\sin 15°} = 90.694\ldots \approx 90.7$ m. The tower, the ground and the sight line QT form a right-angled triangle at the base, in which the height is the side opposite the $43°$ angle at Q: height $= QT \sin 43° = 90.694\ldots \times \sin 43° = 61.853\ldots \approx 62$ m.

Q16. The third angle is $180° - 54° - 63° = 63°$. Label the sail ABC with $A = 54°$, $B = 63°$ and $C = 63°$, so that the lower edge is $c = 5.6$ m, joining the $54°$ and $63°$ corners and opposite one of the $63°$ angles. Because $B = C = 63°$, the sail is isosceles and the side b (opposite B) equals c exactly: $b = \frac{5.6 \sin 63°}{\sin 63°} = 5.6$ m. The side opposite the $54°$ angle is $a = \frac{5.6 \sin 54°}{\sin 63°} = 5.084\ldots \approx 5.1$ m. The perimeter is therefore $5.6 + 5.6 + 5.084\ldots = 16.284\ldots \approx 16.3$ m, as required.

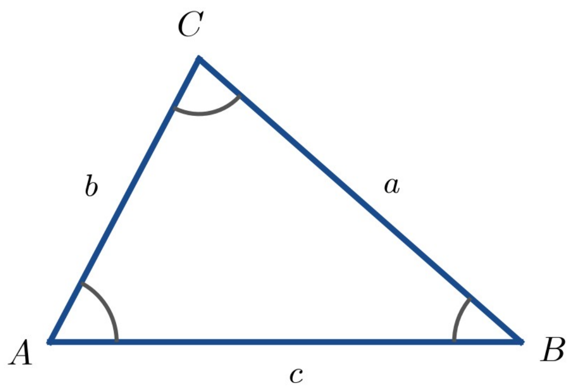
Q17. If such a triangle existed, the sine rule would give $\sin B = \frac{b \sin A}{a} = \frac{10 \sin 40°}{5} = 2 \sin 40° = 1.2855\ldots \approx 1.286$. But the sine of an angle can never exceed 1, so the equation has no solution: the 5 cm side opposite the $40°$ angle is too short ever to close a triangle against a 10 cm side. No such triangle exists. (This is the third possibility in the ambiguous case: two sides and a non-included angle can describe two triangles, one triangle, or none at all.)

Q18. (a) The angle at the fire is $\angle AFB = 180° - 47° - 58° = 75°$, opposite the 600 m baseline — a complete matching pair. AF is opposite the $58°$ angle at B and BF is opposite the $47°$ angle at A: $AF = \frac{600 \sin 58°}{\sin 75°} = 526.778\ldots \approx 527$ m and $BF = \frac{600 \sin 47°}{\sin 75°} = 454.291\ldots \approx 454$ m. (b) The shortest distance from F to the firebreak is the perpendicular distance h. In the right-angled triangle formed by dropping this perpendicular from F, h is opposite the $47°$ angle at A along the sight line AF, so $h = AF \sin 47° = 526.778\ldots \times \sin 47° = 385.261\ldots \approx 385$ m. (As a check, $BF \sin 58° = 454.291\ldots \times \sin 58°$ gives the same 385 m.)

Theory

Many practical triangles contain no right angle, so the trigonometry of right-angled triangles cannot be applied to them directly. The sine rule handles some of these triangles, but it needs a known angle **paired with** its opposite side before it can start. This section develops the **cosine rule**, which works instead from two sides and the angle between them, or from all three sides — exactly the situations the sine rule cannot handle.

Labelling convention. In triangle ABC the angles at the vertices are written with capital letters A, B and C, and the side **opposite** each angle is written with the matching lower-case letter: side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C.



The cosine rule. In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Here A is the **included angle** — the angle sitting between the two sides b and c — and a is the side opposite it. The same relationship can be written from each vertex: $b^2 = a^2 + c^2 - 2ac\cos B$ and $c^2 = a^2 + b^2 - 2ab\cos C$.

Finding a side. When a triangle's two sides and the **included angle** are known (a "side–angle–side" triangle), the cosine rule gives the third side directly: substitute, evaluate the whole right-hand side, then take the square root. Work with unrounded values on the calculator and round only the final answer.

Finding an angle. When all **three sides** are known, rearranging the cosine rule gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and then $A = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$. The side opposite the wanted angle, a, is the one subtracted in the numerator. If the fraction is **negative**, the angle is **obtuse**; $\cos^{-1}$ automatically returns the correct angle between $0°$ and $180°$, so — unlike the sine rule — there is never any ambiguity. As a check, the **largest angle** of a triangle is always opposite the **longest side**.

A generalisation of Pythagoras' theorem. If $A = 90°$ then $\cos A = 0$ and the cosine rule collapses to $a^2 = b^2 + c^2$ — Pythagoras' theorem exactly. The term $-2bc\cos A$ is a correction to Pythagoras for triangles without a right angle: when A is acute, $\cos A$ is positive and the correction **shortens** the third side; when A is obtuse, $\cos A$ is negative and the correction **lengthens** it.

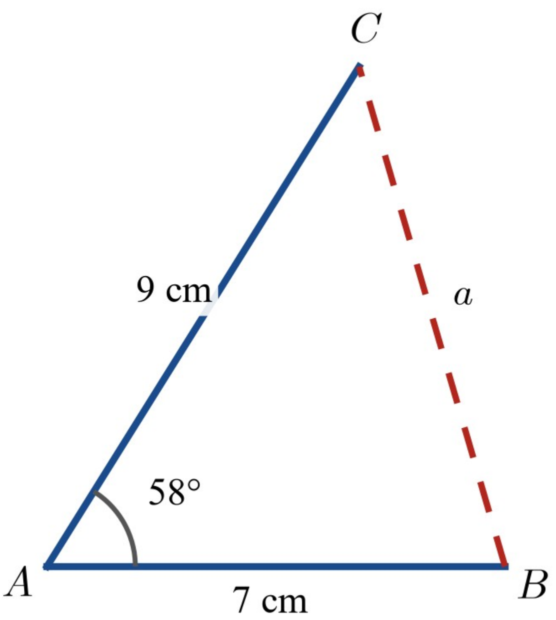
Choosing between the sine rule and the cosine rule. Use the **cosine rule** when you know two sides and the included angle (to find the third side), or all three sides (to find an angle). Use the **sine rule** when the known information contains a matching pair — an angle together with its opposite side — such as two angles and one side, or two sides and a non-included angle. A quick test: if no known angle sits opposite a known side, the sine rule cannot start, so the cosine rule is the one to use.

Using CAS. Check first that the calculator is set to **degree mode** — a radian-mode answer will be wrong without warning. For a side, type the whole right-hand side in one line, for example $9^2+7^2-2\times9\times7\times\cos 58°$, then square-root the answer; for an angle, use the $\cos^{-1}$ key on the whole fraction. Keep the unrounded value on the screen (or store it) and round only when writing the final answer. A CAS can also solve $a^2=b^2+c^2-2bc\cos A$ numerically for any one unknown, but the by-hand substitution below is quick and shows the method clearly.

Worked examples

Example 1 — scaffolded

In triangle ABC, $b=9$ cm, $c=7$ cm and angle $A=58°$. Find the length of side a, correct to two decimal places.

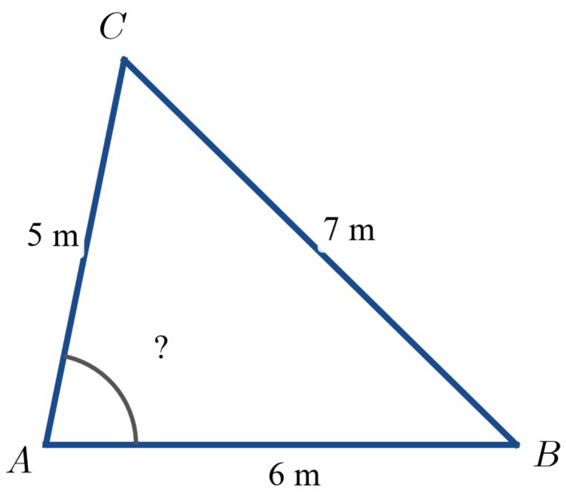


Working	Comment
The known sides $b=9$ and $c=7$ include the known angle $A=58°$, so use the cosine rule.	Two sides and the included angle: a cosine-rule triangle. No known angle is opposite a known side, so the sine rule cannot start.
$a^2=b^2+c^2-2bc\cos A=9^2+7^2-2\times9\times7\times\cos 58°$.	Substitute the values; the calculator must be in degree mode.
$a^2=130-126\cos 58°\approx130-66.7698=63.2302$.	Evaluate the whole right-hand side first — do not take any square roots yet.
$a=\sqrt{63.2302\ldots}\approx7.95$.	Square root last, and round only at this final step.

So $a\approx7.95$ cm.

Example 2 — scaffolded

In triangle ABC, $a=7$ m, $b=5$ m and $c=6$ m. Find the size of angle A, correct to one decimal place.

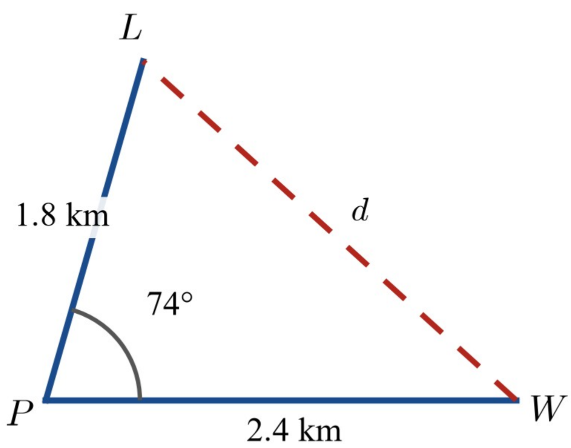


Working	Comment
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.	All three sides are known and no angle is known, so use the rearranged cosine rule.
$\cos A = \dfrac{5^2 + 6^2 - 7^2}{2 \times 5 \times 6} = \dfrac{25 + 36 - 49}{60} = \dfrac{12}{60} = 0.2$.	The side opposite the wanted angle, $a = 7$, is the one subtracted in the numerator.
$A = \cos^{-1}(0.2) \approx 78.5°$.	$\cos A$ is positive, so A is acute. $\cos^{-1}$ returns the angle directly.

So $A \approx 78.5°$. As a check, A is opposite the longest side (7 m), so it should be — and is — the largest angle of the triangle.

Example 3 — unscaffolded

Two straight walking tracks leave the picnic ground P at Messmate Creek Reserve, diverging at an angle of $74°$. One track runs 1.8 km to a lookout at L; the other runs 2.4 km to a waterhole at W. Find the straight-line distance between the lookout and the waterhole, correct to two decimal places.



The two known distances $PL = 1.8$ km and $PW = 2.4$ km include the known angle of $74°$ at P, so the cosine rule gives the third side of triangle PLW directly:

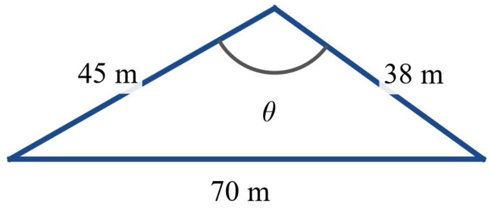
$$LW^2 = 1.8^2 + 2.4^2 - 2 \times 1.8 \times 2.4 \times \cos 74° = 9 - 8.64 \cos 74° \approx 9 - 2.3815 = 6.6185.$$

Hence $LW = \sqrt{6.6185\ldots} \approx 2.57$ km.

$\therefore$ the lookout and the waterhole are approximately 2.57 km apart in a straight line.

Example 4 — unscaffolded

A triangular paddock is enclosed by three straight fences of lengths 45 m, 38 m and 70 m. Find the size of the largest angle between two fences, correct to the nearest degree.



Only the three sides are known, so no angle is available to start the sine rule; the rearranged cosine rule is needed. The largest angle, θ, lies opposite the longest fence, 70 m, so

$$\cos\theta = \frac{45^2+38^2-70^2}{2\times45\times38} = \frac{2025+1444-4900}{3420} = \frac{-1431}{3420} \approx -0.4184.$$

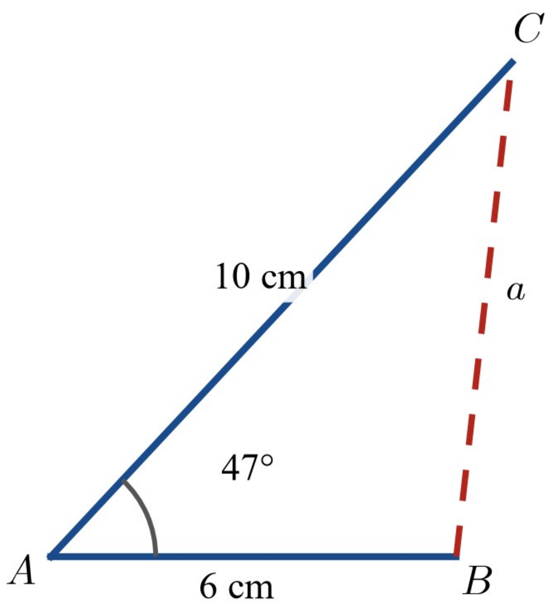
The cosine is negative, so the angle is obtuse — consistent with it being the largest angle. Then $\theta = \cos^{-1}(-0.4184\ldots) \approx 114.73° \approx 115°$.

∴ the largest angle between two fences is approximately $115°$, at the corner opposite the 70 m fence.

Practice questions

Scaffolded

Q1. In triangle ABC, angle $A = 47°$, $b = 10$ cm and $c = 6$ cm.



