General Mathematics Units 1 & 2

Chapter 9 — Measurement, geometry and similarity

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Chapter 9 Measurement, geometry and similarity — 9A Units, accuracy and scientific notation

Theory

Every topic in this chapter — perimeter, area, volume and similarity — begins with measurements, so we first make sure of two basic skills: **converting** a measurement from one metric unit to another, and stating an answer to a sensible level of **accuracy**. Along the way we meet **scientific notation**, the standard shorthand for very large and very small numbers.

Metric units of length. The everyday metric units of **length** are the millimetre (mm), the centimetre (cm), the metre (m) and the kilometre (km), linked by

$$1\,\text{cm} = 10\,\text{mm}\,,\, 1\,\text{m} = 100\,\text{cm}\,,\, 1\,\text{km} = 1000\,\text{m}\,.$$

To convert to a **smaller** unit we **multiply** by the conversion factor, because it takes many small units to make up a large one; to convert to a **larger** unit we **divide**. For example, $3.2\,\text{m} = 3.2 \times 100 = 320$ cm, and $4600\,\text{m} = 4600 \div 1000 = 4.6$ km. A quick reasonableness check: measuring the same object with a smaller unit must give a larger number.

Units of area. Area is measured in squared units. A square of side 1 cm is a 10×10 grid of small squares of side 1 mm, so $1\,\text{cm}^2 = 100\,\text{mm}^2$: when the length factor is 10, the area factor is $10^2 = 100$. In the same way $1\,\text{m}^2 = 100^2\,\text{cm}^2 = 10\,000\,\text{cm}^2$ and $1\,\text{km}^2 = 1000^2\,\text{m}^2 = 1\,000\,000\,\text{m}^2$. For example, $0.6\,\text{m}^2 = 0.6 \times 10\,000 = 6000\,\text{cm}^2$. Australian land areas are usually quoted in **hectares** (ha): one hectare is a $100\,\text{m} \times 100\,\text{m}$ square, so $1\,\text{ha} = 10\,000\,\text{m}^2$ and $1\,\text{km}^2 = 100$ ha.

Units of volume and capacity. Volume is measured in cubed units. A cube of side 1 cm contains $10 \times 10 \times 10 = 1000$ cubes of side 1 mm, so $1\,\text{cm}^3 = 1000\,\text{mm}^3$; likewise $1\,\text{m}^3 = 100^3\,\text{cm}^3 = 1\,000\,000\,\text{cm}^3$ — a length factor of 10 or 100 becomes a volume factor of 10^3 or 100^3. The **capacity** of a container is the volume of fluid it can hold, measured in millilitres (mL), litres (L), kilolitres (kL) and megalitres (ML). The two systems are linked by

$$1\,\text{mL} = 1\,\text{cm}^3\,,\, 1\,\text{L} = 1000\,\text{mL} = 1000\,\text{cm}^3\,,\, 1\,\text{kL} = 1000\,\text{L} = 1\,\text{m}^3\,,$$

so a 600 mL water bottle holds exactly $600\,\text{cm}^3$. Household water bills are issued in kilolitres, and reservoirs are measured in megalitres ($1\,\text{ML} = 1\,000\,000$ L).

Length	Area	Volume and capacity
$1\,\text{cm} = 10\,\text{mm}$	$1\,\text{cm}^2 = 100\,\text{mm}^2$	$1\,\text{cm}^3 = 1000\,\text{mm}^3$
$1\,\text{m} = 100\,\text{cm}$	$1\,\text{m}^2 = 10\,000\,\text{cm}^2$	$1\,\text{m}^3 = 1\,000\,000\,\text{cm}^3$
$1\,\text{km} = 1000\,\text{m}$	$1\,\text{ha} = 10\,000\,\text{m}^2$	$1\,\text{mL} = 1\,\text{cm}^3$; $1\,\text{L} = 1000\,\text{cm}^3$
	$1\,\text{km}^2 = 1\,000\,000\,\text{m}^2 = 100\,\text{ha}$	$1\,\text{kL} = 1000\,\text{L} = 1\,\text{m}^3$

Exact and approximate answers. An answer is **exact** if it records the value with nothing lost: $\frac{5}{6}$ of a metre is exact, but its decimal form $0.8333\ldots$ never terminates, so any decimal we actually write down, such as 0.83, is an **approximation** — we signal this with the symbol ≈. Exact answers are usually fractions (or, from Section 9B on, multiples of π). Every physical measurement is approximate: a length recorded as 27 mm has only been measured to the nearest millimetre. To round a decimal to a given number of **decimal places**, cut off after the required place and look at the next digit: if it is 5 or more, round up; otherwise leave the last kept digit alone.

Scientific notation. A number is in **scientific notation** when it is written in the form

$$a \times 10^n\,,\, 1 \le a < 10\,,\, n\,\text{an integer}\,.$$

For a large number, n counts how many places the decimal point moves to the left: $52\,000\,000 = 5.2\times10^7$. For a small number (between 0 and 1) the power is negative and counts the places the point moves to the right: $0.000\,61 = 6.1\times10^{-4}$. Scientific notation makes size comparisons easy — compare the powers of 10 first, and only if they are equal compare the leading numbers.

Significant figures. The **significant figures** of a number are the digits that carry measurement information. The first significant figure is the first non-zero digit (reading from the left), and every digit from there on is significant: 5083 has four significant figures, 0.0071 has two (the leading zeros merely locate the decimal point), and 2.500 has four (the trailing zeros in a decimal record genuine precision). To round to n significant figures, keep the first n of them, use the next digit to decide whether to round up, and in a whole number replace the discarded digits with zeros to preserve place value: $683\,400$ rounded to two significant figures is $680\,000$. Decimal places and significant figures can behave very differently for small numbers: $0.006\,19$ rounded to two decimal places is the almost useless 0.01, but rounded to two significant figures it is the informative 0.0062.

Using CAS. Your CAS switches to scientific notation by itself when a result is very large or very small — a display such as 5.2E7 means 5.2×10^7. The document settings let you fix the number of displayed decimal places, but the calculator still rounds: when it shows 0.833333 for $5\div6$, the display is an approximation, and the fraction template (or exact mode) is needed for the exact value. Unit conversions can be done on CAS too, but learn the factors in the table above so that you can convert instantly by hand, and always sense-check the result: converting to a smaller unit must give a bigger number.

Worked examples

Example 1 — scaffolded

A landscaper is pricing a new community garden in Geelong and takes these measurements from the site plan and a product catalogue. Convert each to the unit shown.

(a) 3.7 m to centimetres. (b) 5600 m to kilometres. (c) $2.5\,\text{m}^2$ to square centimetres.

(b) $640\,\text{mm}^2$ to square centimetres. (e) 3.2 ha to square metres.

Working	Comment
(a) $3.7\times100=370$ cm.	Metres to centimetres is a smaller unit, so multiply by 100.
(b) $5600\div1000=5.6$ km.	Metres to kilometres is a larger unit, so divide by 1000.
(c) $2.5\times10\,000=25\,000\,\text{cm}^2$.	The area factor is $100^2=10\,000$, not 100.
(d) $640\div100=6.4\,\text{cm}^2$.	$1\,\text{cm}^2=100\,\text{mm}^2$, and cm^2 is the larger unit, so divide.
(e) $3.2\times10\,000=32\,000\,\text{m}^2$.	$1\,\text{ha}=10\,000\,\text{m}^2$.

So the measurements are 370 cm, 5.6 km, $25\,000\,\text{cm}^2$, $6.4\,\text{cm}^2$ and $32\,000\,\text{m}^2$.

Example 2 — scaffolded

A plant nursery in Bendigo installs a rainwater tank with a volume of $2.4\,\text{m}^3$.

(a) Express the tank's volume in cubic centimetres. (b) Express it in litres. (c) Express it in kilolitres. (d) Each seedling receives 35 mL of water — write this in cubic centimetres. (e) A syringe holds $5.6\,\text{cm}^3$ of liquid fertiliser — write this in cubic millimetres.

Working	Comment
(a) $2.4\times1\,000\,000=2\,400\,000\,\text{cm}^3$.	The volume factor from m^3 to cm^3 is $100^3=1\,000\,000$.
(b) $2\,400\,000\div1000=2400$ L.	$1\,\text{L}=1000\,\text{cm}^3$.
(c) $2400\div1000=2.4$ kL.	$1\,\text{kL}=1000\,\text{L}$ — note that $1\,\text{m}^3=1\,\text{kL}$ directly.
(d) $35\,\text{mL}=35\,\text{cm}^3$.	$1\,\text{mL}=1\,\text{cm}^3$: millilitres and cubic centimetres are

Working	Comment
	interchangeable.
(e) $5.6\times1000=5600\,\text{mm}^3$.	The volume factor from cm^3 to mm^3 is $10^3=1000$.

So the tank holds $2\,400\,000\,\text{cm}^3$, which is 2400 L or 2.4 kL; each seedling receives $35\,\text{cm}^3$ of water, and the syringe holds $5600\,\text{mm}^3$.

Example 3 — unscaffolded

For a science assignment, a student collects two measurements: Australia's land area, about $7\,690\,000\,\text{km}^2$, and the thickness of a sheet of aluminium foil, about $0.000\,016$ m. (a) Write both measurements in scientific notation. (b) The student's third measurement appears on her CAS as 9.05×10^4 m. Write it in ordinary notation.

(a) For the land area, place the decimal point after the first significant figure to get 7.69. The point has moved 6 places to the left, so $7\,690\,000\,\text{km}^2=7.69\times10^6\,\text{km}^2$. For the foil, the first significant figure is the 1, which sits 5 places to the right of the decimal point, so $0.000\,016\,\text{m}=1.6\times10^{-5}$ m — the power is negative because the number is less than 1. (b) The power 10^4 moves the decimal point 4 places to the right: $9.05\times10^4=90\,500$ m.

∴ Australia's land area is $7.69\times10^6\,\text{km}^2$, the foil is 1.6×10^{-5} m thick, and the CAS display 9.05×10^4 m means $90\,500$ m.

Example 4 — unscaffolded

A plumber cuts a 4 m length of copper pipe into seven equal pieces. Her CAS displays the length of each piece, in metres, as 0.571429. (a) State the exact length of each piece, and explain why the CAS display is an approximation. (b) Round the length to two decimal places, and to three significant figures. (c) Express the length in centimetres, correct to one decimal place.