(a) Write down the cosine rule in the form that gives a^2.

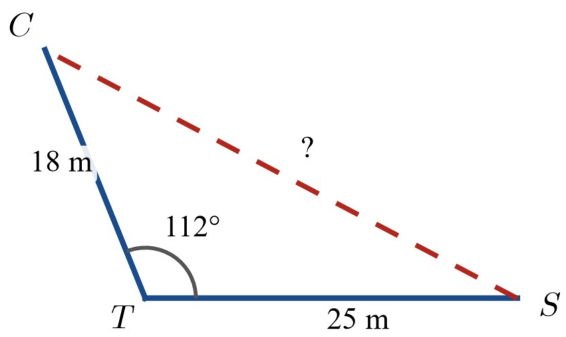
(b) Substitute the given values into the rule.

(c) Show that $a^2 \approx 54.1602$.

(d) Hence find a, correct to two decimal places.

Q2. In triangle ABC, $a = 9$ cm, $b = 7$ cm and $c = 5$ cm. (a) Write down the rearranged cosine rule that gives $\cos A$. (b) Show that $\cos A = -0.1$. (c) Hence find angle A, correct to one decimal place. (d) How could you tell that A is obtuse before working out $\cos^{-1}$?

Q3. In a community garden, two straight gravel paths leave the water tap T: one runs 25 m to the tool shed S and the other runs 18 m to the compost bay C. The angle between the two paths at the tap is $112°$.



(a) Substitute into the cosine rule to write an expression for SC^2.
(b) The term $-2 \times 25 \times 18 \times \cos 112°$ turns out to be **positive**. Explain why.
(c) Hence find the distance SC between the shed and the compost bay, correct to two decimal places.

Q4. In triangle ABC, $b = 8$ m and $c = 6$ m. (a) Use the cosine rule with $A = 90°$ to find a. What familiar rule does the cosine rule reduce to in this case? (b) Use the cosine rule with $A = 60°$ to find a, correct to two decimal places. (c) Use the cosine rule with $A = 120°$ to find a, correct to two decimal places. (d) Compare your answers to (b) and (c) with (a), and explain what the term $-2bc\cos A$ does when A is acute and when A is obtuse.

Q5. A triangle has sides of length 5 cm, 7 cm and 10 cm. (a) Find the size of the largest angle, correct to one decimal place. (b) Find the size of the smallest angle, correct to one decimal place. (c) Find the third angle in two ways: using the cosine rule, and using the angle sum of a triangle. Check that the two answers agree.

Q6. For each triangle described below, state whether the **sine rule** or the **cosine rule** is needed, and give a brief reason. Do **not** solve the triangles. (a) Two sides of 8 cm and 12 cm meet at an angle of $40°$; find the third side. (b) Two angles are $63°$ and $49°$, and the side opposite the $63°$ angle is known; find another side. (c) The three sides are 6 m, 9 m and 11 m; find an angle. (d) Two sides are 14 cm and 17 cm, and the angle opposite the 14 cm side is $35°$; find the angle opposite the 17 cm side.

Unscaffolded

Q7. In triangle DEF, angle $D = 63°$, $e = 14$ cm and $f = 9$ cm. Find d, correct to two decimal places.

Q8. A triangle has sides 9 cm, 12 cm and 15 cm. Use the cosine rule to find the size of its largest angle. What kind of triangle is this? Confirm your conclusion with a check that does not use trigonometry.

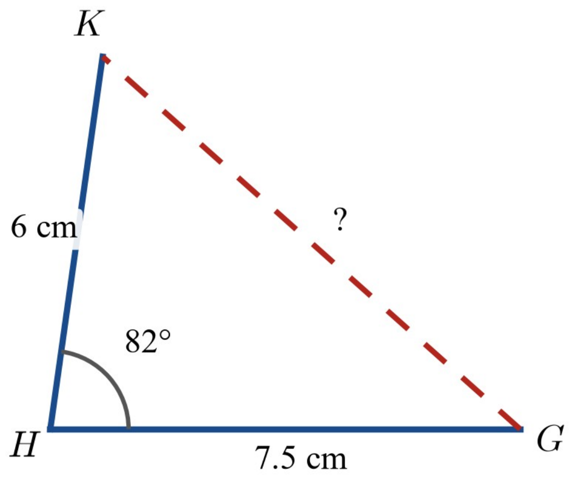
Q9. Two kayakers set off from the same pontoon on Lake Marram, paddling on straight courses that diverge at an angle of $54°$. Ash paddles 650 m and Bree paddles 900 m. How far apart are the two kayakers, to the nearest metre?

Q10. A yacht's triangular sail has sides of length 3.2 m, 4.5 m and 5.9 m. Find the size of the sail's largest angle, correct to the nearest degree.

Q11. A survey drone flies 1.4 km in a straight line, turns, and flies a further 2.0 km in a second straight line; the angle between the two legs of its flight is $133°$. How far is the drone from its launch point, correct to two decimal places?

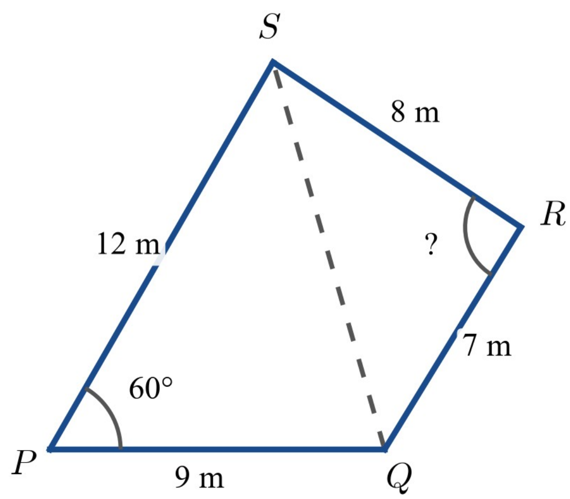
Q12. A school cross-country course is a triangle with legs of 1.2 km, 1.5 km and 2.1 km. Find the angle between the two shorter legs, correct to the nearest degree.

Q13. In triangle GHK, $GH = 7.5$ cm, $HK = 6$ cm and angle $H = 82\,°$. Find GK, correct to two decimal places, and explain why the sine rule could not have been used directly to find it.



Q14. A council's pocket park is a triangle: two equal boundary fences of 30 m meet at a back corner, and the third side is a 48 m street frontage. The plans describe the back corner as "wider than a right angle". Find the angle between the two 30 m fences, correct to one decimal place, and comment on whether the description in the plans is correct.

Q15. A landscaper edges a garden bed shaped as quadrilateral $PQRS$, with $PQ = 9$ m, $QR = 7$ m, $RS = 8$ m and $SP = 12$ m, and lays a straight timber sleeper along the diagonal QS. The angle at P is $60\,°$.



(a) Use triangle PQS to find the length of the sleeper QS, correct to two decimal places.

(b) Hence find the size of angle R, correct to the nearest degree.

Q16. (a) An equilateral triangle has all three sides 10 cm. Use the cosine rule to show that each of its angles is exactly $60\,°$. (b) An isosceles triangle has sides 10 cm, 10 cm and 12 cm. Find all three of its angles, correct to two decimal places, and check that they add to $180\,°$.

Q17. Two straight roads leave the Old Highway roundabout: one runs 34 km to Redgate and the other runs 27 km to Silverleigh. The angle between the two roads at the roundabout is $96\,°$. A direct link road between the two towns is proposed. Find the length of the proposed road, correct to one decimal place, and determine how much shorter the direct trip between the towns would be than driving via the roundabout.

Q18. A puzzle book claims that a triangle has sides 4 m, 6 m and 11 m. Try to use the cosine rule to find the angle opposite the 11 m side, and explain what goes wrong. Then confirm, using the side lengths alone, that no such triangle exists.

Answers

Q1. (a) $a^2=b^2+c^2-2bc\cos A$. (b) $a^2=10^2+6^2-2\times10\times6\times\cos 47°$. (c) $a^2=136-120\cos 47°\approx54.1602$. (d) $a\approx7.36$ cm. **Q2.** (a) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$. (b) $\cos A=\dfrac{49+25-81}{70}=-0.1$. (c) $A\approx95.7°$. (d) $\cos A$ is negative, so A is obtuse. **Q3.** (a) $SC^2=25^2+18^2-2\times25\times18\times\cos 112°$. (b) $\cos 112°$ is negative, so the term is positive. (c) $SC\approx35.86$ m. **Q4.** (a) $a=10$ m; Pythagoras' theorem. (b) $a\approx7.21$ m. (c) $a\approx12.17$ m. (d) Acute A: term subtracts, $a<10$; obtuse A: term adds, $a>10$. **Q5.** (a) $111.8°$. (b) $27.7°$. (c) $40.5°$ by both methods. **Q6.** (a) Cosine rule (two sides and the included angle). (b) Sine rule (an angle is paired with its opposite side). (c) Cosine rule (three sides, no angle). (d) Sine rule (the pair $35°$ and 14 cm is known). **Q7.** $d\approx12.75$ cm. **Q8.** Largest angle $=90°$; right-angled, since $9^2+12^2=225=15^2$. **Q9.** ≈738 m. **Q10.** $\approx99°$. **Q11.** ≈3.13 km. **Q12.** $\approx102°$. **Q13.** $GK\approx8.93$ cm; the only known angle is opposite the unknown side, so every sine-rule equation would contain two unknowns. **Q14.** $\approx106.3°$; the description is correct — the angle is obtuse. **Q15.** (a) $QS\approx10.82$ m. (b) Angle $R\approx92°$. **Q16.** (a) $\cos A=\dfrac{100+100-100}{200}=0.5$, so $A=60°$; the same at every vertex. (b) $73.74°$, $53.13°$, $53.13°$; sum $=180.00°$. **Q17.** ≈45.6 km; about 15.4 km shorter. **Q18.** $\cos\theta=-1.4375$, which is impossible (a cosine lies between -1 and 1); also $4+6=10<11$, so the two shorter sides cannot reach.

Fully-worked solutions

Q1. (a) $a^2=b^2+c^2-2bc\cos A$. (b) Substituting $b=10$, $c=6$ and $A=47°$ gives $a^2=10^2+6^2-2\times10\times6\times\cos 47°$. (c) $a^2=100+36-120\cos 47°=136-120\cos 47°\approx136-81.8398=54.1602$. (d) $a=\sqrt{54.1602\ldots}\approx7.36$ cm.

Q2. (a) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$. (b) $\cos A=\dfrac{7^2+5^2-9^2}{2\times7\times5}=\dfrac{49+25-81}{70}=\dfrac{-7}{70}=-0.1$. (c) $A=\cos^{-1}(-0.1)\approx95.7°$. (d) In (b) the cosine came out **negative**, and a negative cosine always corresponds to an obtuse angle, so A had to be obtuse before $\cos^{-1}$ was evaluated. (Consistently, $a=9$ is the longest side, so A is the triangle's largest angle.)

Q3. (a) In triangle TSC the known sides $TS=25$ and $TC=18$ include the $112°$ angle at the tap, so $SC^2=25^2+18^2-2\times25\times18\times\cos 112°$. (b) The angle $112°$ is obtuse, so $\cos 112°$ is **negative**; multiplying it by the negative sign in $-2\times25\times18\times\cos 112°$ makes the whole term positive. The third side is therefore **longer** than Pythagoras alone would give. (c) $SC^2=625+324-900\cos 112°\approx949+337.1459=1286.1459$, so $SC=\sqrt{1286.1459\ldots}\approx35.86$ m.

Q4. (a) $a^2=8^2+6^2-2\times8\times6\times\cos 90°=100-96\times0=100$, so $a=10$ m. With a $90°$ included angle the cosine term vanishes and the cosine rule reduces to Pythagoras' theorem, $a^2=b^2+c^2$. (b) $a^2=100-96\cos 60°=100-96\times0.5=52$, so $a=\sqrt{52}\approx7.21$ m. (c) $\cos 120°=-0.5$, so $a^2=100-96\times(-0.5)=100+48=148$ and $a=\sqrt{148}\approx12.17$ m. (d) With the same two sides, the right-angled triangle gave $a=10$ m. When $A=60°$ (acute) the term $-2bc\cos A$ is negative, and the third side (7.21 m) is **shorter** than the Pythagoras value; when $A=120°$ (obtuse) the term is positive, and the third side (12.17 m) is **longer**. The cosine rule is Pythagoras' theorem plus a correction term for the angle.

Q5. (a) The largest angle is opposite the longest side, 10 cm: $\cos\theta=\dfrac{5^2+7^2-10^2}{2\times5\times7}=\dfrac{25+49-100}{70}=\dfrac{-26}{70}\approx-0.3714$, so $\theta=\cos^{-1}(-0.3714\ldots)\approx111.8°$. (b) The smallest angle is opposite the shortest side, 5 cm: $\cos\alpha=\dfrac{7^2+10^2-5^2}{2\times7\times10}=\dfrac{124}{140}\approx0.8857$, so $\alpha\approx27.7°$. (c) By the cosine rule, the angle opposite the 7 cm side satisfies $\cos\beta=\dfrac{5^2+10^2-7^2}{2\times5\times10}=\dfrac{76}{100}=0.76$, so $\beta\approx40.5°$. By the angle sum, $\beta\approx180°-111.8°-27.7°=40.5°$. The two methods agree.

Q6. (a) **Cosine rule** — two sides and the **included** angle are known, so the third side comes from $a^2 = b^2 + c^2 - 2bc\cos A$; there is no known angle–opposite-side pair for the sine rule. (b) **Sine rule** — the $63°$ angle is paired with its known opposite side, which is exactly what the sine rule needs (and only one side is known, so the cosine rule cannot start). (c) **Cosine rule** — with three sides and no angle, only the rearranged cosine rule can produce an angle. (d) **Sine rule** — the known pair ($35°$ opposite 14 cm) together with the 17 cm side gives the angle opposite 17 cm directly.