(a) The exact length is the fraction $\frac{4}{7}$ m. As a decimal, $\frac{4}{7}=0.571\,428\,571\ldots$, a recurring decimal that never terminates, so no decimal display can be exact — the CAS has rounded at the sixth decimal place. (b) For two decimal places, the third decimal digit is 1, which is less than 5, so $\frac{4}{7}\approx0.57$ m. For three significant figures, the first three significant figures are 5, 7 and 1 and the next digit is 4, so $\frac{4}{7}\approx0.571$ m. The two roundings keep different amounts of detail even though both begin with 0.57. (c) In centimetres, $\frac{4}{7}\,\text{m}=\frac{400}{7}\,\text{cm}=57.142\,8\ldots$ cm ≈57.1 cm, since the second decimal digit, 4, is less than 5.

∴ each piece is exactly $\frac{4}{7}$ m long, which is 0.57 m to two decimal places, 0.571 m to three significant figures, and 57.1 cm to one decimal place.

Practice questions

Scaffolded

Q1. A carpenter's cutting list for a pergola shows the following measurements. Convert each one to the unit indicated. (a) 5.3 m to centimetres. (b) 470 cm to metres. (c) 2.85 km to metres. (d) 64 mm to centimetres. (e) 9600 m to kilometres.

Q2. Convert each area measurement to the unit indicated. (a) $7\,\text{cm}^2$ to square millimetres. (b) $3.5\,\text{m}^2$ to square centimetres. (c) $42\,000\,\text{cm}^2$ to square metres. (d) 6.8 ha to square metres. (e) $2.4\,\text{km}^2$ to hectares.

Q3. Convert each volume or capacity measurement to the unit indicated. (a) $3\,\text{cm}^3$ to cubic millimetres. (b) $0.75\,\text{m}^3$ to cubic centimetres. (c) $1900\,\text{cm}^3$ to litres. (d) 4.5 L to millilitres. (e) 3200 L to kilolitres. (f) $2.6\,\text{m}^3$ to litres.

Q4. For parts (a) to (d), write the number in scientific notation; for parts (e) and (f), write the number in ordinary notation. (a) $86\,000$ (b) $5\,100\,000$ (c) 0.0043 (d) $0.000\,072$ (e) 9.05×10^3 (f) 1.8×10^{-4}

Q5. (a) State the number of significant figures in each of 240.7, 0.0052 and 30.60. (b) Round $46\,178$ to two significant figures. (c) Round $46\,178$ to three significant figures. (d) Round $0.030\,461$ to two significant figures. (e) Round $0.030\,461$ to three significant figures.

Q6. At a school fete, 2 L of juice is shared equally among three glasses. (a) Write the exact amount of juice in each glass in millilitres, as a fraction. (b) A CAS displays the amount as 666.666667 mL. Is this display exact or approximate? Explain. (c) How much juice is in each glass, to the nearest millilitre? (d) A student claims that if every glass really held the rounded amount from part (c), the three glasses would need more than the 2 L available. Do the multiplication and explain what this shows about calculations that use rounded values.

Unscaffolded

Q7. Convert: (a) $15\,400$ m to kilometres; (b) 2.35 m to millimetres; (c) 78 mm to centimetres; (d) 0.65 km to metres.

Q8. A grazing property near Hamilton covers 348 ha. Express the property's area (a) in square metres and (b) in square kilometres, and (c) write your answer to part (a) in scientific notation.

Q9. A backyard swimming pool holds $48\,\text{m}^3$ of water when full. (a) Express the pool's capacity in kilolitres and in litres. (b) One dose of pool chlorine is 250 mL. Write this dose in cubic centimetres, and then in litres.

Q10. (a) Australia's population is about $27\,800\,000$ people. Write this number in scientific notation. (b) The road distance from Melbourne to Perth is about 3400 km. Write this distance in scientific notation, first in kilometres and then in metres.

Q11. A red blood cell has a diameter of about $0.000\,0076$ m, and a grain of wattle pollen has a diameter of about $0.000\,03$ m. (a) Write both diameters in scientific notation. (b) An ant is 4.2×10^{-3} m long. Write its length in ordinary notation in metres, and then in millimetres.

Q12. Four regional water storages currently hold 8.9×10^4 kL, 3.2×10^5 kL, 9.9×10^4 kL and 1.05×10^5 kL. Write the four amounts in order from smallest to largest, and explain how scientific notation lets you order them without writing any of the numbers out in full.

Q13. Round: (a) $2\,649\,000$ to two significant figures; (b) $0.004\,906$ to two significant figures; (c) $0.004\,906$ to three significant figures; (d) 19.978 to three significant figures; (e) 19.978 to two decimal places.

Q14. A survey drone's flight log records altitude in kilometres. One entry reads 0.085 km. (a) Express the altitude in metres. (b) Express it in centimetres. (c) Write your answer to part (b) in scientific notation.

Q15. A Ballarat household used 285 kL of water last year (365 days). (a) Express the year's usage in litres. (b) Write your answer to part (a) in scientific notation. (c) Find the mean daily usage in litres, correct to three significant figures. (d) Is your answer to part (c) exact or approximate? Explain what the figure tells the household.

Q16. For each of the following, state the most appropriate metric unit and give a brief reason. (a) The area of an outback sheep station. (b) One dose of cough medicine. (c) The driving distance from Bendigo to Mildura. (d) The thickness of a 10-cent coin.

Q17. A CAS gives the thickness of a machined steel plate as $0.048\,265$ m. (a) Round the thickness to three decimal places. (b) Round it to three significant figures. (c) Which of the two rounded values keeps more of the measured information? Explain. (d) Write your answer to part (b) in millimetres.

Q18. The Moon's average distance from Earth is about 3.844×10^5 km. (a) Write the distance in ordinary notation in kilometres. (b) Write it in metres, in scientific notation. (c) Round the distance to two significant figures, in kilometres. (d) Is the quoted value 3.844×10^5 km exact or approximate? Explain.

Answers

Q1. (a) 530 cm. (b) 4.7 m. (c) 2850 m. (d) 6.4 cm. (e) 9.6 km. **Q2.** (a) $700\,\text{mm}^2$. (b) $35\,000\,\text{cm}^2$. (c) $4.2\,\text{m}^2$. (d) $68\,000\,\text{m}^2$. (e) 240 ha. **Q3.** (a) $3000\,\text{mm}^3$. (b) $750\,000\,\text{cm}^3$. (c) 1.9 L. (d) 4500 mL. (e) 3.2 kL. (f) 2600 L. **Q4.** (a) 8.6×10^4. (b) 5.1×10^6. (c) 4.3×10^{-3}. (d) 7.2×10^{-5}. (e) 9050. (f) 0.000 18. **Q5.** (a) Four; two; four. (b) 46 000. (c) 46 200. (d) 0.030. (e) 0.0305. **Q6.** (a) $\frac{2000}{3} = 666\frac{2}{3}$ mL. (b) Approximate — the exact value is a recurring decimal, so the display has been rounded. (c) 667 mL. (d) $3 \times 667 = 2001$ mL > 2000 mL; totals computed from rounded values need not equal the exact total. **Q7.** (a) 15.4 km. (b) 2350 mm. (c) 7.8 cm. (d) 650 m. **Q8.** (a) $3\,480\,000\,\text{m}^2$. (b) $3.48\,\text{km}^2$. (c) $3.48 \times 10^6\,\text{m}^2$. **Q9.** (a) 48 kL $= 48\,000$ L. (b) $250\,\text{cm}^3 = 0.25$ L. **Q10.** (a) 2.78×10^7. (b) 3.4×10^3 km $= 3.4 \times 10^6$ m. **Q11.** (a) 7.6×10^{-6} m and 3×10^{-5} m. (b) 0.0042 m $= 4.2$ mm. **Q12.** $8.9 \times 10^4 < 9.9 \times 10^4 < 1.05 \times 10^5 < 3.2 \times 10^5$ (kL); compare the powers of 10 first, then the leading numbers of any with equal powers. **Q13.** (a) 2 600 000. (b) 0.0049. (c) 0.00491. (d) 20.0. (e) 19.98. **Q14.** (a) 85 m. (b) 8500 cm. (c) 8.5×10^3 cm. **Q15.** (a) 285 000 L. (b) 2.85×10^5 L. (c) 781 L per day. (d) Approximate — the division does not terminate and has been rounded; it is the average day's use, not any actual day's use. **Q16.** (a) Hectares. (b) Millilitres. (c) Kilometres. (d) Millimetres. (Reasons in the solutions.) **Q17.** (a) 0.048 m. (b) 0.0483 m. (c) The three-significant-figure value — the three-decimal-place value keeps only two significant figures. (d) 48.3 mm. **Q18.** (a) 384 400 km. (b) 3.844×10^8 m. (c) 380 000 km. (d) Approximate — the distance varies around the orbit, and the average has itself been rounded (to four significant figures).

Fully-worked solutions

Q1. (a) Metres to centimetres is a smaller unit, so multiply: $5.3 \times 100 = 530$ cm. (b) Centimetres to metres is a larger unit, so divide: $470 \div 100 = 4.7$ m. (c) $2.85 \times 1000 = 2850$ m. (d) $64 \div 10 = 6.4$ cm. (e) $9600 \div 1000 = 9.6$ km.

Q2. Area conversions use the squares of the length factors. (a) $7 \times 100 = 700\,\text{mm}^2$. (b) $3.5 \times 10\,000 = 35\,000\,\text{cm}^2$. (c) $42\,000 \div 10\,000 = 4.2\,\text{m}^2$. (d) $6.8 \times 10\,000 = 68\,000\,\text{m}^2$, since $1\,\text{ha} = 10\,000\,\text{m}^2$. (e) $2.4 \times 100 = 240$ ha, since $1\,\text{km}^2 = 100$ ha.

Q3. Volume conversions use the cubes of the length factors. (a) $3 \times 1000 = 3000\,\text{mm}^3$. (b) $0.75 \times 1\,000\,000 = 750\,000\,\text{cm}^3$. (c) $1900\,\text{cm}^3 = 1900$ mL, and $1900 \div 1000 = 1.9$ L. (d) $4.5 \times 1000 = 4500$ mL. (e) $3200 \div 1000 = 3.2$ kL. (f) $1\,\text{m}^3 = 1$ kL $= 1000$ L, so $2.6 \times 1000 = 2600$ L.

Q4. (a) The decimal point moves 4 places left: $86\,000 = 8.6 \times 10^4$. (b) The point moves 6 places left: $5\,100\,000 = 5.1 \times 10^6$. (c) The first significant figure is 3 places right of the point: $0.0043 = 4.3 \times 10^{-3}$. (d) The first significant figure is 5 places right of the point: $0.000\,072 = 7.2 \times 10^{-5}$. (e) Move the point 3 places right: $9.05 \times 10^3 = 9050$. (f) Move the point 4 places left: $1.8 \times 10^{-4} = 0.000\,18$.