Q7. In triangle DEF the known sides $e=14$ and $f=9$ include the known angle $D=63°$, and d is opposite D: $d^2 = e^2 + f^2 - 2ef\cos D = 14^2 + 9^2 - 2\times14\times9\times\cos63° = 277 - 252\cos63° \approx 277 - 114.4056 = 162.5944$. Hence $d = \sqrt{162.5944\ldots} \approx 12.75$ cm.

Q8. The largest angle θ is opposite the 15 cm side: $\cos\theta = \dfrac{9^2+12^2-15^2}{2\times9\times12} = \dfrac{81+144-225}{216} = \dfrac{0}{216} = 0$, so $\theta = \cos^{-1}(0) = 90°$. The triangle is **right-angled**. Check without trigonometry: $9^2 + 12^2 = 81 + 144 = 225 = 15^2$, so the sides satisfy Pythagoras' theorem exactly (this is the 3–4–5 triangle scaled by 3), confirming the right angle.

Q9. The kayakers' distances 650 m and 900 m include the $54°$ angle at the pontoon, so their distance apart d satisfies
$d^2 = 650^2 + 900^2 - 2\times650\times900\times\cos54° = 1232500 - 1170000\cos54° \approx 1232500 - 687708.75 = 544791.25$. Hence $d = \sqrt{544791.25\ldots} \approx 738.10$, so the kayakers are approximately 738 m apart.

Q10. The largest angle θ is opposite the longest side, 5.9 m:
$\cos\theta = \dfrac{3.2^2+4.5^2-5.9^2}{2\times3.2\times4.5} = \dfrac{10.24+20.25-34.81}{28.8} = \dfrac{-4.32}{28.8} = -0.15$, so $\theta = \cos^{-1}(-0.15) \approx 98.63° \approx 99°$.

Q11. The two legs 1.4 km and 2.0 km include the $133°$ angle at the turn, so the distance d from the launch point satisfies $d^2 = 1.4^2 + 2.0^2 - 2\times1.4\times2.0\times\cos133° = 5.96 - 5.6\cos133° \approx 5.96 + 3.8192 = 9.7792$, because $\cos133°$ is negative. Hence $d = \sqrt{9.7792\ldots} \approx 3.13$ km.

Q12. The angle between the two shorter legs is opposite the longest leg, 2.1 km:
$\cos\theta = \dfrac{1.2^2+1.5^2-2.1^2}{2\times1.2\times1.5} = \dfrac{1.44+2.25-4.41}{3.6} = \dfrac{-0.72}{3.6} = -0.2$, so $\theta = \cos^{-1}(-0.2) \approx 101.54° \approx 102°$.

Q13. The known sides $GH=7.5$ and $HK=6$ include the known angle $H=82°$, and GK is opposite H: $GK^2 = 7.5^2 + 6^2 - 2\times7.5\times6\times\cos82° = 92.25 - 90\cos82° \approx 92.25 - 12.5256 = 79.7244$, so $GK = \sqrt{79.7244\ldots} \approx 8.93$ cm. The sine rule could not be used directly because the only known angle, $82°$, is opposite the **unknown** side GK: every sine-rule equation, such as $\dfrac{GK}{\sin82°} = \dfrac{6}{\sin G}$, would contain two unknowns.

Q14. The angle θ between the two 30 m fences is opposite the 48 m frontage:
$\cos\theta = \dfrac{30^2+30^2-48^2}{2\times30\times30} = \dfrac{900+900-2304}{1800} = \dfrac{-504}{1800} = -0.28$, so $\theta = \cos^{-1}(-0.28) \approx 106.3°$. Since $106.3° > 90°$, the back corner is indeed obtuse — the plans' description "wider than a right angle" is correct.

Q15. (a) In triangle PQS the sides $PQ=9$ and $PS=12$ include the $60°$ angle at P: $QS^2 = 9^2 + 12^2 - 2\times9\times12\times\cos60° = 225 - 216\times0.5 = 117$, so $QS = \sqrt{117} \approx 10.82$ m. (b) In triangle QRS all three sides are now known: $QR=7$, $RS=8$ and QS, with $QS^2=117$ exactly (using the unrounded value avoids rounding error). Angle R is opposite QS: $\cos R = \dfrac{7^2+8^2-117}{2\times7\times8} = \dfrac{49+64-117}{112} = \dfrac{-4}{112} \approx -0.0357$, so $R = \cos^{-1}(-0.0357\ldots) \approx 92.05° \approx 92°$.

Q16. (a) Every angle sits between two sides of 10 cm and opposite the third: $\cos A = \dfrac{10^2+10^2-10^2}{2\times10\times10} = \dfrac{100}{200} = 0.5$, so $A = \cos^{-1}(0.5) = 60°$ exactly, and by the same calculation each angle of the triangle is $60°$. (b) The apex angle is opposite the 12 cm side: $\cos\theta = \dfrac{10^2+10^2-12^2}{2\times10\times10} = \dfrac{56}{200} = 0.28$, so $\theta \approx 73.74°$. Each base angle is opposite a 10 cm side: $\cos\beta = \dfrac{10^2+12^2-10^2}{2\times10\times12} = \dfrac{144}{240} = 0.6$, so $\beta \approx 53.13°$; by symmetry the two base angles are equal, as expected in an isosceles triangle. Check: $73.74° + 53.13° + 53.13° = 180.00°$.

Q17. The two roads of 34 km and 27 km include the $96°$ angle at the roundabout, so the link road d satisfies $d^2 = 34^2 + 27^2 - 2\times34\times27\times\cos96° = 1885 - 1836\cos96° \approx 1885 + 191.9143 = 2076.9143$, because $\cos 96°$ is negative. Hence $d = \sqrt{2076.9143\ldots} \approx 45.6$ km. Driving via the roundabout covers $34 + 27 = 61$ km, so the direct road would be about $61 - 45.6 = 15.4$ km shorter.

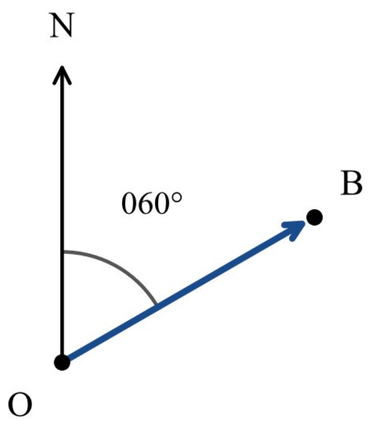
Q18. If such a triangle existed, the angle θ opposite the 11 m side would satisfy $\cos\theta = \dfrac{4^2+6^2-11^2}{2\times4\times6} = \dfrac{16+36-121}{48} = \dfrac{-69}{48} = -1.4375$. But the cosine of any angle lies between -1 and 1, so no angle has a cosine of -1.4375 — the calculator's $\cos^{-1}$ key returns an error, and the “triangle” is impossible. The side lengths alone confirm this: $4+6=10<11$, so the two shorter sides together are not long enough to reach the ends of the 11 m side (the triangle inequality fails).

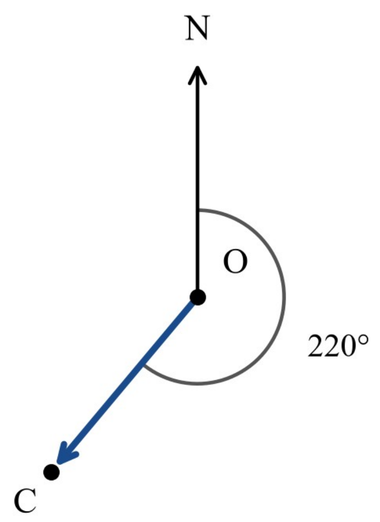
Theory

Bushwalkers, sailors, pilots and surveyors all need a precise way to describe direction. Compass words such as “roughly north-east” are too vague to navigate by, so navigation uses bearings, and the triangles they create are solved with the right-angled trigonometry of 10A and the sine and cosine rules of 10B and 10C.

Three-figure (true) bearings. A **three-figure bearing** (also called a **true bearing**) is the direction of one point from another, measured **clockwise from north** and written with three digits, from 000° up to 360°. Due north is 000°, due east is 090°, due south is 180° and due west is 270°. A direction halfway between south and west (south-west) is 225°. Navigators sometimes add a T, writing 042°T, to emphasise that the angle is measured from true north.

Drawing a bearings diagram. Every bearings problem should start with a diagram. At the point the bearing is measured **from**, draw a **north arrow**, then measure the bearing clockwise from that arrow. The diagrams below show bearings of 060° and 220° from a point O. Each point of a journey gets its own north arrow, and all the north arrows are parallel.





Right-angled triangles in navigation. A single leg of a journey can be split into a distance north (or south) and a distance east (or west). For a leg of d km on a bearing θ between 000° and 090°, the leg is the hypotenuse of a right-angled triangle, so

$$\text{distance north} = d\cos\theta\,,\text{distance east} = d\sin\theta\,.$$

For bearings beyond 090°, draw the diagram and work with the acute angle between the leg and the north–south line. When two legs run at right angles (for example due east then due south), Pythagoras’ theorem gives the direct distance and $\tan^{-1}$ gives the angle, which is then converted to a bearing by measuring clockwise from north on the diagram.

Back bearings. The **back bearing** (or return bearing) is the bearing of the return journey — the direction from the finishing point back to the starting point. Outward and back bearings always differ by exactly 180°: if the outward bearing is **less than 180°, add 180°**; if it is **180° or more, subtract 180°**. For example, an outward bearing of 060° has back bearing 240°, and an outward bearing of 310° has back bearing 130°.

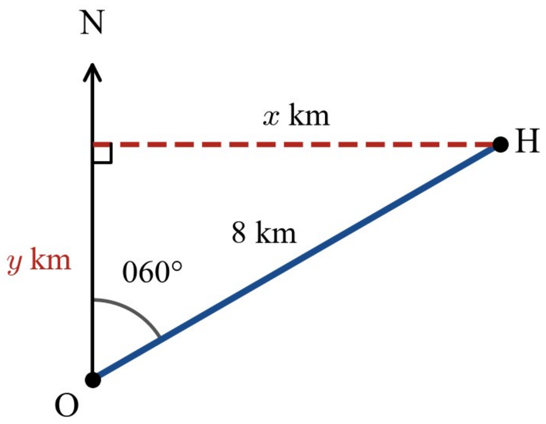
Multi-leg journeys. When the legs of a journey are not at right angles, the two legs and the direct path form a triangle that is solved with the sine and cosine rules. The key step is the **angle between the legs at the turning point**: find the bearing of the starting point from the turning point (the back bearing of the first leg), then find the angle between that direction and the bearing of the second leg. The cosine rule then gives the direct distance, and the sine rule gives the angle needed to state a bearing. The same triangle methods locate a landmark that has been sighted on two different bearings from two known positions — navigators call this a **fix**.

Using CAS. Check that your CAS is set to **degree mode** before any bearings work — radian mode is the most common source of wrong answers in navigation problems. Use sin, cos and tan and their inverses exactly as in 10A–10C, store the unrounded value, and round only at the final step. A CAS cannot see your diagram, so always check that a bearing answer lies in the correct quadrant: a point that your diagram shows to the south-west, for instance, must have a bearing between 180° and 270°.

Worked examples

Example 1 — scaffolded

On a hike in the Grampians, Mia walks 8 km from her camp O on a bearing of 060° to a waterfall H.



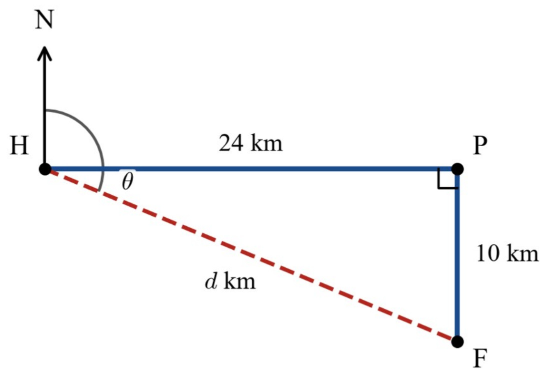
Find how far north and how far east the waterfall is from the camp (correct to two decimal places where necessary), and state the bearing Mia must walk to return directly to camp.

Working	Comment
The bearing 060° is measured clockwise from the north arrow at O, so the leg OH makes an angle of 60° with the north line.	Always start from a diagram with a north arrow at the point the bearing is measured from.
Distance north: $y = 8\cos 60° = 4$, so the waterfall is 4 km north of the camp.	The northward side is adjacent to the 60° angle in the right-angled triangle, and the 8 km leg is the hypotenuse.
Distance east: $x = 8\sin 60° \approx 6.93$, so the waterfall is about 6.93 km east of the camp.	The eastward side is opposite the 60° angle.
Return bearing $= 060° + 180° = 240°$.	The outward bearing is less than 180°, so add 180° to find the back bearing.

So the waterfall is 4 km north and about 6.93 km east of the camp, and the return walk is on a bearing of 240°.

Example 2 — scaffolded

A fishing boat leaves the harbour H at Lakes Entrance and motors 24 km due east to a reef P, then 10 km due south to a fishing spot F.



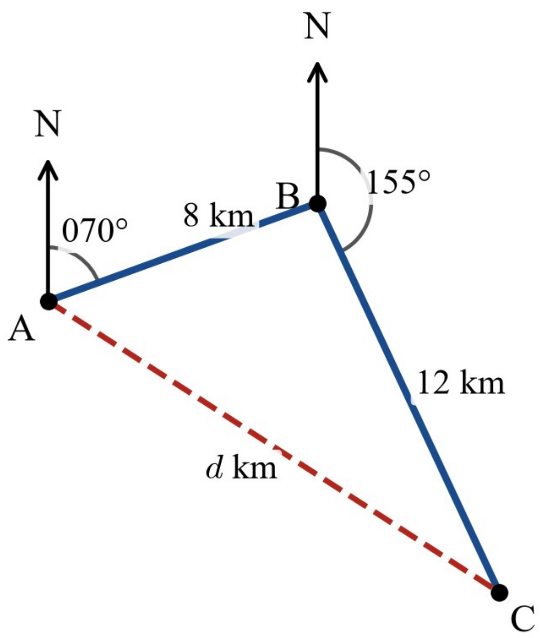
Find the distance from the harbour to F, and the bearing θ of F from the harbour, to the nearest degree.

Working	Comment
The legs run due east then due south, so triangle HPF has a right angle at P.	Perpendicular legs mean Pythagoras' theorem and basic trigonometry apply — no sine or cosine rule is needed.
$HF = \sqrt{24^2 + 10^2} = \sqrt{676} = 26$, so F is 26 km from the harbour.	Pythagoras' theorem with the two legs as the shorter sides.
Angle between the due-east leg HP and the line HF: $\tan^{-1}\left(\frac{10}{24}\right) \approx 22.62°$.	In the right-angled triangle, the side opposite this angle is 10 and the adjacent side is 24.
Bearing of F from H: $\theta = 90° + 22.62° = 112.62° \approx 113°$.	Due east is 090°, and HF lies a further 22.62° clockwise (south of east).

So the fishing spot is 26 km from the harbour on a bearing of approximately 113°.

Example 3 — unscaffolded

In an orienteering event at the You Yangs, Sanjay runs 8 km on a bearing of 070° from the start A to a checkpoint B, then 12 km on a bearing of 155° to the finish C. Find the direct distance from the start to the finish, correct to two decimal places, and the bearing of the finish from the start, to the nearest degree.



At the checkpoint B, the bearing of A from B is the back bearing of the first leg, $070° + 180° = 250°$. The second leg leaves B on a bearing of 155°, so the angle between the legs is $\angle ABC = 250° - 155° = 95°$. By the cosine rule,

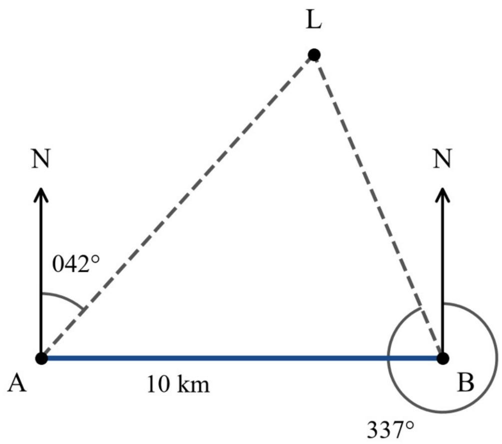
$$AC^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 95° = 64 + 144 - 192 \cos 95° \approx 224.73,$$

so $AC \approx 14.99$ km. To find the bearing of C from A, use the sine rule in triangle ABC: $\dfrac{\sin A}{12} = \dfrac{\sin 95°}{14.99}$, so $\sin A \approx 0.7974$ and $\angle BAC \approx 52.89°$. Sanjay turned clockwise (to his right) at B, so the finish lies 52.89° clockwise of the first leg's direction: the bearing of C from A is $070° + 52.89° = 122.89°$.

∴ the finish is approximately 14.99 km from the start, on a bearing of 123° (to the nearest degree).

Example 4 — unscaffolded

From a ship at A, the skipper sights the Cape Otway lighthouse L on a bearing of 042°. The ship sails 10 km due east to B, from where the lighthouse bears 337°. Find the distance from the lighthouse to each position of the ship, correct to two decimal places, and the shortest distance between the lighthouse and the ship's course.



At A the course runs due east (090°) and the lighthouse bears 042°, so $\angle LAB = 90° - 42° = 48°$. At B the bearing of A is due west, 270°, and the lighthouse bears 337°, so $\angle LBA = 337° - 270° = 67°$. The angles of triangle ABL sum to 180°, so $\angle ALB = 180° - 48° - 67° = 65°$. By the sine rule with $AB = 10$:

$$BL = \frac{10 \sin 48°}{\sin 65°} \approx 8.20, \quad AL = \frac{10 \sin 67°}{\sin 65°} \approx 10.16.$$

The shortest distance from the lighthouse to the course is along the perpendicular from L to the line AB, which forms a right-angled triangle with hypotenuse AL: distance = $AL \sin 48° \approx 10.16 \times \sin 48° \approx 7.55$ km.

∴ the lighthouse is approximately 10.16 km from A and 8.20 km from B, and the ship passes within about 7.55 km of the lighthouse on this course.

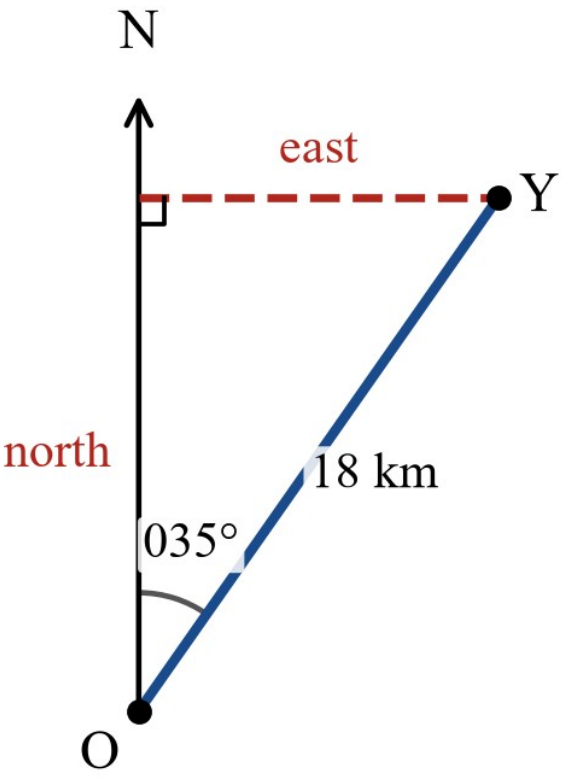
Practice questions

Scaffolded

Q1. A ranger at a trail junction in the Dandenong Ranges records the direction of each track. Write each direction as a three-figure bearing. (a) due east (b) due south (c) south-west (d) north-west (e) 20° east of due north (f) 30° west of due south

Q2. For each outward bearing, state the back bearing (the bearing of the return journey). (a) 065° (b) 130° (c) 218° (d) 340°

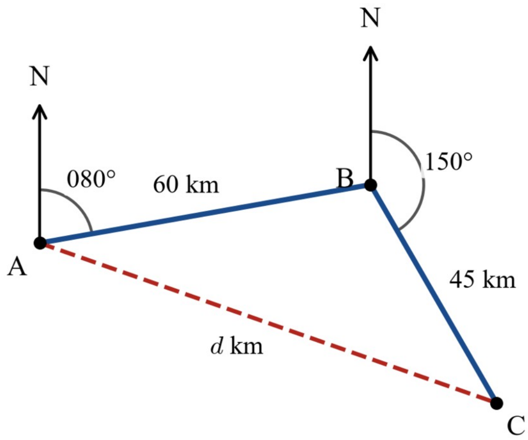
Q3. A yacht leaves Portarlington O and sails 18 km on a bearing of 035° to a racing mark Y, as shown.



(a) Find how far north of Portarlington the mark is, correct to two decimal places.
(b) Find how far east of Portarlington the mark is, correct to two decimal places.
(c) State the bearing the yacht must sail to return directly to Portarlington.

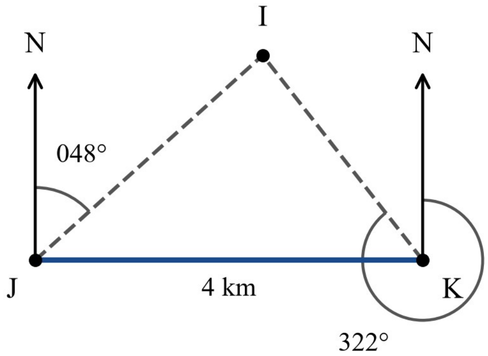
Q4. From a carpark in the Cathedral Ranges, Lena hikes 9 km due north along a ridge, then 5 km due east to a lookout. (a) Draw a bearings diagram of the two legs, with a north arrow at the carpark. (b) Find the direct distance from the carpark to the lookout, correct to two decimal places. (c) Find the bearing of the lookout from the carpark, to the nearest degree. (d) Find the bearing of the carpark from the lookout, to the nearest degree.

Q5. A light plane takes off from Mildura A, flies 60 km on a bearing of 080° to B, then turns and flies 45 km on a bearing of 150° to C, as shown.



(a) Show that $\angle ABC = 110°$.
(b) Use the cosine rule to find the direct distance AC, correct to two decimal places.
(c) Use the sine rule to find $\angle BAC$, and hence find the bearing of C from A, to the nearest degree.

Q6. From a jetty J on the Gippsland Lakes, a small island I bears 048°. From a point K, 4 km due east of the jetty, the island bears 322°, as shown.



(a) Explain why $\angle IJK = 42°$ and $\angle IKJ = 52°$.
(b) Find $\angle JIK$.
(c) Use the sine rule to find the distance from the island to the jetty and to K, each correct to two decimal places.

Unscaffolded

Q7. A ferry crosses Port Phillip from Queenscliff towards Sorrento on a bearing of 118°. (a) State the bearing of the return crossing. (b) On another route the outward bearing is 296°. State the bearing of that return crossing.

Q8. A bushwalker leaves a picnic ground and walks 6.5 km on a bearing of 122°. How far south and how far east of the picnic ground is she? Give both distances correct to two decimal places.

Q9. From a boat ramp at Inverloch, a tinnie motors 12 km due west, then 9 km due north into an inlet. Find the tinnie's direct distance from the ramp, and the bearing of the tinnie from the ramp, to the nearest degree.

Q10. A rescue helicopter is hovering 30 km north and 16 km west of its base in the Latrobe Valley. Find its direct distance from the base, the bearing of the helicopter from the base, and the bearing the pilot must fly to return directly to base, with bearings to the nearest degree.

Q11. In a night orienteering event, Keiko runs 2.4 km on a bearing of 065° to a checkpoint, then 3.1 km on a bearing of 140° to the finish. Find her direct distance from the start, correct to two decimal places, and the bearing of the start from the finish, to the nearest degree.

Q12. On a cattle station near Broken Hill, a quad bike travels 7 km on a bearing of 030°, then 5 km on a bearing of 100° to reach a dam. Rani says the direct distance from the start to the dam can be found using Pythagoras' theorem. Explain why she is incorrect, state which rule should be used instead, and find the direct distance correct to two decimal places.

Q13. Fire-spotting tower Q is 11 km due east of fire-spotting tower P. From P a smoke plume bears 038°, and from Q the same plume bears 312°. Find the distance from each tower to the plume, correct to two decimal places, and state which tower's crew is closer to the fire.

Q14. A farm ute in the Wimmera leaves the homestead and drives 5 km due north, then 7 km due east, then 2 km due south, stopping at a bore. Find the direct distance from the homestead to the bore, correct to two decimal places, and the bearing of the bore from the homestead, to the nearest degree.

Q15. The bearing of a campsite from a lookout is 250°. Priya claims that the bearing of the lookout from the campsite is 110°, because $360° - 250° = 110°$. Explain the error in Priya's reasoning and find the correct bearing.

Q16. A sea kayaker at Apollo Bay paddles 1.8 km on a bearing of 210°, then 2.6 km on a bearing of 285°. Find the direct distance back to her starting point, correct to two decimal places, and the bearing she must paddle to return directly, to the nearest degree.

Q17. From the Point Lonsdale lighthouse, boat A is 8 km away on a bearing of 075° and boat B is 11 km away on a bearing of 158°. Find the distance between the two boats, correct to two decimal places.

Q18. A charter pilot flies from an airfield near Horsham 90 km on a bearing of 340°, then 70 km on a bearing of 250°. (a) Show that the angle between the two legs at the turning point is 90°. (b) Hence find the direct distance from the plane back to the airfield, correct to two decimal places. (c) Find the bearing the pilot must fly to return straight to the airfield, to the nearest degree.

Answers

Q1. (a) 090°. (b) 180°. (c) 225°. (d) 315°. (e) 020°. (f) 210°. **Q2.** (a) 245°. (b) 310°. (c) 038°. (d) 160°. **Q3.** (a) 14.74 km north. (b) 10.32 km east. (c) 215°. **Q4.** (b) 10.30 km. (c) 029°. (d) 209°. **Q5.** (a) $260° - 150° = 110°$. (b) $AC \approx 86.44$ km. (c) $\angle BAC \approx 29.29°$; bearing $\approx 109°$. **Q6.** (b) $\angle JIK = 86°$. (c) $JI \approx 3.16$ km, $KI \approx 2.68$ km. **Q7.** (a) 298°. (b) 116°. **Q8.** About 3.44 km south and 5.51 km east. **Q9.** 15 km; bearing $\approx 307°$. **Q10.** 34 km; bearing of helicopter $\approx 332°$; bearing to fly home $\approx 152°$. **Q11.** Distance ≈ 4.38 km; bearing of start from finish $\approx 288°$. **Q12.** The legs meet at 110°, not 90°, so the triangle is not right-angled; use the cosine rule; distance ≈ 9.90 km. **Q13.** $PF \approx 7.38$ km, $QF \approx 8.69$ km; tower P's crew is closer. **Q14.** Distance ≈ 7.62 km; bearing $\approx 067°$. **Q15.** Back bearings differ by 180°, not to 360°; correct bearing $= 250° - 180° = 070°$. **Q16.** Distance ≈ 3.52 km; bearing to paddle home $\approx 075°$. **Q17.** $AB \approx 12.79$ km. **Q18.** (a) $250° - 160° = 90°$. (b) $\sqrt{13000} \approx 114.02$ km. (c) $\approx 122°$.