Q5. (a) In 240.7 every digit is significant: four. In 0.0052 the leading zeros only locate the decimal point, so only the 5 and 2 count: two. In 30.60 the zero between the 3 and the 6 and the trailing zero after the 6 are both significant: four. (b) The first two significant figures of 46 178 are 4 and 6; the next digit is 1, which is less than 5, so round down: 46 000. (c) The first three are 4, 6 and 1; the next digit is 7, so round up: 46 200. (d) The first two significant figures of 0.030 461 are 3 and 0; the next digit is 4, so round down: 0.030 (the final zero is kept to show two significant figures). (e) The first three are 3, 0 and 4; the next digit is 6, so round up: 0.0305.

Q6. (a) Each glass receives $2000 \div 3 = \frac{2000}{3} = 666\frac{2}{3}$ mL exactly. (b) Approximate. The exact value $666.666\,666\ldots$ is a recurring decimal that never terminates, so the CAS has rounded it at the sixth decimal place to fit the display. (c) $666.66\ldots$ rounds up to 667 mL. (d) $3 \times 667 = 2001$ mL, which exceeds the 2000 mL available. Each rounded share is $\frac{1}{3}$ mL too large, and the three small rounding errors accumulate to a full millilitre. A total computed from rounded values need not equal the exact total, so keep values exact (or unrounded) during working and round only the final answer.

Q7. (a) $15\,400 \div 1000 = 15.4$ km. (b) $2.35 \times 1000 = 2350$ mm. (c) $78 \div 10 = 7.8$ cm. (d) $0.65 \times 1000 = 650$ m.

Q8. (a) $348 \times 10\,000 = 3\,480\,000\,\text{m}^2$. (b) $3\,480\,000 \div 1\,000\,000 = 3.48\,\text{km}^2$ (equivalently, $348 \div 100 = 3.48$, using $1\,\text{km}^2 = 100$ ha). (c) The decimal point moves 6 places left, so $3\,480\,000\,\text{m}^2 = 3.48 \times 10^6\,\text{m}^2$.

Q9. (a) Since $1\,\text{m}^3 = 1\,\text{kL}$, the pool holds 48 kL, and $48 \times 1000 = 48\,000$ L. (b) Since $1\,\text{mL} = 1\,\text{cm}^3$, the dose is $250\,\text{cm}^3$; and $250 \div 1000 = 0.25$ L.

Q10. (a) The decimal point moves 7 places left: $27\,800\,000 = 2.78 \times 10^7$. (b) The point moves 3 places left: $3400 = 3.4 \times 10^3$ km. Converting to metres, $3400 \times 1000 = 3\,400\,000$ m, and the point now moves 6 places left: 3.4×10^6 m. (Equivalently, multiplying by $1000 = 10^3$ raises the power from 10^3 to 10^6.)

Q11. (a) For the red blood cell the first significant figure, 7, is 6 places right of the decimal point: $0.000\,0076 = 7.6 \times 10^{-6}$ m. For the pollen grain the 3 is 5 places right of the point: $0.000\,03 = 3 \times 10^{-5}$ m. (b) Move the point 3 places left: $4.2 \times 10^{-3} = 0.0042$ m. Then $0.0042 \times 1000 = 4.2$ mm.

Q12. Compare the powers of 10 first: the storages with 10^4 hold less than the storages with 10^5. Among the 10^4 pair, $8.9 < 9.9$; among the 10^5 pair, $1.05 < 3.2$. Hence

$$8.9 \times 10^4 < 9.9 \times 10^4 < 1.05 \times 10^5 < 3.2 \times 10^5.$$

In full the amounts are $89\,000$, $99\,000$, $105\,000$ and $320\,000$ kL, which confirms the order. Note that $9.9 \times 10^4 < 1.05 \times 10^5$ even though $9.9 > 1.05$: the power of 10 dominates, which is exactly why scientific notation makes comparisons quick.

Q13. (a) The first two significant figures of $2\,649\,000$ are 2 and 6; the next digit is 4, so round down and pad with zeros: $2\,600\,000$. (b) The first two significant figures of $0.004\,906$ are 4 and 9; the next digit is 0, so round down: 0.0049. (c) The first three are 4, 9 and 0; the next digit is 6, so round up: 0.00491. (d) The first three significant figures of 19.978 are 1, 9 and 9; the next digit is 7, so rounding up carries: $19.9 \to 20.0$. The trailing zero is kept to show three significant figures. (e) The second decimal place of 19.978 is 7 and the next digit is 8, so round up: 19.98.

Q14. (a) $0.085 \times 1000 = 85$ m. (b) $85 \times 100 = 8500$ cm. (c) The decimal point moves 3 places left: $8500 = 8.5 \times 10^3$ cm.

Q15. (a) $285 \times 1000 = 285\,000$ L. (b) The decimal point moves 5 places left: $285\,000 = 2.85 \times 10^5$ L. (c) $285\,000 \div 365 = 780.821\ldots$; the first three significant figures are 7, 8 and 0, and the next digit is 8, so round up: 781 L per day. (d) Approximate: the division gives a non-terminating decimal that has been rounded to three significant figures. It is also an average — the household did not use exactly 781 L on any particular day, but over the year its use averaged about 781 L per day, which is useful for comparing with other households or with water-saving targets.

Q16. (a) Hectares (or square kilometres): a sheep station covers an enormous area, and in square metres the number would be unwieldy — hectares are the standard Australian unit for land. (b) Millilitres: a single dose is a small volume of liquid, typically around 10 to 20 mL. (c) Kilometres: the towns are a few hundred kilometres apart, so kilometres give convenient numbers. (d) Millimetres: a coin is only a few millimetres thick, so millimetres suit the small length.

Q17. (a) The third decimal place of $0.048\,265$ is 8 and the next digit is 2, which is less than 5, so round down: 0.048 m. (b) The first three significant figures are 4, 8 and 2; the next digit is 6, so round up: 0.0483 m. (c) The three-significant-figure value keeps more of the measured information. Rounding to three decimal places leaves only the two significant figures 4 and 8, because the leading zeros carry no measurement detail; rounding to three significant figures keeps a third meaningful digit. For small numbers, significant figures describe precision better than decimal places. (d) $0.0483 \times 1000 = 48.3$ mm.

Q18. (a) The power 10^5 moves the decimal point 5 places right: $3.844 \times 10^5 = 384\,400$ km. (b) $384\,400$ km $\times\, 1000 = 384\,400\,000$ m; the decimal point moves 8 places left, so the distance is 3.844×10^8 m. (Equivalently, multiplying by 10^3 raises the power from 10^5 to 10^8.) (c) The first two significant figures are 3 and 8; the next digit is 4, so round down: $380\,000$ km (that is, 3.8×10^5 km). (d) Approximate. The Moon's distance from Earth changes continually as it orbits, so the figure is an average, and that average has itself been rounded to four significant figures.

Theory

Builders, landscapers, painters and farmers all ask the same two questions about a flat shape: how far is it *around* the shape, and how much surface does the shape *cover*? The first is answered by the perimeter, the second by the area, and almost every practical measurement job — fencing a paddock, paving a courtyard, painting a wall, laying turf — comes down to one of the two.

Perimeter. The **perimeter** P of a two-dimensional shape is the total distance around its boundary. For a shape with straight sides, simply add the lengths of all the sides, making sure every length is in the same unit first. Perimeter is a length, so it is measured in length units: mm, cm, m or km. Jobs priced *per metre* — fencing, garden edging, picture framing — are perimeter problems.

Area. The **area** A of a shape is the amount of surface it covers, measured in square units such as mm^2, cm^2, m^2 or km^2. Jobs priced *per square metre* — paving, painting, carpet, instant turf — are area problems: multiply the area by the rate. The area formulas for the standard shapes are:

Shape	Area
square, side l	$A=l^2$
rectangle, length l and width w	$A=lw$
triangle, base b and height h	$A=\frac{1}{2}bh$
parallelogram, base b and height h	$A=bh$
trapezium, parallel sides a and b, height h	$A=\frac{1}{2}(a+b)h$
rhombus, diagonals x and y	$A=\frac{1}{2}xy$

In every formula the height h is the **perpendicular height** — the distance between the base and the opposite side (or vertex), measured at right angles to the base. It is never the slant side.

Heron's rule. Sometimes the three side lengths a, b and c of a triangle are known but its height is not — a surveyed paddock, for instance. **Heron's rule** finds the area directly from the sides. First compute the **semi-perimeter** $s=\frac{a+b+c}{2}$, half the perimeter; then

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

Circles. A circle of radius r (or diameter $d=2r$) has **circumference** (the perimeter of a circle)

$$C=2\pi r=\pi d \text{ and area } A=\pi r^2.$$

Because π is not an exact decimal, an answer such as $C=3\pi$ m is called **exact**, while $C\approx 9.42$ m is a **rounded** (approximate) answer. Give the exact answer where asked, and round only at the final step.

Arcs and sectors. A **sector** is a pizza-slice region of a circle bounded by two radii and a part of the circumference called an **arc**. If the angle between the radii at the centre is θ, measured in degrees, then the sector is the fraction $\frac{\theta}{360}$ of the whole circle, so

$$\text{arc length } l=\frac{\theta}{360}\times 2\pi r \text{ and sector area } A=\frac{\theta}{360}\times \pi r^2.$$

A quarter circle, for example, has a 90° angle at the centre and is the fraction $\frac{90}{360}=\frac{1}{4}$ of the circle. Take care with the **perimeter of a sector**: it is the arc *plus the two radii*, $P=l+2r$, because the boundary of the slice includes both straight edges.

Composite shapes. A **composite shape** is built from simpler shapes. To find its area, either **split** it into rectangles, triangles, sectors and so on and *add* the pieces, or start from a larger simple shape and *subtract* the parts that are missing (such as a hole or a cut-out corner). For the perimeter, trace once around the boundary and add only the edges you actually pass along — internal construction lines used to split the shape are not part of the perimeter.

Exact and rounded answers. Keep answers exact (in terms of π, or as a surd from Heron's rule) for as long as possible and round once, at the end — rounding early and then multiplying magnifies the error. Money is rounded to the nearest cent or dollar as asked. Materials are different: fencing wire, turf rolls and paint are sold in fixed quantities, so round *up* to the amount you must buy — 20.11 m of netting means buying 21 whole metres.

Using CAS. Your CAS has a π key, so enter expressions such as $2\pi\times2.4$ directly and let the calculator carry full precision; only round when you write the final answer. Heron's rule can be evaluated in one line once s is stored. It is still worth estimating by hand with $\pi\approx3$ — if your CAS says a circle of radius 1.5 m has area $70\,\mathrm{m}^2$ when the estimate is about $3\times1.5^2\approx7\,\mathrm{m}^2$, you have mistyped something.

Worked examples

Example 1 — scaffolded

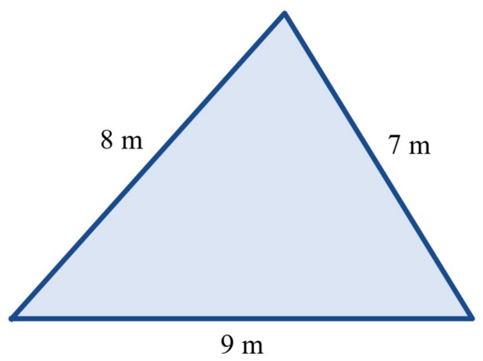
The paved courtyard behind a Ballarat café is a rectangle 6.4 m long and 3.5 m wide. Find the perimeter of the courtyard and its area, and hence the cost of repaving it at $48 per square metre.

Working	Comment
$P=2(l+w)=2(6.4+3.5)=2\times9.9=19.8$ m.	A rectangle's perimeter is twice the sum of its length and width.
$A=lw=6.4\times3.5=22.4\,\mathrm{m}^2$.	Area of a rectangle = length × width, in square metres.
Cost $=22.4\times48=\$1075.20$.	Paving is priced per square metre, so multiply the area by the rate.

So the courtyard has perimeter 19.8 m and area $22.4\,\mathrm{m}^2$, and repaving it costs $1075.20.

Example 2 — scaffolded

A council gardener is planting out a triangular garden bed on a corner block in Shepparton. Its sides measure 7 m, 8 m and 9 m. Find the length of timber edging needed for the perimeter, and use Heron's rule to find the area of the bed, correct to two decimal places.

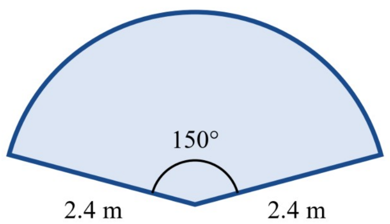


Working	Comment
$P=7+8+9=24$ m.	The perimeter is the sum of the three sides.
$s=\frac{P}{2}=\frac{24}{2}=12$.	Heron's rule needs the semi-perimeter first.
$A=\sqrt{12(12-7)(12-8)(12-9)}$	Substitute into $A=\sqrt{s(s-a)(s-b)(s-c)}$.
$=\sqrt{12\times5\times4\times3}=\sqrt{720}$	The three brackets are 5, 4 and 3.
$=12\sqrt{5}\approx26.83\,\text{m}^2$.	$720=144\times5$, so $\sqrt{720}=12\sqrt{5}$. Round only at the end.

So the bed needs 24 m of edging and has area $12\sqrt{5}\approx26.83\,\text{m}^2$.

Example 3 — unscaffolded

A landscaper is building a sector-shaped garden bed of radius 2.4 m, with an angle of 150° between the two straight edges at the centre. Find, exactly and correct to two decimal places, the length of the curved edge, the area of the bed, and the total length of edging needed to surround the bed.

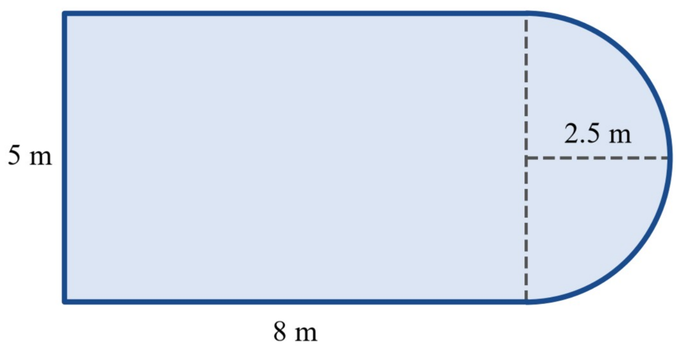


The sector is the fraction $\frac{150}{360}=\frac{5}{12}$ of a full circle of radius 2.4 m. The curved edge is the arc: $l=\frac{5}{12}\times2\pi\times2.4=\frac{5}{12}\times4.8\pi=2\pi\approx6.28$ m. The area is the same fraction of the circle's area: $A=\frac{5}{12}\times\pi\times2.4^2=\frac{5}{12}\times5.76\pi=2.4\pi\approx7.54\,\text{m}^2$. The edging must run along the arc *and* both straight edges, which are radii, so $P=l+2r=2\pi+2\times2.4=2\pi+4.8\approx11.08$ m.

∴ the curved edge is $2\pi\approx6.28$ m long, the bed has area $2.4\pi\approx7.54\,\text{m}^2$, and $2\pi+4.8\approx11.08$ m of edging is needed.

Example 4 — unscaffolded

The outdoor dining area at a Bendigo pub is a composite shape: a rectangle 8 m long and 5 m wide, with a semicircular end of diameter 5 m attached to one of the short sides. Find the area and the perimeter of the dining area, correct to two decimal places, and the cost of paving it at $85 per square metre, to the nearest dollar.



Split the shape into the rectangle and the semicircle. The rectangle has area $8 \times 5 = 40\,\text{m}^2$. The semicircle has radius $r = 5 \div 2 = 2.5$ m, so its area is half a circle: $\frac{1}{2} \times \pi \times 2.5^2 = 3.125\pi \approx 9.82\,\text{m}^2$. The total area is $40 + 3.125\pi \approx 49.82\,\text{m}^2$. The boundary consists of three sides of the rectangle (the fourth is swallowed by the semicircle) plus the semicircular arc: $8 + 5 + 8 = 21$ m of straight edge, and the arc is $\frac{1}{2} \times 2\pi \times 2.5 = 2.5\pi \approx 7.85$ m, so $P = 21 + 2.5\pi \approx 28.85$ m. Costing uses the exact area: $85 \times (40 + 3.125\pi) = 3400 + 265.625\pi \approx \$4234.49$, which is \$4234 to the nearest dollar.

∴ the dining area is $40 + 3.125\pi \approx 49.82\,\text{m}^2$ with perimeter $21 + 2.5\pi \approx 28.85$ m, and paving it costs \$4234 to the nearest dollar.

Practice questions

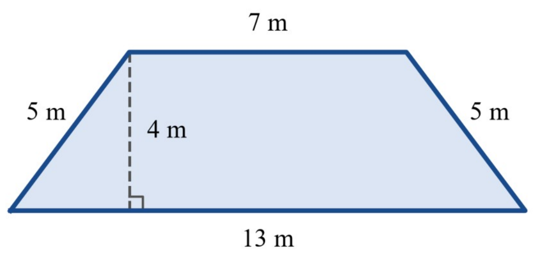
Scaffolded

Q1. A rectangular backyard lawn in Wodonga measures 9.5 m by 6.4 m. (a) Find the perimeter of the lawn. (b) Find the area of the lawn. (c) Instant turf costs \$14 per square metre, laid. Find the cost of returfing the whole lawn.

Q2. A triangular garden bed fits into the corner of a courtyard. The two sides that meet at the right angle measure 6 m and 8 m, and the longest side measures 10 m. (a) Find the area of the bed. (b) Find the perimeter of the bed. (c) Timber edging costs \$7.50 per metre. Find the cost of edging the whole bed.

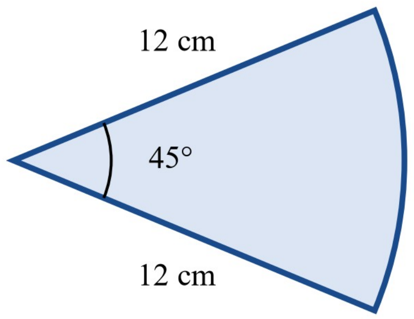
Q3. A circular fish pond has radius 1.5 m. (a) Find the circumference of the pond, exactly and correct to two decimal places. (b) Find the area of the pond's surface, exactly and correct to two decimal places. (c) To keep ibis away, three strands of wire are to be stretched around the pond's edge, one above the other. Find the total length of wire needed, correct to two decimal places.

Q4. The diagram shows a trapezium-shaped garden bed. Its parallel sides measure 13 m and 7 m and are 4 m apart, and each sloping side measures 5 m.



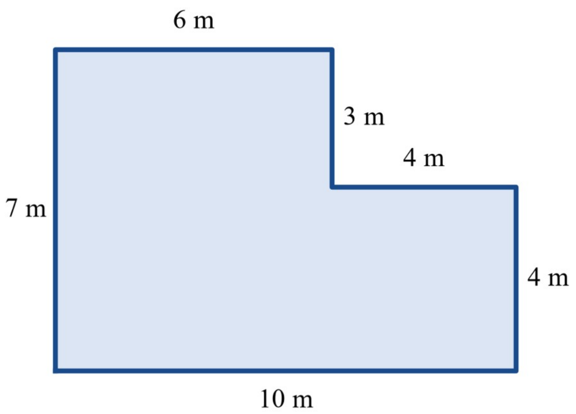
(a) Find the area of the bed.
(b) Find the perimeter of the bed.
(c) A gravel topping costs \$12 per square metre. Find the cost of topping the whole bed.

Q5. A circular pizza of radius 12 cm is cut into equal slices, each with an angle of 45° at the centre.



(a) What fraction of the whole pizza is one slice?
(b) Find the length of the crust (the arc) on one slice, exactly and correct to two decimal places.
(c) Find the area of one slice, exactly and correct to two decimal places.
(d) Find the perimeter of one slice, correct to two decimal places.

Q6. The diagram shows an L-shaped courtyard: a 10 m by 7 m rectangle with a 4 m by 3 m rectangular corner removed.



(a) Find the area of the courtyard by subtracting the missing corner from the full rectangle.
(b) Using the side lengths marked on the diagram, find the perimeter of the courtyard.
(c) Paving the courtyard costs $65 per square metre. Find the total cost.

Unscaffolded

Q7. A square paved entertaining area has sides of length 7.5 m. Find its perimeter and its area, and the cost of sealing the paving at $32 per square metre.

Q8. A parallelogram-shaped decorative wall tile has a base of 24 cm and a slant side of 18 cm, and the perpendicular height to the base is 15 cm. Find the perimeter and the area of the tile.

Q9. A rhombus-shaped tile has diagonals of length 12 cm and 9 cm. Find the area of the tile. Then, using the fact that the diagonals of a rhombus bisect each other at right angles, find the side length of the tile and hence its perimeter.

Q10. A triangular pocket park in inner Melbourne is bounded by three streets, with frontages of 50 m, 60 m and 70 m. Find the perimeter of the park and, using Heron's rule, its area, exactly and correct to two decimal places.