Fully-worked solutions

Q1. Bearings are measured clockwise from north. (a) Due east is a quarter-turn clockwise from north: 090°. (b) Due south is a half-turn: 180°. (c) South-west is halfway between south (180°) and west (270°): 225°. (d) North-west is halfway between west (270°) and north (360°): 315°. (e) 20° east of due north is 20° clockwise from 000°: 020°. (f) 30° west of due south is 30° clockwise from 180°: 210°.

Q2. Outward and back bearings differ by 180°. (a) $065° < 180°$, so add: $065° + 180° = 245°$. (b) $130° + 180° = 310°$. (c) $218° \geq 180°$, so subtract: $218° - 180° = 038°$. (d) $340° - 180° = 160°$.

Q3. (a) The leg makes an angle of 35° with the north line at O, and the 18 km leg is the hypotenuse, so the distance north is $18\cos 35° \approx 14.74$ km. (b) The distance east is $18\sin 35° \approx 10.32$ km. (c) The outward bearing 035° is less than 180°, so the return bearing is $035° + 180° = 215°$.

Q4. (a) The diagram shows a north arrow at the carpark, a 9 km leg straight up (due north), then a 5 km leg to the right (due east) ending at the lookout, with the direct path as the hypotenuse of a right-angled triangle. (b) The legs are perpendicular, so by Pythagoras' theorem the direct distance is $\sqrt{9^2 + 5^2} = \sqrt{106} \approx 10.30$ km. (c) The angle at the carpark between north and the direct path satisfies $\tan\theta = \frac{5}{9}$, so $\theta = \tan^{-1}\left(\frac{5}{9}\right) \approx 29.05°$ and the bearing of the lookout is approximately 029°. (d) The back bearing is $29.05° + 180° = 209.05° \approx 209°$.

Q5. (a) At B, the bearing of A is the back bearing of the first leg, $080° + 180° = 260°$. The second leg leaves B on a bearing of 150°, so $\angle ABC = 260° - 150° = 110°$. (b) By the cosine rule, $AC^2 = 60^2 + 45^2 - 2 \times 60 \times 45 \times \cos 110° = 3600 + 2025 - 5400\cos 110° \approx 7471.91$, so $AC \approx 86.44$ km. (c) By the sine rule, $\frac{\sin(\angle BAC)}{45} = \frac{\sin 110°}{86.44}$, so $\sin(\angle BAC) \approx 0.4892$ and $\angle BAC \approx 29.29°$. The plane turned clockwise at B, so C lies 29.29° clockwise of the first leg's direction: the bearing of C from A is $080° + 29.29° = 109.29° \approx 109°$.

Q6. (a) At J, the direction of K is due east, 090°, and the island bears 048°, so $\angle IJK = 90° - 48° = 42°$. At K, the direction of J is due west, 270°, and the island bears 322°, so $\angle IKJ = 322° - 270° = 52°$. (b) The angles of triangle IJK sum to 180°, so $\angle JIK = 180° - 42° - 52° = 86°$. (c) By the sine rule with $JK = 4$: $JI = \frac{4\sin 52°}{\sin 86°} \approx 3.16$ km and $KI = \frac{4\sin 42°}{\sin 86°} \approx 2.68$ km.

Q7. (a) $118° < 180°$, so the return bearing is $118° + 180° = 298°$. (b) $296° \geq 180°$, so the return bearing is $296° - 180° = 116°$.

Q8. The bearing 122° lies between 090° and 180°, so the walker finishes south of east. The leg makes an angle of $180° - 122° = 58°$ with the southward direction. With the 6.5 km leg as the hypotenuse: distance south $= 6.5\cos 58° \approx 3.44$ km and distance east $= 6.5\sin 58° \approx 5.51$ km.

Q9. The legs are perpendicular, so the direct distance is $\sqrt{12^2+9^2}=\sqrt{225}=15$ km. The tinnie finishes west and north of the ramp. The angle between north and the direct line, measured towards the west, is $\tan^{-1}\left(\frac{12}{9}\right)\approx 53.13°$, so the bearing of the tinnie from the ramp is $360°-53.13°=306.87°\approx 307°$.

Q10. The helicopter is 30 km north and 16 km west of the base, so its direct distance is $\sqrt{30^2+16^2}=\sqrt{1156}=34$ km. The angle between north and the line to the helicopter, measured towards the west, is $\tan^{-1}\left(\frac{16}{30}\right)\approx 28.07°$, so the bearing of the helicopter from the base is $360°-28.07°=331.93°\approx 332°$. The bearing to fly home is the back bearing: $331.93°-180°=151.93°\approx 152°$.

Q11. At the checkpoint, the bearing of the start is the back bearing of the first leg, $065°+180°=245°$, and the second leg leaves on a bearing of 140°, so the angle between the legs is $245°-140°=105°$. By the cosine rule, the direct distance d satisfies $d^2=2.4^2+3.1^2-2\times 2.4\times 3.1\times\cos 105°\approx 19.22$, so $d\approx 4.38$ km. By the sine rule, the angle at the start between the first leg and the direct line satisfies $\sin A=\frac{3.1\sin 105°}{4.38}\approx 0.6830$, so $A\approx 43.08°$. Keiko turned clockwise at the checkpoint, so the bearing of the finish from the start is $065°+43.08°=108.08°$, and the bearing of the start from the finish is the back bearing, $108.08°+180°=288.08°\approx 288°$.

Q12. Pythagoras' theorem applies only to right-angled triangles. At the turning point, the bearing of the start is $030°+180°=210°$ and the second leg leaves on a bearing of 100°, so the legs meet at $210°-100°=110°$, not 90° — the triangle is not right-angled, so Rani is incorrect. The cosine rule should be used instead: $d^2=7^2+5^2-2\times 7\times 5\times\cos 110°=74-70\cos 110°\approx 97.94$, so the direct distance is $d\approx 9.90$ km.

Q13. At P, the direction of Q is due east, 090°, and the plume bears 038°, so the angle at P is $90°-38°=52°$. At Q, the direction of P is due west, 270°, and the plume bears 312°, so the angle at Q is $312°-270°=42°$. The angle at the plume F is $180°-52°-42°=86°$. By the sine rule with $PQ=11$: $PF=\frac{11\sin 42°}{\sin 86°}\approx 7.38$ km and $QF=\frac{11\sin 52°}{\sin 86°}\approx 8.69$ km. Since $7.38<8.69$, tower P's crew is closer to the fire.

Q14. The ute finishes $5-2=3$ km north and 7 km east of the homestead. The direct distance is $\sqrt{7^2+3^2}=\sqrt{58}\approx 7.62$ km. The angle between north and the direct line, measured towards the east, is $\tan^{-1}\left(\frac{7}{3}\right)\approx 66.80°$, so the bearing of the bore from the homestead is approximately 067°.

Q15. A bearing and its back bearing describe the same line travelled in opposite directions, so they differ by exactly 180° — they do not add to 360°. Priya has subtracted from 360°, which gives the angle measured anticlockwise from north, not a bearing of the return direction. Since $250°\geq 180°$, the correct bearing of the lookout from the campsite is $250°-180°=070°$.

Q16. At the turning point, the bearing of the start is the back bearing of the first leg, $210°-180°=030°$. Turning clockwise from the direction of the start (030°) to the second leg (285°) sweeps $285°-30°=255°$, so the interior angle between the legs is $360°-255°=105°$. By the cosine rule, the direct distance d back to the start satisfies $d^2=1.8^2+2.6^2-2\times 1.8\times 2.6\times\cos 105°\approx 12.42$, so $d\approx 3.52$ km. By the sine rule, the angle at the finish between the line back to the turning point and the line back to the start satisfies $\sin C=\frac{1.8\sin 105°}{3.52}\approx 0.4933$, so $C\approx 29.56°$. From the finish, the turning point lies on the back bearing of 285°, which is $285°-180°=105°$. The start lies on the northern side of that line (the kayaker travelled south-west then west-north-west, so home is back to the north-east), so the bearing to paddle home is $105°-29.56°=75.44°\approx 075°$.

Q17. The angle at the lighthouse between the two sight lines is $158°-75°=83°$. By the cosine rule, $AB^2=8^2+11^2-2\times 8\times 11\times\cos 83°=185-176\cos 83°\approx 163.55$, so the boats are $AB\approx 12.79$ km apart.

Q18. (a) At the turning point, the bearing of the airfield is the back bearing of the first leg, $340° - 180° = 160°$. The second leg leaves on a bearing of 250°, so the angle between the legs is $250° - 160° = 90°$. (b) The triangle is right-angled at the turning point, so by Pythagoras' theorem the direct distance is $\sqrt{90^2 + 70^2} = \sqrt{13000} \approx 114.02$ km. (c) The angle at the airfield between the first leg and the direct line satisfies $\tan\theta = \frac{70}{90}$, so $\theta \approx 37.87°$. The plane turned anticlockwise (the second leg swings from 340° back towards the south-west), so the plane's bearing from the airfield is $340° - 37.87° = 302.13°$, and the bearing to fly home is the back bearing: $302.13° - 180° = 122.13° \approx 122°$.

Theory

In Section 9B two ways of finding the area of a triangle were met. When the base and the **perpendicular height** are known, the area is

$$A = 1/2\,b\,h,$$

and when all **three side lengths** are known but no height is, **Heron's rule** $A = \sqrt{s(s-a)(s-b)(s-c)}$ (with semi-perimeter $s = 1/2(a+b+c)$) gives the area directly. Neither rule helps with the very common surveying situation in which **two sides and the angle between them** are known but no height and no third side has been measured. This section develops a third area formula for exactly that case.

The included angle. In triangle ABC the side opposite each angle takes the matching lower-case letter, so side a is opposite A, side b opposite B and side c opposite C. The angle **between** two given sides is called the **included angle**: the angle C sits between the sides a and b, because those two sides meet at the vertex C.

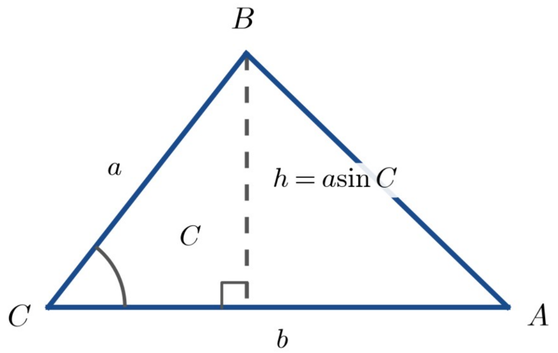
The area rule. In any triangle,

$$A = 1/2\,a\,b\sin C,$$

where a and b are two sides and C is the included angle between them. The rule can be written from any vertex, always as *half the product of two sides times the sine of the angle between them*:

$$A = 1/2\,b\,c\sin A = 1/2\,a\,c\sin B = 1/2\,a\,b\sin C.$$

Where the rule comes from. Take the side b as the base and drop the perpendicular height h from the opposite vertex onto it.



The height is the far side of a right-angled triangle whose hypotenuse is the side a and whose angle at the base is the included angle C, so

$$h = a\sin C.$$

Substituting into $A = 1/2 \times \text{base} \times \text{height} = 1/2\,b\,h$ gives $A = 1/2\,b(a\sin C) = 1/2\,a\,b\sin C$. The rule is simply the base–height formula with the height replaced by $a\sin C$, so no separate height measurement is needed.

Finding an area. Identify the two known sides and check that the known angle lies **between** them, then substitute into $A = 1/2\,a\,b\sin C$ and evaluate. If the angle given is *not* the included one, the rule cannot be used with those two sides.

Finding an unknown side. When the area and one side are known together with the included angle, substitute and rearrange for the unknown side. From $A = 1/2\,a\,b\sin C$,

$$a = \frac{2A}{b\sin C}.$$

Finding the included angle. When the area and both sides are known, rearrange for the sine of the angle:

$$\sin C = \frac{2A}{ab},$$

and then $C = \sin^{-1}\left(\frac{2A}{ab}\right)$. As with the sine rule, there is an **ambiguity** to watch for: because $\sin C = \sin(180° - C)$, an acute angle and its obtuse supplement have the **same sine**, so a given area can be produced by **two** different included angles — a sharp (acute) corner and a wide (obtuse) corner. Unless the context rules one out, **both** answers should be reported. (A useful consequence: since $\sin C$ can never exceed 1, the largest possible area for two given sides is $1/2\,ab$, reached when $C = 90°$; if a required area needs $\sin C > 1$, no such triangle exists.)

Choosing the area method. Which of the three area methods to use depends only on the information given:

Information known	Method	Formula
base and perpendicular height	base–height (9B)	$A = 1/2\,bh$
all three sides	Heron's rule (9B)	$A = \sqrt{s(s-a)(s-b)(s-c)}$
two sides and the included angle	the area rule (this section)	$A = 1/2\,ab\sin C$

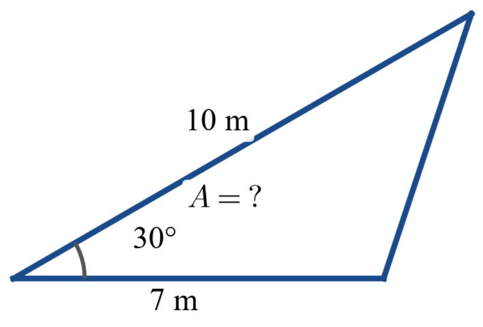
If a perpendicular height is available, $1/2\,bh$ is quickest; if only the three sides are known, use Heron's rule; if two sides and the angle wedged between them are known, use the area rule.