Q11. The circular helipad at a regional hospital has diameter 18 m. Find its circumference and its area, exactly and correct to two decimal places, and the cost of resurfacing it with non-slip paint at $6.50 per square metre, to the nearest dollar.

Q12. A garden sprinkler sprays water over a sector of radius 6 m as it turns through 210°. Find the area of lawn that is watered, and the length of the outer curved edge of the watered region, each exactly and correct to two decimal places.

Q13. For each triangle below, state which method you would use to find its area — the base–height formula or Heron's rule — and why, and then find the area. (a) A triangle with base 10 cm and perpendicular height 7 cm. (b) A triangular paddock in which only the three side lengths are known: 9 m, 10 m and 17 m. (c) A right-angled triangle whose two shorter sides measure 9 cm and 12 cm.

Q14. A rectangular steel plate measuring 40 cm by 25 cm has a circular hole of radius 6 cm cut through it. Find the area of steel remaining, exactly and correct to two decimal places.

Q15. A feature window is a composite shape: a rectangle 1.2 m wide and 1.5 m tall, topped by a semicircle of diameter 1.2 m. Find the area of glass, exactly and correct to two decimal places, and the total length of the window frame (the perimeter), exactly and correct to two decimal places.

Q16. The running track at a Geelong athletics club consists of two 85 m straights joined by two semicircular ends, each of diameter 60 m. Find the distance of one lap around the track and the area enclosed by the track, each exactly and correct to two decimal places.

Q17. A circular chicken run of radius 3.2 m is to be enclosed with wire netting. Find the circumference of the run, correct to two decimal places. The netting is sold only in whole-metre lengths at $18.50 per metre. How many metres must be bought, and why? What does the netting cost?

Q18. A circular lawn of radius 4 m is surrounded by a gravel path 1 m wide. Find the area of the path, exactly and correct to two decimal places, and the cost of gravelling it at $22 per square metre, laid, to the nearest dollar.

Answers

Q1. (a) $P=31.8$ m. (b) $A=60.8\,\mathrm{m}^2$. (c) \$851.20. **Q2.** (a) $A=24\,\mathrm{m}^2$. (b) $P=24$ m. (c) \$180. **Q3.** (a) $C=3\pi\approx9.42$ m. (b) $A=2.25\pi\approx7.07\,\mathrm{m}^2$. (c) $9\pi\approx28.27$ m. **Q4.** (a) $A=40\,\mathrm{m}^2$. (b) $P=30$ m. (c) \$480. **Q5.** (a) $\frac{1}{8}$. (b) $3\pi\approx9.42$ cm. (c) $18\pi\approx56.55\,\mathrm{cm}^2$. (d) $3\pi+24\approx33.42$ cm. **Q6.** (a) $A=58\,\mathrm{m}^2$. (b) $P=34$ m. (c) \$3770. **Q7.** $P=30$ m, $A=56.25\,\mathrm{m}^2$; cost \$1800. **Q8.** $P=84$ cm, $A=360\,\mathrm{cm}^2$. **Q9.** $A=54\,\mathrm{cm}^2$; side $=7.5$ cm, $P=30$ cm. **Q10.** $P=180$ m; $A=600\sqrt{6}\approx1469.69\,\mathrm{m}^2$. **Q11.** $C=18\pi\approx56.55$ m; $A=81\pi\approx254.47\,\mathrm{m}^2$; cost \$1654. **Q12.** $A=21\pi\approx65.97\,\mathrm{m}^2$; arc $=7\pi\approx21.99$ m. **Q13.** (a) Base–height; $35\,\mathrm{cm}^2$. (b) Heron's rule; $36\,\mathrm{m}^2$. (c) Base–height with the two shorter sides; $54\,\mathrm{cm}^2$. **Q14.** $A=1000-36\pi\approx886.90\,\mathrm{cm}^2$. **Q15.** $A=1.8+0.18\pi\approx2.37\,\mathrm{m}^2$; frame $=4.2+0.6\pi\approx6.08$ m. **Q16.** Lap $=170+60\pi\approx358.50$ m; area $=5100+900\pi\approx7927.43\,\mathrm{m}^2$. **Q17.** $C=6.4\pi\approx20.11$ m; buy 21 m (round up — 20 m is not enough); cost \$388.50. **Q18.** $A=9\pi\approx28.27\,\mathrm{m}^2$; cost \$622.

Fully-worked solutions

Q1. (a) $P=2(l+w)=2(9.5+6.4)=2\times15.9=31.8$ m. (b) $A=lw=9.5\times6.4=60.8\,\mathrm{m}^2$. (c) Cost $=60.8\times14=\$851.20$.

Q2. (a) The two sides that meet at the right angle are perpendicular, so one is a base and the other is the height: $A=\frac{1}{2}\times6\times8=24\,\mathrm{m}^2$. (b) $P=6+8+10=24$ m. (c) Cost $=24\times7.50=\$180$.

Q3. (a) $C=2\pi r=2\pi\times1.5=3\pi\approx9.42$ m. (b) $A=\pi r^2=\pi\times1.5^2=2.25\pi\approx7.07\,\mathrm{m}^2$. (c) Each strand is one circumference, so the total is $3\times3\pi=9\pi\approx28.27$ m.

Q4. (a) $A=\frac{1}{2}(a+b)h=\frac{1}{2}(13+7)\times4=\frac{1}{2}\times20\times4=40\,\mathrm{m}^2$. Note that the height used is the perpendicular distance 4 m between the parallel sides, not the 5 m slant sides. (b) $P=13+7+5+5=30$ m. (c) Cost $=40\times12=\$480$.

Q5. (a) One slice is $\frac{45}{360}=\frac{1}{8}$ of the pizza. (b) Arc $l=\frac{1}{8}\times2\pi\times12=\frac{1}{8}\times24\pi=3\pi\approx9.42$ cm. (c) $A=\frac{1}{8}\times\pi\times12^2=\frac{1}{8}\times144\pi=18\pi\approx56.55\,\mathrm{cm}^2$. (d) The perimeter is the arc plus the two straight (radius) edges: $P=3\pi+2\times12=3\pi+24\approx33.42$ cm.

Q6. (a) The full rectangle has area $10\times7=70\,\mathrm{m}^2$ and the missing corner has area $4\times3=12\,\mathrm{m}^2$, so $A=70-12=58\,\mathrm{m}^2$. (b) Going around the boundary: $10+4+4+3+6+7=34$ m. (This equals the perimeter $2(10+7)=34$ m of the full rectangle: the two cut edges of 4 m and 3 m exactly replace the removed pieces of the original sides.) (c) Cost $=58\times65=\$3770$.

Q7. $P=4\times7.5=30$ m and $A=l^2=7.5^2=56.25\,\mathrm{m}^2$. The sealing cost is $56.25\times32=\$1800$.

Q8. The perimeter uses the two pairs of sides: $P=2(24+18)=2\times42=84$ cm. The area uses the perpendicular height, not the slant side: $A=bh=24\times15=360\,\mathrm{cm}^2$.

Q9. Using the diagonals, $A=\frac{1}{2}xy=\frac{1}{2}\times12\times9=54\,\mathrm{cm}^2$. The diagonals bisect each other at right angles, so half-diagonals of 6 cm and 4.5 cm form a right-angled triangle whose hypotenuse is a side of the rhombus: side $=\sqrt{6^2+4.5^2}=\sqrt{36+20.25}=\sqrt{56.25}=7.5$ cm. Hence $P=4\times7.5=30$ cm.

Q10. $P = 50 + 60 + 70 = 180$ m, so $s = \frac{180}{2} = 90$. By Heron's rule,
$A = \sqrt{90(90-50)(90-60)(90-70)} = \sqrt{90 \times 40 \times 30 \times 20} = \sqrt{2\,160\,000}$. Since $2\,160\,000 = 360\,000 \times 6$, this is $A = 600\sqrt{6} \approx 1469.69\,\text{m}^2$.

Q11. The radius is $r = 18 \div 2 = 9$ m. Then $C = 2\pi \times 9 = 18\pi \approx 56.55$ m and $A = \pi \times 9^2 = 81\pi \approx 254.47\,\text{m}^2$. Costing from the exact area, $81\pi \times 6.50 = 526.5\pi \approx \$1654.05$, which is \$1654 to the nearest dollar.

Q12. The sector is $\frac{210}{360} = \frac{7}{12}$ of a circle of radius 6 m. The watered area is
$A = \frac{7}{12} \times \pi \times 6^2 = \frac{7}{12} \times 36\pi = 21\pi \approx 65.97\,\text{m}^2$, and the outer curved edge is the arc
$l = \frac{7}{12} \times 2\pi \times 6 = \frac{7}{12} \times 12\pi = 7\pi \approx 21.99$ m.

Q13. (a) The base and the perpendicular height are both given, so use the base–height formula:
$A = \frac{1}{2} \times 10 \times 7 = 35\,\text{cm}^2$. (b) Only the three sides are known and no height, so use Heron's rule: $s = \frac{9+10+17}{2} = 18$, and $A = \sqrt{18(18-9)(18-10)(18-17)} = \sqrt{18 \times 9 \times 8 \times 1} = \sqrt{1296} = 36\,\text{m}^2$. (c) In a right-angled triangle the two shorter sides are perpendicular, so they serve as base and height: $A = \frac{1}{2} \times 9 \times 12 = 54\,\text{cm}^2$.

Q14. The plate has area $40 \times 25 = 1000\,\text{cm}^2$ and the hole has area $\pi \times 6^2 = 36\pi \approx 113.10\,\text{cm}^2$. Subtracting the hole, the steel remaining is $A = 1000 - 36\pi \approx 886.90\,\text{cm}^2$.

Q15. The rectangle has area $1.2 \times 1.5 = 1.8\,\text{m}^2$. The semicircle has radius $r = 1.2 \div 2 = 0.6$ m, so its area is $\frac{1}{2} \times \pi \times 0.6^2 = 0.18\pi \approx 0.57\,\text{m}^2$. The glass area is $A = 1.8 + 0.18\pi \approx 2.37\,\text{m}^2$. The frame runs along the two vertical sides and the sill, $1.5 + 1.5 + 1.2 = 4.2$ m, plus the semicircular arc $\frac{1}{2} \times 2\pi \times 0.6 = 0.6\pi \approx 1.88$ m, so the frame length is $P = 4.2 + 0.6\pi \approx 6.08$ m.

Q16. The two semicircular ends together make one full circle of radius $60 \div 2 = 30$ m, with circumference $2\pi \times 30 = 60\pi \approx 188.50$ m. One lap is the two straights plus the two ends: $2 \times 85 + 60\pi = 170 + 60\pi \approx 358.50$ m. The enclosed region is the central 85×60 rectangle plus the two semicircular ends, which together make one full circle of radius 30 m: $A = 85 \times 60 + \pi \times 30^2 = 5100 + 900\pi \approx 7927.43\,\text{m}^2$.

Q17. $C = 2\pi \times 3.2 = 6.4\pi \approx 20.11$ m. Netting is sold only in whole metres, and 20 m is less than the 20.11 m needed, so the netting must be rounded *up*: buy 21 m. (Rounding to the nearest metre would leave the run short of netting.) The cost is $21 \times 18.50 = \$388.50$.

Q18. The outer edge of the path is a circle of radius $4 + 1 = 5$ m. The path is the region between the two circles, so its area is $A = \pi \times 5^2 - \pi \times 4^2 = 25\pi - 16\pi = 9\pi \approx 28.27\,\text{m}^2$. Costing from the exact area, $9\pi \times 22 = 198\pi \approx \$622.04$, which is \$622 to the nearest dollar.

Chapter 9 Measurement, geometry and similarity — 9C Surface area and volume of solids

Theory

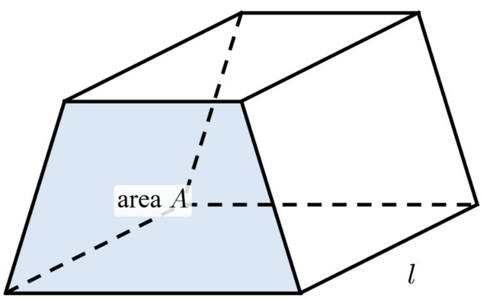
Many practical problems in measurement involve three-dimensional solids — water tanks, packing cartons, stormwater pipes and grain silos. Two measurements describe a solid. Its **volume** is the amount of space it occupies, measured in cubic units such as cm^3 or m^3; volume determines how much a container can hold. Its **surface area** is the total area of all the faces and curved surfaces that enclose it, measured in square units such as cm^2 or m^2; surface area determines how much material is needed to make, paint or wrap the solid.

Prisms and cylinders. A **prism** is a solid with the same **cross-section** all the way along its length, like a loaf of sliced bread in which every slice is identical. Its volume is

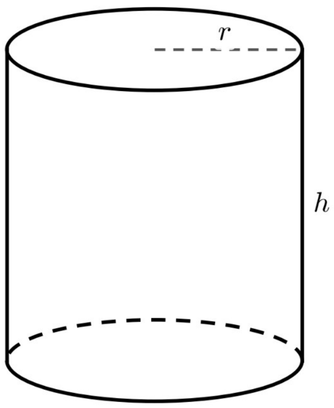
$$V = \text{area of cross-section} \times \text{length}.$$

For a **rectangular prism** (a box) with length l, width w and height h the cross-section is a rectangle, giving $V = l\,w\,h$. A **cylinder** has a circular cross-section of radius r, so the same rule gives $V = \pi r^2 h$.

Prism: $V = A \times l$



Cylinder: $V = \pi r^2 h$



Surface area of prisms and cylinders. The surface area of a prism is found by adding the areas of all its faces; sketching the **net** of the solid helps to make sure that no face is missed. A closed cylinder has two circular ends, and its curved surface unrolls into a rectangle whose width is the circumference $2\pi r$ and whose height is h, so

$$\text{surface area of a closed cylinder} = 2\pi r^2 + 2\pi r h.$$

Pyramids and cones. A **pyramid** has a flat base and triangular faces that meet at a point called the **apex**; a **cone** is the circular version. Each holds exactly one-third of the prism or cylinder with the same base and height:

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}, V_{\text{cone}} = \frac{1}{3}\pi r^2 h.$$

The **slant height** ℓ of a cone is the distance from the apex down the sloping surface to the edge of the base. The radius, the height and the slant height form a right-angled triangle inside the cone, so by Pythagoras' theorem $\ell = \sqrt{r^2 + h^2}$. The curved surface of a cone has area $\pi r \ell$, so a closed cone has total surface area

$$\text{surface area of a closed cone} = \pi r^2 + \pi r \ell.$$

For a square-based pyramid with base edge b and height h, the slant height of each triangular face is $s = \sqrt{h^2 + \left(\frac{b}{2}\right)^2}$, and the surface area is the base b^2 plus the four triangular faces, $4 \times \frac{1}{2} b s$.

Spheres. A **sphere** of radius r has

$$V = \frac{4}{3}\pi r^3 \text{ and surface area} = 4\pi r^2.$$

A **hemisphere** is half a sphere, so halve the volume; its curved surface is half the sphere's surface area, $2\pi r^2$.

Pythagoras in three dimensions. Right-angled triangles occur inside solids as well as on flat pages. The longest thin rod that fits inside a box lies along its **space diagonal**, from one corner to the opposite corner. Apply Pythagoras' theorem twice: the diagonal of the base is $\sqrt{l^2 + w^2}$, and this base diagonal together with the height gives

$$d = \sqrt{l^2 + w^2 + h^2}.$$

The same idea gives the slant height of a cone or a pyramid from its radius (or half-base) and its height.

Composite solids. Many everyday objects are **composite solids**, made by joining or removing simpler solids: a silo is a cylinder with a cone on top, and a pipe is a large cylinder with a smaller cylinder removed. To find the volume, **add** the volumes of joined parts, or **subtract** the volume that has been removed. For surface area, count only the surfaces that can actually be seen or touched — a face where two parts join is not part of the surface.

Capacity. The **capacity** of a container is the volume of liquid it can hold, and it links cubic units to litres:

$$1\,\text{cm}^3 = 1\,\text{mL},\ 1000\,\text{cm}^3 = 1\,\text{L},\ 1\,\text{m}^3 = 1000\,\text{L}.$$

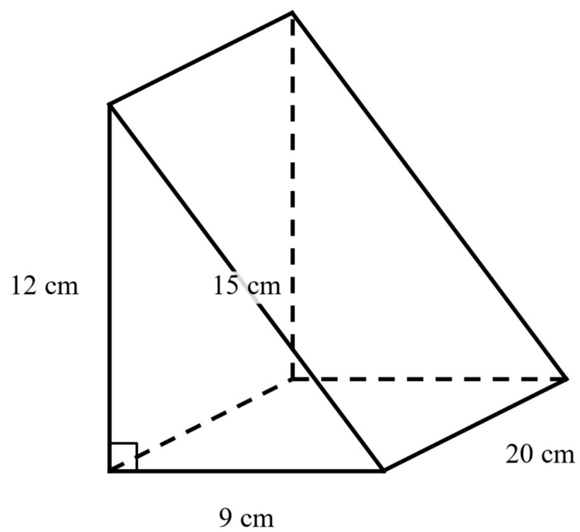
Keep π exact for as long as possible — carry an answer such as $200\pi\,\text{cm}^3$ through the working and round only at the final step.

Using CAS. General Mathematics is a technology-active, open-book course, and once the correct formula is chosen the arithmetic is a single line on your CAS. Use the calculator's π key rather than an approximation such as 3.14, and store the exact result before rounding — especially in composite problems, where one volume is added to or subtracted from another and early rounding can shift the final answer. A quick estimate with $\pi \approx 3$ is a good check that the calculator's answer is reasonable; the by-hand working below shows exactly what the calculator is computing.

Worked examples

Example 1 — scaffolded

A hardwood door wedge is a triangular prism 20 cm long. Its cross-section is a right-angled triangle with base 9 cm, height 12 cm and sloping side 15 cm. Find the volume of the wedge and the total surface area of timber to be sealed.

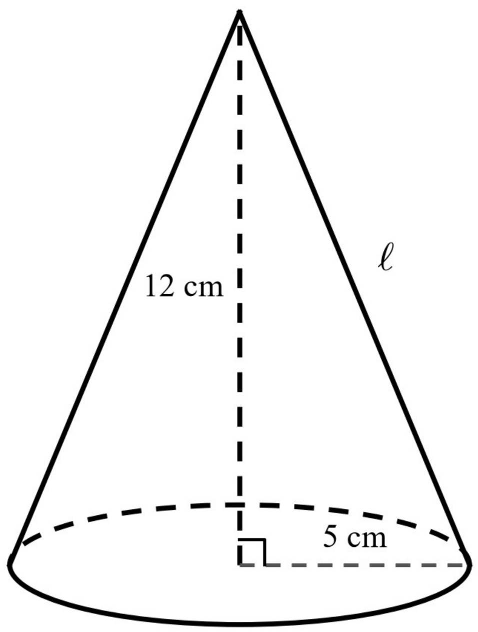


Working	Comment
Area of cross-section $= \frac{1}{2} \times 9 \times 12 = 54\,\text{cm}^2$.	The cross-section is a right-angled triangle with base 9 cm and height 12 cm.
$V = \text{area of cross-section} \times \text{length} = 54 \times 20 = 1080\,\text{cm}^3$.	The volume of any prism is its cross-section area times its length.
The two triangular ends: $2 \times 54 = 108\,\text{cm}^2$.	A triangular prism has five faces: two triangles and three rectangles.
The three rectangles: $(9+12+15) \times 20 = 36 \times 20 = 720\,\text{cm}^2$.	Each rectangle is 20 cm long and as wide as one side of the triangle, so together they cover perimeter × length.
Surface area $= 108 + 720 = 828\,\text{cm}^2$.	Add the areas of all five faces.

So the wedge has volume $1080\,\text{cm}^3$ and surface area $828\,\text{cm}^2$.

Example 2 — scaffolded

A solid glass paperweight is a cone of radius 5 cm and height 12 cm. Find its slant height, its volume and its total surface area, rounding to two decimal places where necessary.



Working	Comment
$\ell = \sqrt{r^2 + h^2} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$ cm.	The radius, the height and the slant height form a right-angled triangle — Pythagoras in three dimensions.
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 5^2 \times 12 = 100\pi \approx 314.16\,\text{cm}^3$.	A cone holds one-third of the matching cylinder. Keep the exact form 100π until the final rounding.
Surface area $= \pi r^2 + \pi r \ell = 25\pi + \pi \times 5 \times 13 = 25\pi + 65\pi = 90\pi \approx 28$.	The base circle plus the curved surface; the curved part uses the slant height, not the vertical height.

So the slant height is 13 cm, the volume is $100\pi \approx 314.16\,\text{cm}^3$ and the surface area is $90\pi \approx 282.74\,\text{cm}^2$.