Using CAS. General Mathematics is a technology-active, open-book course. Set the calculator to **degree mode** before you start — a radian-mode answer will be wrong without warning. The area rule can be evaluated in one line, and the equation $A = 1/2\,ab\sin C$ can be entered and solved for an unknown side or angle directly; remember that the calculator's $\sin^{-1}$ returns only the acute angle, so in the angle case you must form the obtuse supplement $180° - C$ yourself and decide whether it also fits. Keep unrounded values on the calculator and round only the final answer.

Worked examples

Example 1 — scaffolded

A triangular garden bed is formed by two straight edges of 10 m and 7 m meeting at an included angle of $30°$. Find the area of the bed.

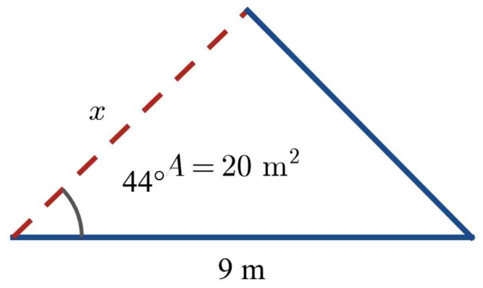


Working	Comment
The two sides are $a = 10$ m and $b = 7$ m, and the angle between them is $C = 30°$.	The $30°$ angle lies where the two edges meet, so it is the included angle.
$A = 1/2\,ab\sin C = 1/2 \times 10 \times 7 \times \sin 30°$.	Substitute into the area rule.
$\sin 30° = 0.5$, so $A = 1/2 \times 70 \times 0.5 = 17.5$.	With a special angle the arithmetic is exact.

So the garden bed has an area of $17.5\,\text{m}^2$.

Example 2 — scaffolded

A triangular sail has an area of $20\,\text{m}^2$. One edge measures 9 m, and the angle between that edge and the adjoining edge is $44\,°$. Find the length of the adjoining edge, correct to two decimal places.

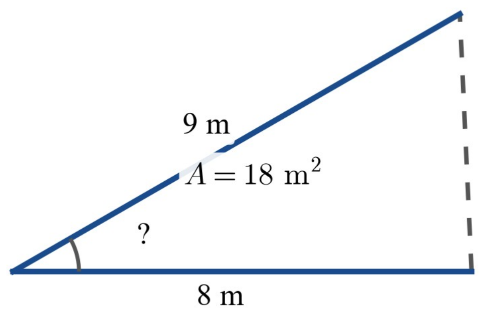


Working	Comment
The known side is 9 m and the included angle is $44\,°$; the area is $20\,\text{m}^2$. Let the adjoining edge be x m.	Two sides and the included angle, but here it is a side that is unknown and the area that is given.
$A=1/2\,a\,b\sin C$ becomes $20=1/2\times 9\times x\times\sin 44\,°$.	Substitute the known area, side and included angle.
$20=4.5\,x\sin 44\,°$, so $x=\dfrac{20}{4.5\sin 44\,°}$.	$1/2\times 9=4.5$; divide both sides by $4.5\sin 44\,°$.
$x=6.398\ldots\approx 6.40$ m.	Evaluate with CAS in degree mode; round only at the end.

So the adjoining edge is approximately 6.40 m long.

Example 3 — unscaffolded

A triangular courtyard garden is planted between two straight walls of length 9 m and 8 m that meet at one corner. The planted region has an area of $18\,\text{m}^2$. Find the size of the angle between the two walls.



The two walls are the known sides $a=9$ m and $b=8$ m, and the angle between them is the included angle C. Rearranging the area rule for $\sin C$,

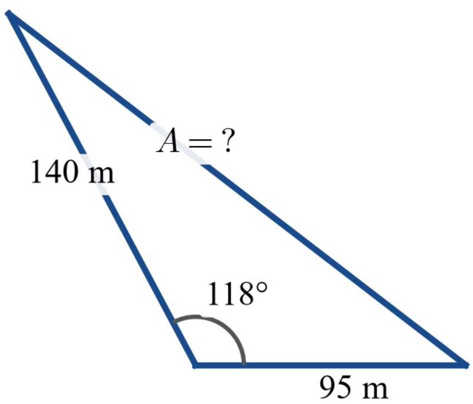
$$\sin C=\frac{2\,A}{a\,b}=\frac{2\times 18}{9\times 8}=\frac{36}{72}=0.5\,.$$

The acute solution is $C = \sin^{-1}(0.5) = 30°$, but because $\sin C = \sin(180° - C)$ the obtuse angle $180° - 30° = 150°$ has the same sine and gives the same area. Both are genuinely possible: the two walls can meet at a sharp $30°$ corner or at a wide $150°$ corner, and each encloses exactly $18\,\mathrm{m}^2$. Nothing in the description rules either out.

∴ the angle between the walls is $30°$ or $150°$.

Example 4 — unscaffolded

A surveyor needs the area of a triangular paddock. Two of its fences, of length 140 m and 95 m, meet at a corner where the surveyor measures the angle between them to be $118°$; the third fence has not been measured. Find the area of the paddock, to the nearest square metre, and explain why the area rule is the appropriate method.



Only two sides and the angle between them are known. There is no perpendicular height, so the base–height formula $1/2\,b\,h$ cannot be used, and the third side is unknown, so Heron's rule cannot be used either; the known angle is exactly the **included** angle between the two known sides, which is precisely what the area rule needs. Substituting $a = 140$, $b = 95$ and $C = 118°$,

$$A = 1/2\,a\,b\sin C = 1/2 \times 140 \times 95 \times \sin 118° = 6650 \sin 118° = 5871.60\ldots$$

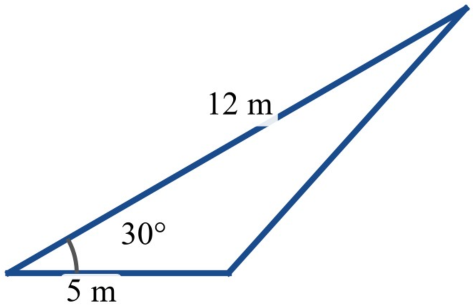
Rounding to the nearest square metre gives $5872\,\mathrm{m}^2$. (As a check, the corner is obtuse, so $\sin 118° < 1$ and the area is a little under the maximum $1/2 \times 140 \times 95 = 6650\,\mathrm{m}^2$ that these two fences could enclose.)

∴ the paddock has an area of approximately $5872\,\mathrm{m}^2$.

Practice questions

Scaffolded

Q1. A triangular garden bed is bounded by two straight edges of 12 m and 5 m that meet at an included angle of $30°$.

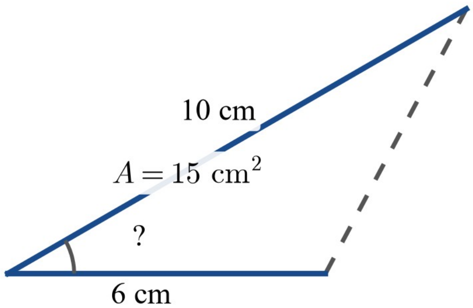


(a) Write down the two sides and the included angle.
(b) Write down the area rule $A = 1/2\,a\,b\sin C$ with the values substituted.
(c) Find the area of the bed.

Q2. A triangular shade sail has two edges of 6.5 m and 4.8 m meeting at an included angle of $63°$. (a) Identify the two sides and the included angle. (b) Substitute into $A = 1/2\,a\,b\sin C$. (c) Find the area of the sail, correct to two decimal places.

Q3. A triangular blade of a wind sculpture has an area of $24\,\text{m}^2$. One side measures 16 m and the included angle at the end of that side is $30°$. (a) Write the area rule with the unknown side x in it. (b) Substitute the known area, side and angle. (c) Solve for x, the length of the other side.

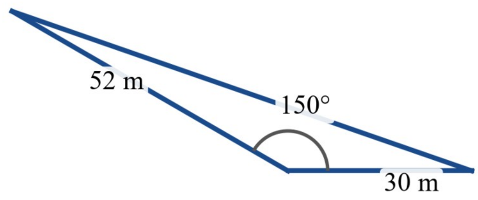
Q4. Two straight fences of length 10 cm and 6 cm on a scale model meet at a corner, enclosing a triangular region of area $15\,\text{cm}^2$.



(a) Use $A = 1/2\,a\,b\sin C$ to form an equation for $\sin C$.
(b) Show that $\sin C = 0.5$.
(c) Write down the two possible sizes of the included angle C.
(d) Explain why both answers are valid.

Q5. For each triangle below, state which of the three area methods — the base–height formula $1/2\,b\,h$, Heron's rule, or the area rule $1/2\,a\,b\sin C$ — you would use and why, then find the area. (a) A triangle with base 14 m and perpendicular height 9 m. (b) A triangular paddock with three known side lengths 6 m, 8 m and 10 m. (c) A triangle with two sides 9 m and 10 m meeting at an included angle of $40°$ (answer correct to two decimal places).

Q6. A triangular block of land is advertised with two boundaries of 52 m and 30 m that meet at a corner of $150°$.



(a) Find the area of the block.
(b) Instant turf is laid over the whole block at $8 per square metre. Find the total cost of the turf.

Unscaffolded

Q7. A triangular courtyard has two sides of 7.2 m and 5.5 m meeting at an included angle of $68°$. Find its area, correct to two decimal places.

Q8. A triangular metal bracket has two edges of 14 cm and 9 cm meeting at an included angle of $120°$. Find its area, both as an exact value and correct to two decimal places.

Q9. A triangular racing flag has an area of $30\,\mathrm{m}^2$. One edge is 15 m long and the included angle at the end of that edge is $30°$. Find the length of the other edge.

Q10. A triangular garden bed has an area of $40\,\mathrm{m}^2$. One side measures 11 m and the included angle at the end of that side is $55°$. Find the length of the other side, correct to two decimal places.

Q11. Two straight paths of length 12 cm and 10 cm on a landscape plan meet at a corner and enclose a triangular lawn of area $40\,\mathrm{cm}^2$. Find the two possible sizes of the angle between the paths, correct to one decimal place.

Q12. A farmer wants to fence a triangular holding pen using two existing fences of length 8 m and 6 m that meet at one corner, and asks whether the pen can be given an area of $30\,\mathrm{m}^2$. By finding the largest area these two fences could possibly enclose, explain whether this is possible.

Q13. For each of the following, name the area method you would use and find the area. (a) A triangle with two sides 8 m and 11 m and an included angle of $35°$ (correct to two decimal places). (b) A triangular reserve whose three boundaries measure 13 m, 14 m and 15 m. (c) A triangle with base 20 m and perpendicular height 6.5 m.

Q14. A boat's triangular sail has two edges of 5.2 m and 3.8 m that meet at the head of the sail at an angle of $48°$. (a) Find the area of the sail, correct to two decimal places. (b) Sailcloth costs $65 per square metre. Find the cost of the cloth for the sail, to the nearest cent.

Q15. A triangular parkland is bounded by two straight paths of 85 m and 60 m that leave a single corner at an angle of $74°$. Find the area of the parkland, to the nearest square metre.

Q16. A triangular garden bed of area $20\,\mathrm{m}^2$ is to be built from two straight timber edges of 8 m and 7 m meeting at one corner. (a) Find the two possible sizes of the angle between the edges, correct to one decimal place. (b) Describe, in the context of the garden bed, how the two possible beds differ.

Q17. An isosceles triangular pennant has two equal sides of 10 cm meeting at the apex. (a) Find the area of the pennant if the apex angle is $40°$, correct to two decimal places. (b) Find the area if instead the apex angle is $140°$, correct to two decimal places. (c) Comment on your two answers, and explain the connection using the sines of the two angles.

Q18. A triangular paddock has two fences of 50 m and 35 m meeting at one corner, and encloses an area of $700\,\mathrm{m}^2$. (a) Find the two possible sizes of the angle between the two fences, correct to one decimal place. (b) Explain, using the value of $\sin C$, why it would be impossible for these two fences to enclose an area of $900\,\mathrm{m}^2$.