Example 3 — unscaffolded

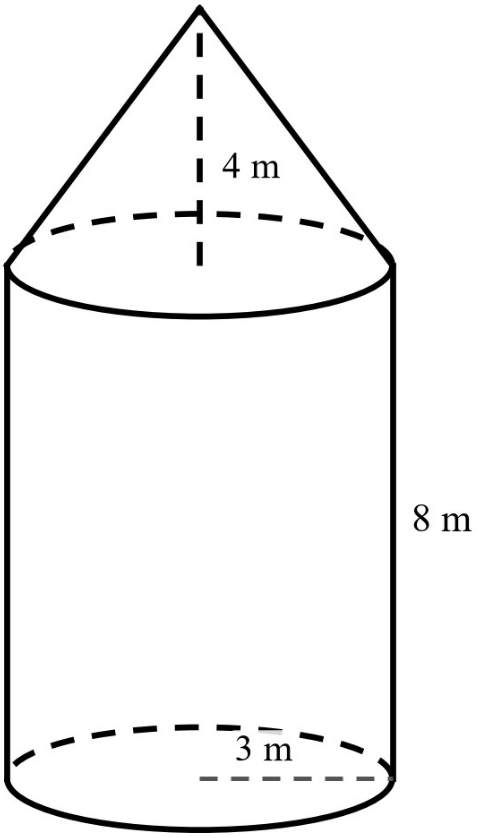
A spherical glass terrarium has radius 12 cm. Find the area of glass in its surface and the volume inside, each correct to two decimal places, and its capacity in litres, correct to two decimal places.

The surface area of a sphere is $4\pi r^2 = 4\times\pi\times 12^2 = 576\pi \approx 1809.56\,\text{cm}^2$. The volume is $\frac{4}{3}\pi r^3 = \frac{4}{3}\times\pi\times 12^3 = \frac{4}{3}\pi\times 1728 = 2304\pi \approx 7238.23\,\text{cm}^3$. Since $1\,\text{cm}^3$ holds 1 mL, the terrarium holds about 7238.23 mL, and dividing by 1000 converts this to litres: $7238.23 \div 1000 \approx 7.24$ L.

∴ the glass has area $576\pi \approx 1809.56\,\text{cm}^2$, and the volume of $2304\pi \approx 7238.23\,\text{cm}^3$ gives a capacity of about 7.24 L.

Example 4 — unscaffolded

A grain silo on a farm near Horsham consists of a cylinder of radius 3 m and height 8 m, topped by a cone of height 4 m. Find the total volume of the silo, correct to two decimal places, and its capacity to the nearest litre.



The cylinder has volume $\pi r^2 h = \pi\times 3^2\times 8 = 72\pi\,\text{m}^3$ and the cone has volume $\frac{1}{3}\pi r^2 h = \frac{1}{3}\times\pi\times 3^2\times 4 = 12\pi\,\text{m}^3$. Working exactly and adding, the total volume is $72\pi + 12\pi = 84\pi \approx 263.89\,\text{m}^3$. Each cubic metre holds 1000 L, so the capacity is $84\pi\times 1000 \approx 263894$ L — only now is the answer rounded. (A quick check with $\pi \approx 3$ gives $84\times 3 = 252\,\text{m}^3$, so $263.89\,\text{m}^3$ is reasonable.)

∴ the silo's volume is $84\pi \approx 263.89\,\text{m}^3$, which is a capacity of about 263 894 L.

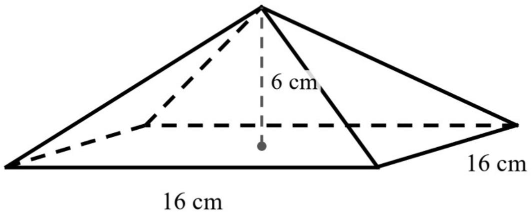
Practice questions

Scaffolded

Q1. A removalist's packing carton is a closed rectangular prism 40 cm long, 25 cm wide and 30 cm high. (a) Find the volume of the carton. (b) Find its capacity in litres. (c) Find the total area of cardboard in the carton's six faces.

Q2. A closed cylindrical paint tin has radius 10 cm and height 25 cm. (a) Find its volume, correct to two decimal places. (b) Find its capacity in litres, correct to two decimal places. (c) Find its total surface area, correct to two decimal places.

Q3. A candle is a square-based pyramid with base edge 16 cm and height 6 cm.

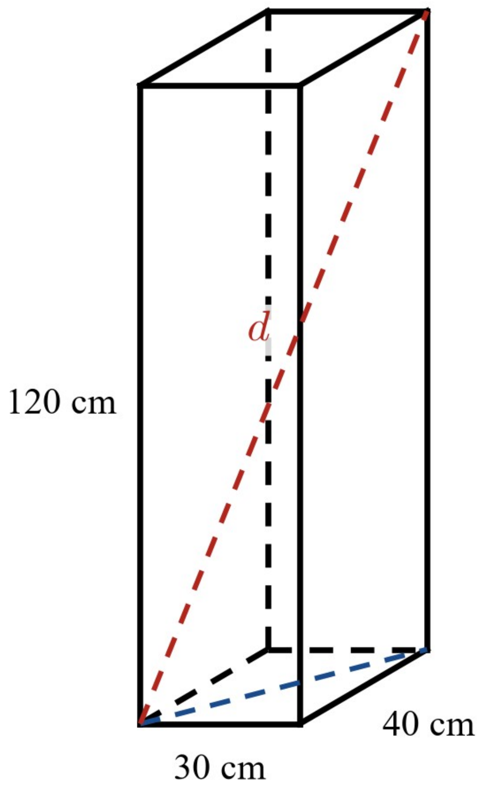


(a) Find the volume of wax in the candle.
(b) Use Pythagoras' theorem to find the slant height of each triangular face.
(c) Find the total surface area of the candle, including its base.

Q4. A solid timber cone in a toddler's stacking toy has radius 8 cm and height 15 cm. (a) Find its slant height. (b) Find its volume, correct to two decimal places. (c) Find its total surface area, correct to two decimal places.

Q5. A leather medicine ball is a sphere of radius 9 cm. (a) Find the area of leather in its surface, correct to two decimal places. (b) Find its volume, correct to two decimal places. (c) Find the capacity of the ball in litres, correct to two decimal places.

Q6. A school sports locker is a rectangular prism 30 cm wide, 40 cm deep and 120 cm tall.



(a) Find the length of the diagonal of the locker's base.
(b) Hence find the length of the space diagonal — the longest thin rod that fits inside the locker.
(c) A student's collapsed tent pole is 125 cm long. Will it fit in the locker? Explain.

Unscaffolded

Q7. A camping tent is a triangular prism 3 m long. Its cross-section is an isosceles triangle with base 2.4 m, vertical height 0.9 m and two equal sloping sides of 1.5 m. Find the volume of the tent and the total area of canvas, including the floor and both triangular ends.

Q8. A cylindrical rainwater tank has radius 1 m and height 1.4 m. Find its volume in cubic metres, correct to two decimal places, and its capacity to the nearest litre.

Q9. A cardboard party hat is an open cone (it has no base) with radius 7 cm and vertical height 24 cm. Find its slant height and the area of cardboard in the hat, correct to two decimal places. Explain why the formula $\pi r^2 + \pi r \ell$ would give the wrong answer here.

Q10. A hemispherical salad bowl has radius 15 cm. Find the volume it holds, correct to two decimal places, its capacity in litres, correct to two decimal places, and the area of its curved outer surface, correct to two decimal places.

Q11. A concrete garden ornament is a square-based pyramid with base edge 18 cm and height 12 cm. Find the volume of concrete in the ornament and its total surface area, including the base.

Q12. A vitamin capsule consists of a cylinder of radius 4 mm and length 10 mm with a hemisphere of radius 4 mm on each end. Find the volume of the capsule, correct to two decimal places.

Q13. A concrete stormwater pipe is 2 m long. Its outer diameter is 60 cm and its inner diameter is 50 cm. Find the volume of concrete in the pipe in cm^3, correct to two decimal places, and convert your answer to m^3, correct to two decimal places.

Q14. A cereal company is choosing between two closed cartons. Carton A is a cube with edge 10 cm; carton B is a rectangular prism 20 cm long, 10 cm wide and 5 cm high. Show that the two cartons hold exactly the same amount, find the area of cardboard in each, and explain which carton the company should choose in order to use less cardboard.

Q15. A juice popper is a rectangular prism 4 cm wide, 6 cm deep and 12 cm tall. Find the length of the longest straw that fits entirely inside the popper. A 20 cm straw is then inserted through a top corner so that its lower end sits in the opposite bottom corner. How far does the straw stick out of the popper?

Q16. A school’s rectangular swimming pool is 25 m long and 10 m wide, with a uniform depth of 1.6 m. Find the volume of the pool and its capacity in litres, and find how long the pool takes to fill from empty at 250 L per minute. Give the time in hours and minutes.

Q17. A square concrete paver is 40 cm by 40 cm and 20 cm thick. A cylindrical hole of radius 10 cm is cast right through its thickness so that a garden pot can sit in it. Find the volume of concrete remaining, correct to two decimal places.

Q18. A garden shed is 6 m long and 4 m wide, with vertical walls 2.5 m high and a gable roof whose ridge runs the length of the shed, 1.5 m above the tops of the walls. Find the volume of air inside the shed, the slant width of each roof section, and the total area of roofing sheet required for the two rectangular roof sections.

Answers

Q1. (a) $V = 30000\,\text{cm}^3$. (b) 30 L. (c) $5900\,\text{cm}^2$. **Q2.** (a) $V = 2500\pi \approx 7853.98\,\text{cm}^3$. (b) ≈ 7.85 L. (c) $700\pi \approx 2199.11\,\text{cm}^2$. **Q3.** (a) $V = 512\,\text{cm}^3$. (b) $s = 10$ cm. (c) $576\,\text{cm}^2$. **Q4.** (a) $\ell = 17$ cm. (b) $V = 320\pi \approx 1005.31\,\text{cm}^3$. (c) $200\pi \approx 628.32\,\text{cm}^2$. **Q5.** (a) $324\pi \approx 1017.88\,\text{cm}^2$. (b) $V = 972\pi \approx 3053.63\,\text{cm}^3$. (c) ≈ 3.05 L. **Q6.** (a) 50 cm. (b) 130 cm. (c) Yes — 125 cm < 130 cm, so the pole fits along the space diagonal. **Q7.** $V = 3.24\,\text{m}^3$; canvas area $= 18.36\,\text{m}^2$. **Q8.** $V = 1.4\pi \approx 4.40\,\text{m}^3$; capacity ≈ 4398 L. **Q9.** $\ell = 25$ cm; cardboard $= 175\pi \approx 549.78\,\text{cm}^2$; the hat has no base, so the πr^2 term must be left out — only the curved surface $\pi r \ell$ is cardboard. **Q10.** $V = 2250\pi \approx 7068.58\,\text{cm}^3$; capacity ≈ 7.07 L; curved surface $= 450\pi \approx 1413.72\,\text{cm}^2$. **Q11.** $V = 1296\,\text{cm}^3$; slant height 15 cm; surface area $= 864\,\text{cm}^2$. **Q12.** $V = \dfrac{736\pi}{3} \approx 770.74\,\text{mm}^3$. **Q13.** $55000\pi \approx 172787.60\,\text{cm}^3 \approx 0.17\,\text{m}^3$. **Q14.** Both hold $1000\,\text{cm}^3 = 1$ L; carton A: $600\,\text{cm}^2$; carton B: $700\,\text{cm}^2$; choose carton A (the cube) — the same capacity with less cardboard. **Q15.** Longest straw $= 14$ cm; the 20 cm straw sticks out 6 cm. **Q16.** $V = 400\,\text{m}^3$; capacity 400 000 L; 1600 minutes $= 26$ hours 40 minutes. **Q17.** $32000 - 2000\pi \approx 25716.81\,\text{cm}^3$. **Q18.** $V = 78\,\text{m}^3$; slant width 2.5 m; roofing area $= 30\,\text{m}^2$.