Answers

Q1. (a) $a = 12$ m, $b = 5$ m, $C = 30°$. (b) $A = 1/2 \times 12 \times 5 \times \sin 30°$. (c) $A = 15\,\mathrm{m}^2$. **Q2.** (a) 6.5 m, 4.8 m, included angle $63°$. (b) $A = 1/2 \times 6.5 \times 4.8 \times \sin 63°$. (c) $A \approx 13.90\,\mathrm{m}^2$. **Q3.** (a) $24 = 1/2 \times 16 \times x \times \sin 30°$. (b) $24 = 4x$. (c) $x = 6$ m. **Q4.** (a) $15 = 1/2 \times 10 \times 6 \times \sin C$. (b) $30 \sin C = 15$, so $\sin C = \frac{15}{30} = 0.5$. (c) $C = 30°$ or $C = 150°$. (d) $\sin 150° = \sin 30° = 0.5$, so both corners give the area $15\,\mathrm{cm}^2$. **Q5.** (a) Base–height, $A = 63\,\mathrm{m}^2$. (b) Heron's rule, $A = 24\,\mathrm{m}^2$. (c) Area rule, $A \approx 28.93\,\mathrm{m}^2$. **Q6.** (a) $A = 390\,\mathrm{m}^2$. (b) \$3120. **Q7.** $A \approx 18.36\,\mathrm{m}^2$. **Q8.** $A = \frac{63\sqrt{3}}{2} \approx 54.56\,\mathrm{cm}^2$. **Q9.** 8 m. **Q10.** ≈ 8.88 m. **Q11.** $C \approx 41.8°$ or $C \approx 138.2°$. **Q12.** No — the maximum area is $1/2 \times 8 \times 6 = 24\,\mathrm{m}^2$ (at $90°$), which is less than $30\,\mathrm{m}^2$; equivalently $\sin C = 30/24 = 1.25 > 1$, which is impossible. **Q13.** (a) Area rule, $\approx 25.24\,\mathrm{m}^2$. (b) Heron's rule, $84\,\mathrm{m}^2$. (c) Base–height, $65\,\mathrm{m}^2$. **Q14.** (a) $\approx 7.34\,\mathrm{m}^2$. (b) $\approx \$477.25$. **Q15.** $\approx 2451\,\mathrm{m}^2$. **Q16.** (a) $\approx 45.6°$ or $\approx 134.4°$. (b) One bed has a sharp (acute) $45.6°$ corner and the other a wide (obtuse) $134.4°$ corner, yet both enclose $20\,\mathrm{m}^2$. **Q17.** (a) $\approx 32.14\,\mathrm{cm}^2$. (b) $\approx 32.14\,\mathrm{cm}^2$. (c) The two areas are equal, because $\sin 140° = \sin 40°$. **Q18.** (a) $C \approx 53.1°$ or $C \approx 126.9°$. (b) An area of $900\,\mathrm{m}^2$ needs $\sin C = \frac{2 \times 900}{50 \times 35} = \frac{1800}{1750} \approx 1.03 > 1$, which no angle can satisfy (the greatest possible area is $875\,\mathrm{m}^2$).

Fully-worked solutions

Q1. (a) The two edges are the sides, $a = 12$ m and $b = 5$ m, and the angle where they meet is the included angle, $C = 30°$. (b) $A = 1/2\, a\, b \sin C = 1/2 \times 12 \times 5 \times \sin 30°$. (c) Since $\sin 30° = 0.5$, $A = 1/2 \times 60 \times 0.5 = 15\,\mathrm{m}^2$.

Q2. (a) The sides are 6.5 m and 4.8 m, and the included angle between them is $63°$. (b) $A = 1/2 \times 6.5 \times 4.8 \times \sin 63°$. (c) $A = 1/2 \times 31.2 \times \sin 63° = 15.6 \sin 63° = 13.899\ldots \approx 13.90\,\mathrm{m}^2$.

Q3. (a) With the unknown side x, $A = 1/2 \times 16 \times x \times \sin 30°$. (b) Substituting $A = 24$ and $\sin 30° = 0.5$: $24 = 1/2 \times 16 \times x \times 0.5 = 4x$. (c) Dividing, $x = \frac{24}{4} = 6$ m.

Q4. (a) $A = 1/2\, a\, b \sin C$ gives $15 = 1/2 \times 10 \times 6 \times \sin C$. (b) The right side is $30 \sin C$, so $30 \sin C = 15$ and $\sin C = \frac{15}{30} = 0.5$. (c) $C = \sin^{-1}(0.5) = 30°$, or the obtuse supplement $C = 180° - 30° = 150°$. (d) Because $\sin 150° = \sin 30° = 0.5$, both angles give $1/2 \times 10 \times 6 \times 0.5 = 15\,\mathrm{cm}^2$; the two fences can meet at either a $30°$ or a $150°$ corner and still enclose the same area.

Q5. (a) The base and the perpendicular height are given, so use the **base–height** formula: $A = 1/2\, b\, h = 1/2 \times 14 \times 9 = 63\,\mathrm{m}^2$. (b) Only the three sides are known, so use **Heron's rule**: $s = 1/2(6 + 8 + 10) = 12$, so $A = \sqrt{12(12-6)(12-8)(12-10)} = \sqrt{12 \times 6 \times 4 \times 2} = \sqrt{576} = 24\,\mathrm{m}^2$. (c) Two sides and the included angle are known, so use the **area rule**: $A = 1/2 \times 9 \times 10 \times \sin 40° = 45 \sin 40° = 28.925\ldots \approx 28.93\,\mathrm{m}^2$.

Q6. (a) The two boundaries are the sides and $150°$ is the included angle: $A = 1/2 \times 52 \times 30 \times \sin 150° = 780 \times 0.5 = 390\,\mathrm{m}^2$. (b) Cost $= 390 \times 8 = \$3120$.

Q7. $A = 1/2 \times 7.2 \times 5.5 \times \sin 68° = 19.8 \sin 68° = 18.358\ldots \approx 18.36\,\mathrm{m}^2$.

Q8. $A = 1/2 \times 14 \times 9 \times \sin 120° = 63 \sin 120°$. Since $\sin 120° = \frac{\sqrt{3}}{2}$, the exact area is $A = \frac{63\sqrt{3}}{2}\,\mathrm{cm}^2 \approx 54.56\,\mathrm{cm}^2$.

Q9. With $A = 30$, one side 15 m and included angle $30°$, let the other side be x: $30 = 1/2 \times 15 \times x \times \sin 30° = 1/2 \times 15 \times x \times 0.5 = 3.75x$, so $x = \frac{30}{3.75} = 8$ m.

Q10. Let the other side be x: $40 = 1/2 \times 11 \times x \times \sin 55°$, so $x = \dfrac{2 \times 40}{11 \sin 55°} = \dfrac{80}{11 \sin 55°} = 8.878\ldots \approx 8.88$ m.

Q11. $40 = 1/2 \times 12 \times 10 \times \sin C = 60 \sin C$, so $\sin C = \dfrac{40}{60} = \dfrac{2}{3} = 0.6667$ (4 d.p.). The acute solution is $C = \sin^{-1}(0.6667\ldots) = 41.810\ldots \approx 41.8°$, and the obtuse solution is $180° - 41.810\ldots = 138.189\ldots \approx 138.2°$. Both are valid, since two fixed sides can meet at an acute or an obtuse corner.

Q12. Rearranging, an area of $30\,\mathrm{m}^2$ would require $\sin C = \dfrac{2 \times 30}{8 \times 6} = \dfrac{60}{48} = 1.25$. But $\sin C$ can never exceed 1, so no such triangle exists. The largest area is reached when $C = 90°$ (where $\sin C = 1$), namely $A = 1/2 \times 8 \times 6 = 24\,\mathrm{m}^2$; since $30 > 24$, the pen cannot be given an area of $30\,\mathrm{m}^2$ with these two fences.

Q13. (a) Two sides and the included angle are known, so use the **area rule**: $A = 1/2 \times 8 \times 11 \times \sin 35° = 44 \sin 35° = 25.237\ldots \approx 25.24\,\mathrm{m}^2$. (b) Only the three sides are known, so use **Heron's rule**: $s = 1/2(13 + 14 + 15) = 21$, so $A = \sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7056} = 84\,\mathrm{m}^2$. (c) The base and perpendicular height are given, so use the **base–height** formula: $A = 1/2 \times 20 \times 6.5 = 65\,\mathrm{m}^2$.

Q14. (a) $A = 1/2 \times 5.2 \times 3.8 \times \sin 48° = 9.88 \sin 48° = 7.342\ldots \approx 7.34\,\mathrm{m}^2$. (b) Using the unrounded area, the cost is $7.342\ldots \times 65 = 477.247\ldots \approx \$477.25$.

Q15. The two paths are the sides and $74°$ is the included angle: $A = 1/2 \times 85 \times 60 \times \sin 74° = 2550 \sin 74° = 2451.217\ldots \approx 2451\,\mathrm{m}^2$.

Q16. (a) $20 = 1/2 \times 8 \times 7 \times \sin C = 28 \sin C$, so $\sin C = \dfrac{20}{28} = \dfrac{5}{7} = 0.7143$ (4 d.p.). The acute solution is $C = \sin^{-1}(0.7143\ldots) = 45.584\ldots \approx 45.6°$, and the obtuse solution is $180° - 45.584\ldots = 134.415\ldots \approx 134.4°$. (b) One garden bed has a sharp corner of about $45.6°$ (a long, narrow triangle) and the other a wide corner of about $134.4°$ (a short, splayed triangle); both use the same two 8 m and 7 m edges and both enclose exactly $20\,\mathrm{m}^2$.

Q17. (a) The two equal sides are 10 cm and the apex is the included angle: $A = 1/2 \times 10 \times 10 \times \sin 40° = 50 \sin 40° = 32.139\ldots \approx 32.14\,\mathrm{cm}^2$. (b) $A = 1/2 \times 10 \times 10 \times \sin 140° = 50 \sin 140° = 32.139\ldots \approx 32.14\,\mathrm{cm}^2$. (c) The two areas are equal. This is because $140°$ and $40°$ are supplementary ($140° = 180° - 40°$), so $\sin 140° = \sin 40°$, and the area rule uses only the sine of the included angle.

Q18. (a) $700 = 1/2 \times 50 \times 35 \times \sin C = 875 \sin C$, so $\sin C = \dfrac{700}{875} = 0.8$. The acute solution is $C = \sin^{-1}(0.8) = 53.130\ldots \approx 53.1°$, and the obtuse solution is $180° - 53.130\ldots = 126.869\ldots \approx 126.9°$. (b) An area of $900\,\mathrm{m}^2$ would require $\sin C = \dfrac{2 \times 900}{50 \times 35} = \dfrac{1800}{1750} = 1.028\ldots$, which is greater than 1 and so impossible for any angle. The greatest area these two fences can enclose is $1/2 \times 50 \times 35 = 875\,\mathrm{m}^2$ (at $C = 90°$), which is already less than $900\,\mathrm{m}^2$.

Topic 2 Test A — Multiple-choice

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 3 minutes. Writing time: 20 minutes. 12 multiple-choice questions, 12 marks. Choose the response that is ***correct*** *for the question.*

Question 1

A sports scientist investigates whether the resting heart rate (in beats per minute) of an athlete can be predicted from the number of hours the athlete trains each week. In this study,

A. the explanatory variable is resting heart rate and the response variable is training hours

B. the explanatory variable is training hours and the response variable is resting heart rate

C. both the explanatory and response variables are training hours

D. both variables are categorical

Question 2

A scatterplot of two numerical variables shows the points lying close to a straight line that falls from the upper left to the lower right. This association is best described as

A. a weak, negative, linear association

B. a strong, positive, linear association

C. a strong, negative, linear association

D. a strong, negative, non-linear association

Question 3

A line of good fit drawn by eye for the price of a used car, price (in dollars), against its age (in years) is

$$\text{price} = 24\,000 - 1800 \times \text{age}.$$

According to this line,

A. the price increases by \$1800 for each additional year of age

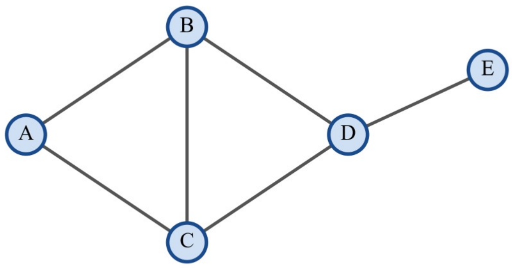
B. the price decreases by \$24 000 for each additional year of age

C. a car of age 0 years is worth \$1800

D. the price decreases by \$1800 for each additional year of age

Question 4

The graph below shows the walking tracks joining five lookouts A, B, C, D and E in a national park.



The sum of the degrees of the five vertices is

A. 5

B. 6

C. 12

D. 24

Question 5

A connected planar graph has 9 vertices and 14 edges. Using Euler's formula $v-e+f=2$, the number of faces (including the infinite outer region) is

A. 7

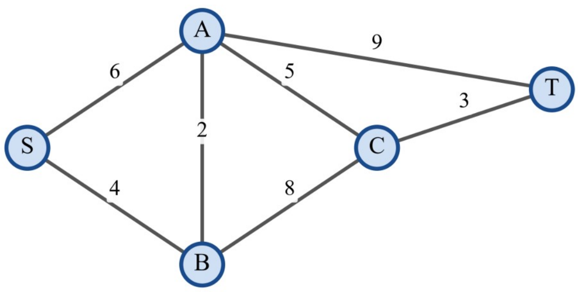
B. 6

C. 5

D. 23

Question 6

The network below shows the walking times, in minutes, along the tracks between points S, A, B, C and T.



The shortest travelling time from S to T, in minutes, is

A. 12

B. 14

C. 15

D. 22

Question 7

The time t taken to fill a swimming pool varies inversely with the number of identical pumps p used. When 3 pumps are used, the pool fills in 40 minutes. Using 5 pumps, the time to fill the pool is

A. 8 minutes

B. 66.7 minutes

C. 200 minutes

D. 24 minutes

Question 8

A quantity is recorded as $Q = 3.2 \times 10^9$. On a base-10 logarithmic scale, its value $\log_{10} Q$ is closest to

A. 9.32

B. 8.51

C. 9.51

D. 10.51

Question 9

Two similar cylindrical water tanks have heights in the ratio $2:3$. The ratio of their volumes is

A. 8 : 27

B. 4 : 9

C. 2 : 3

D. 27 : 8

Question 10

A sector of a circle has radius 12 cm and a sector angle of $60°$. The area of the sector is closest to

A. 37.7 cm²

B. 75.4 cm²

C. 12.6 cm²

D. 150.8 cm²

Question 11

In triangle ABC, $a = 8$ cm, $b = 11$ cm and the included angle $C = 40°$. The area of the triangle is closest to

A. 44.0 cm²

B. 56.6 cm²

C. 33.7 cm²

D. 28.3 cm²

Question 12

In triangle ABC, $b=7$ m, $c=10$ m and angle $A=55°$. The length of side a is closest to

A. 8.3 m

B. 8.5 m

C. 6.9 m

D. 11.7 m

End of Topic 2 Test A

Test A — answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12
Answer	B	C	D	C	A	B	D	C	A	B	D	A

Q1. B. The explanatory variable does the predicting (training hours); the response variable is what is predicted (resting heart rate). Both are numerical.

Q2. C. Points lying close to a straight line that falls from left to right show a strong, negative, linear association.

Q3. D. The slope -1800 is the change in price per extra year, so on average the price falls by $1800 for each additional year of age.

Q4. C. The degrees are $A=2$, $B=3$, $C=3$, $D=3$, $E=1$; their sum is 12 (which is twice the 6 edges).

Q5. A. Rearranging Euler's formula, $f=2-v+e=2-9+14=7$.

Q6. B. The quickest way from S to T takes 14 minutes, along $S–A–C–T=6+5+3=14$ (the route $S–B–A–C–T=4+2+5+3$ also takes 14 minutes). Every other route takes longer, for example $S–B–C–T=4+8+3=15$ and $S–A–T=6+9=15$.

Q7. D. Inverse variation gives $t=\frac{k}{p}$ with $k=3\times 40=120$, so with 5 pumps $t=\frac{120}{5}=24$ minutes.

Q8. C. $\log_{10}(3.2\times 10^9)=9+\log_{10}3.2\approx 9+0.51=9.51$.

Q9. A. For similar solids the volumes scale as the cube of the linear ratio: $2^3:3^3=8:27$.

Q10. B. Area $=\frac{60}{360}\times\pi\times 12^2=24\pi\approx 75.4$ cm².

Q11. D. Area $=1/2\,a\,b\sin C=1/2\times 8\times 11\times\sin 40°\approx 28.3$ cm².

Q12. A. $a^2=b^2+c^2-2bc\cos A=7^2+10^2-2\times 7\times 10\times\cos 55°\approx 68.70$, so $a\approx 8.3$ m.

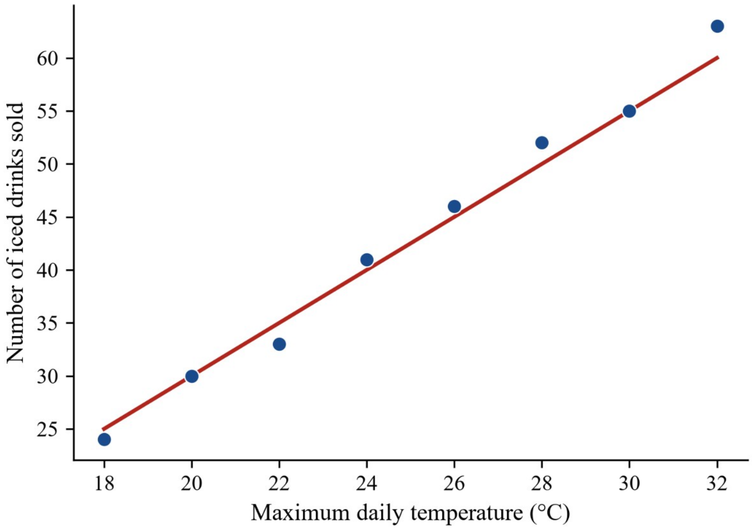
Topic 2 Test B — Short-answer

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 5 minutes. Writing time: 40 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (7 marks)

A café owner records, on eight days, the maximum daily temperature (in °C) and the number of iced drinks sold. The data are shown in the table and scatterplot below, with temperature as the explanatory variable.

Maximum temperature (°C)	18	20	22	24	26	28	30	32
Iced drinks sold	24	30	33	41	46	52	55	63



a. Describe the association between the two variables in terms of direction, form and strength. 2 marks

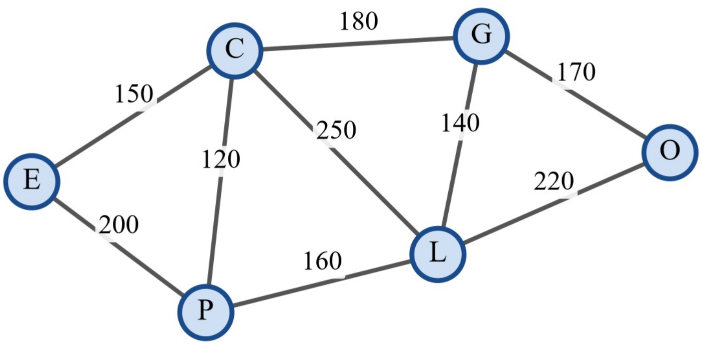
b. A line of good fit is drawn by eye passing through the points $(20, 30)$ and $(30, 55)$. Find the equation of this line in the form $\text{drinks} = a + b \times \text{temp}$. 2 marks

c. Interpret the slope of the line in the context of this problem. 1 mark

d. Use the line of good fit to predict the number of iced drinks sold when the maximum daily temperature is 27 °C. State whether this prediction is an interpolation or an extrapolation, and why. 2 marks

Question 2 (8 marks)

A council is planning sealed paths between six locations in a park: the Entrance E, Café C, Playground P, Lake L, Gardens G and Oval O. The network below shows the possible paths and their lengths in metres.



a. Write down the degree of vertex L and the total number of edges in the network. 2 marks

b. The network is drawn as a planar graph. State the values of v and e, and use Euler's formula $v-e+f=2$ to find the number of faces f. 2 marks

c. Find the shortest path from the Entrance E to the Oval O, and state its length. 2 marks

d. The council will seal the least total length of path needed so that all six locations are connected. Find the edges of this minimum spanning tree and its total length. 2 marks

Question 3 (6 marks)

When a car brakes, the distance d (in metres) it travels before stopping varies directly with the square of its speed v (in km/h); that is, $d=kv^2$. At a speed of 40 km/h the braking distance is 16 m.

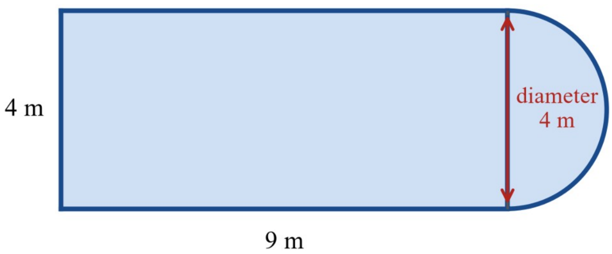
a. Find the constant of proportionality k, and hence write down the rule for d in terms of v. 2 marks

b. Find the braking distance when the car is travelling at 100 km/h. 2 marks

c. Find the speed of the car when its braking distance is 64 m. 2 marks

Question 4 (7 marks)

A garden bed has the shape of a rectangle 9 m long and 4 m wide, with a semicircle of diameter 4 m attached to one end, as shown.



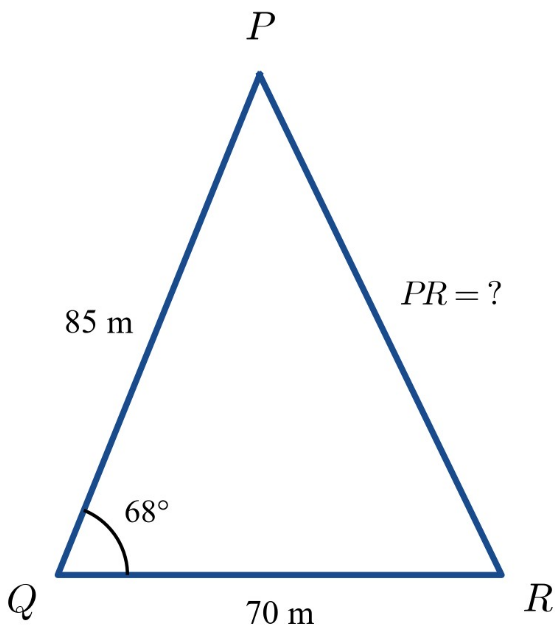
a. Find the perimeter of the garden bed (its outer boundary), correct to one decimal place. 3 marks

b. Find the area of the garden bed, correct to one decimal place. 2 marks

c. Mulch is sold in bags that each cover 4 m². Find the smallest number of whole bags needed to cover the garden bed. 2 marks

Question 5 (8 marks)

A triangular reserve PQR has $PQ=85$ m, $QR=70$ m and the angle at Q equal to $68°$, as shown.



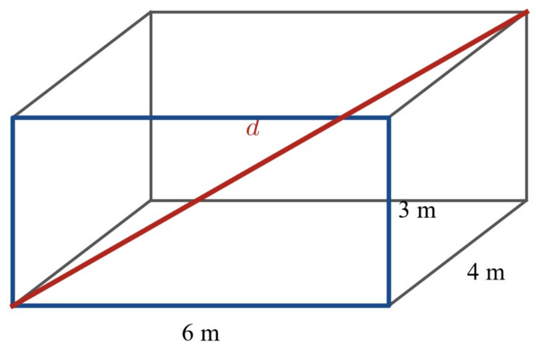
a. Find the length of PR, correct to one decimal place. 3 marks

b. Find the area of the reserve, using the rule $A = 1/2\, a\, b \sin C$, correct to the nearest square metre. 2 marks

c. Find the size of the angle at R, correct to the nearest degree. 3 marks

Question 6 (4 marks)

A storage shed is a rectangular box 6 m long, 4 m wide and 3 m high. The diagonal d runs from a bottom corner to the opposite top corner, as shown.



a. Find the length of the diagonal d, correct to two decimal places. 2 marks

b. Find the angle that this diagonal makes with the floor of the shed, correct to the nearest degree. 2 marks

End of Topic 2 Test B

Test B — answers and solutions

Q1. (a) As the maximum temperature increases the number of iced drinks sold increases, so the association is **positive**. The points lie close to a straight line, so the form is **linear** and the association is **strong**. (b) The slope is $b=\frac{55-30}{30-20}=\frac{25}{10}=2.5$. Substituting $(20,30)$: $30=a+2.5\times 20=a+50$, so $a=-20$. The line of good fit is

$$\text{drinks}=-20+2.5\times\text{temp}.$$

(c) The slope 2.5 means that, on average, about 2.5 extra iced drinks are sold for each 1 °C increase in the maximum daily temperature. (d) At a temperature of 27 °C, $\text{drinks}=-20+2.5\times 27=-20+67.5=47.5$, i.e. about 47 or 48 drinks. Because 27 °C lies within the range of the data (18 °C to 32 °C), this is an **interpolation**, so the prediction is reliable.

Q2. (a) Vertex L is joined to P, C, G and O, so its degree is 4. The network has 9 edges. (b) Here $v=6$ and $e=9$, so $f=2-v+e=2-6+9=5$. There are 5 faces — four bounded regions and the outer region. (c) Comparing the routes from E to O by inspection, the three shortest are

$$E-C-G-O=150+180+170=500\,\text{m},$$

$$E-P-L-O=200+160+220=580\,\text{m}, E-C-L-O=150+250+220=620\,\text{m}.$$

Every remaining route is longer still, so the shortest path is *E–C–G–O*, of length 500 m. (d) Building the minimum spanning tree by repeatedly choosing the shortest available edge that does not form a cycle gives *C–P* (120), *L–G* (140), *E–C* (150), *P–L* (160) and *G–O* (170). These five edges connect all six locations, with total length

$$120+140+150+160+170=740\,\text{m}.$$

Q3. (a) $k=\frac{d}{v^2}=\frac{16}{40^2}=\frac{16}{1600}=0.01$, so $d=0.01\,v^2$. (b) At $v=100$: $d=0.01\times 100^2=0.01\times 10\,000=100$ m. (c) $64=0.01\,v^2\Rightarrow v^2=\frac{64}{0.01}=6400\Rightarrow v=80$ km/h (taking the positive root).

Q4. (a) The boundary consists of the two long sides ($9+9=18$ m), the straight 4 m end, and the semicircular arc of radius 2 m. The arc length is $\pi r=\pi\times 2\approx 6.28$ m, so the perimeter is $18+4+6.28=28.28\approx 28.3$ m. (b) Area = rectangle + semicircle $=9\times 4+1/2\,\pi r^2=36+1/2\,\pi(2)^2=36+6.28=42.28\approx 42.3$ m². (c) $\frac{42.28}{4}=10.57\ldots$ bags. Since only whole bags can be bought and 10 bags would not be enough, 11 bags are needed.

Q5. (a) By the cosine rule,

$$PR^2=PQ^2+QR^2-2\times PQ\times QR\times\cos Q=85^2+70^2-2\times 85\times 70\times\cos 68°\approx 12125-4457.8=7667.2,$$

so $PR\approx 87.6$ m. (b) Area $=1/2\times PQ\times QR\times\sin Q=1/2\times 85\times 70\times\sin 68°\approx 2975\times 0.9272\approx 2758$ m². (c) By the sine rule, $\frac{\sin R}{PQ}=\frac{\sin Q}{PR}$, so

$$\sin R=\frac{85\times\sin 68°}{87.6}\approx 0.900, R\approx 64°.$$

Q6. (a) Using Pythagoras' theorem in three dimensions,

$$d=\sqrt{6^2+4^2+3^2}=\sqrt{36+16+9}=\sqrt{61}\approx 7.81\,\text{m}.$$

(b) The diagonal of the floor is $\sqrt{6^2+4^2}=\sqrt{52}\approx 7.21$ m. The diagonal d rises 3 m above this floor diagonal, so the angle θ it makes with the floor satisfies

$$\tan\theta=\frac{3}{\sqrt{52}}=\frac{3}{7.21}\approx 0.4160, \theta\approx 23°.$$