Fully-worked solutions

Q1. (a) $V = lwh = 40 \times 25 \times 30 = 30000\,\text{cm}^3$. (b) Since $1000\,\text{cm}^3 = 1\,\text{L}$, the capacity is $30000 \div 1000 = 30$ L. (c) The three different faces have areas $40 \times 25 = 1000$, $40 \times 30 = 1200$ and $25 \times 30 = 750\,\text{cm}^2$, and each occurs twice, so the surface area is $2(1000 + 1200 + 750) = 2 \times 2950 = 5900\,\text{cm}^2$.

Q2. (a) $V = \pi r^2 h = \pi \times 10^2 \times 25 = 2500\pi \approx 7853.98\,\text{cm}^3$. (b) $2500\pi\,\text{cm}^3$ is 2500π mL $= 2.5\pi$ L ≈ 7.85 L. (c) Surface area $= 2\pi r^2 + 2\pi r h = 2\pi \times 100 + 2\pi \times 10 \times 25 = 200\pi + 500\pi = 700\pi \approx 2199.11\,\text{cm}^2$.

Q3. (a) $V = \dfrac{1}{3} \times 16^2 \times 6 = \dfrac{1}{3} \times 256 \times 6 = 512\,\text{cm}^3$. (b) The slant height of a face runs from the apex to the midpoint of a base edge. Half the base edge is 8 cm, so $s = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$ cm. (c) Surface area = base + four triangular faces $= 16^2 + 4 \times \dfrac{1}{2} \times 16 \times 10 = 256 + 320 = 576\,\text{cm}^2$.

Q4. (a) $\ell = \sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$ cm. (b) $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3} \times \pi \times 64 \times 15 = 320\pi \approx 1005.31\,\text{cm}^3$. (c) Surface area $= \pi r^2 + \pi r \ell = 64\pi + \pi \times 8 \times 17 = 64\pi + 136\pi = 200\pi \approx 628.32\,\text{cm}^2$.

Q5. (a) Surface area $= 4\pi r^2 = 4\pi \times 81 = 324\pi \approx 1017.88\,\text{cm}^2$. (b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi \times 729 = 972\pi \approx 3053.63\,\text{cm}^3$. (c) $972\pi\,\text{cm}^3 \approx 3053.63$ mL, and $3053.63 \div 1000 \approx 3.05$ L.

Q6. (a) Base diagonal $= \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$ cm. (b) The space diagonal is the hypotenuse of the right-angled triangle formed by the base diagonal and the height: $d = \sqrt{50^2 + 120^2} = \sqrt{2500 + 14400} = \sqrt{16900} = 130$ cm. The longest thin rod that fits in the locker is 130 cm. (c) Yes. The pole is 125 cm long and $125 < 130$, so it fits when placed along the space diagonal — even though it is longer than the locker's 120 cm height.

Q7. The cross-section has area $\dfrac{1}{2} \times 2.4 \times 0.9 = 1.08\,\text{m}^2$, so the volume is $1.08 \times 3 = 3.24\,\text{m}^3$. The canvas consists of the floor ($2.4 \times 3 = 7.2\,\text{m}^2$), the two sloping sides ($2 \times 1.5 \times 3 = 9\,\text{m}^2$) and the two triangular ends ($2 \times 1.08 = 2.16\,\text{m}^2$), a total of $7.2 + 9 + 2.16 = 18.36\,\text{m}^2$.

Q8. $V = \pi r^2 h = \pi \times 1^2 \times 1.4 = 1.4\pi \approx 4.40\,\text{m}^3$. Each cubic metre holds 1000 L, so the capacity is $1.4\pi \times 1000 = 1400\pi \approx 4398$ L.

Q9. $\ell=\sqrt{7^2+24^2}=\sqrt{49+576}=\sqrt{625}=25$ cm. The cardboard is only the curved surface, with area $\pi r \ell=\pi\times 7\times 25=175\pi\approx 549.78\,\text{cm}^2$. The formula $\pi r^2+\pi r \ell$ is for a **closed** cone: its πr^2 term is the area of the base circle, and a party hat has no base, so including it would overstate the cardboard by $49\pi\approx 153.94\,\text{cm}^2$.

Q10. A full sphere of radius 15 cm has volume $\frac{4}{3}\pi\times 15^3=\frac{4}{3}\pi\times 3375=4500\pi\,\text{cm}^3$, so the hemisphere holds $\frac{1}{2}\times 4500\pi=2250\pi\approx 7068.58\,\text{cm}^3$, which is 2250π mL ≈ 7.07 L. The curved surface is half the sphere's surface area: $\frac{1}{2}\times 4\pi\times 15^2=2\pi\times 225=450\pi\approx 1413.72\,\text{cm}^2$.

Q11. $V=\frac{1}{3}\times 18^2\times 12=\frac{1}{3}\times 324\times 12=1296\,\text{cm}^3$. Half the base edge is 9 cm, so each face has slant height $s=\sqrt{12^2+9^2}=\sqrt{144+81}=\sqrt{225}=15$ cm. Surface area $=324+4\times\frac{1}{2}\times 18\times 15=324+540=864\,\text{cm}^2$.

Q12. The cylindrical part has volume $\pi r^2 h=\pi\times 16\times 10=160\pi\,\text{mm}^3$. The two hemispheres together make one whole sphere of radius 4 mm, with volume $\frac{4}{3}\pi\times 64=\frac{256\pi}{3}\,\text{mm}^3$. Adding exactly, $V=160\pi+\frac{256\pi}{3}=\frac{480\pi+256\pi}{3}=\frac{736\pi}{3}\approx 770.74\,\text{mm}^3$.

Q13. The outer radius is 30 cm, the inner radius is 25 cm and the length is 2 m $=200$ cm. The cross-section is a ring of area $\pi\left(30^2-25^2\right)=\pi\left(900-625\right)=275\pi\,\text{cm}^2$, so the volume of concrete is $275\pi\times 200=55000\pi\approx 172787.60\,\text{cm}^3$. Since $1\,\text{m}^3=1000000\,\text{cm}^3$, this is $\approx 0.17\,\text{m}^3$.

Q14. Carton A: $V=10^3=1000\,\text{cm}^3$. Carton B: $V=20\times 10\times 5=1000\,\text{cm}^3$. Both hold $1000\,\text{cm}^3=1$ L, exactly the same amount. Carton A's surface area is $6\times 10^2=600\,\text{cm}^2$. Carton B's faces are $20\times 10=200$, $20\times 5=100$ and $10\times 5=50\,\text{cm}^2$, each occurring twice: $2\left(200+100+50\right)=700\,\text{cm}^2$. The company should choose carton A: it holds the same 1 L but uses $100\,\text{cm}^2$ less cardboard. Of all rectangular prisms with a given volume, the cube has the least surface area.

Q15. The longest straw lies along the space diagonal: $d=\sqrt{4^2+6^2+12^2}=\sqrt{16+36+144}=\sqrt{196}=14$ cm. A 20 cm straw placed along this diagonal therefore has $20-14=6$ cm outside the popper.

Q16. $V=25\times 10\times 1.6=400\,\text{m}^3$. Each cubic metre is 1000 L, so the capacity is $400\times 1000=400000$ L, i.e. 400 000 L. At 250 L per minute the pool takes $400000\div 250=1600$ minutes to fill, and $1600\div 60=26$ remainder 40, so the time is 26 hours and 40 minutes.

Q17. The paver has volume $40\times 40\times 20=32000\,\text{cm}^3$. The hole is a cylinder of radius 10 cm and height 20 cm, with volume $\pi\times 10^2\times 20=2000\pi\approx 6283.19\,\text{cm}^3$. The concrete remaining is $32000-2000\pi\approx 25716.81\,\text{cm}^3$.

Q18. The walls enclose a rectangular prism: $6\times 4\times 2.5=60\,\text{m}^3$. The roof space is a triangular prism whose cross-section has base 4 m and height 1.5 m, so its area is $\frac{1}{2}\times 4\times 1.5=3\,\text{m}^2$ and its volume is $3\times 6=18\,\text{m}^3$. The total volume of air is $60+18=78\,\text{m}^3$. Each roof section slopes from the ridge to the top of a wall: half the width is 2 m and the rise is 1.5 m, so the slant width is $\sqrt{2^2+1.5^2}=\sqrt{4+2.25}=\sqrt{6.25}=2.5$ m. The roofing consists of two rectangles, each 6 m long and 2.5 m wide: $2\times 6\times 2.5=30\,\text{m}^2$.

Theory

A street directory, a house plan, a photo pinched out to zoom on a phone, a model aeroplane — each shows a familiar shape enlarged or reduced without being distorted. Two figures are **similar** if one is an enlargement or a reduction of the other: they have exactly the same shape, but not necessarily the same size. In similar figures, **corresponding angles are equal** and **corresponding sides are in the same ratio**. For triangles we write $ABC \sim PQR$, the order of the letters showing the matching: vertex A corresponds to P, B to Q and C to R, and so side AB corresponds to side PQ, side BC to side QR, and side CA to side RP.

The linear scale factor. The common ratio of the corresponding sides is the **linear scale factor**

$$k=\frac{\text{length in the second figure}}{\text{matching length in the first figure}}.$$

If $k>1$ the second figure is an **enlargement** of the first; if $0<k<1$ it is a **reduction**; if $k=1$ the two figures are congruent — the same size as well as the same shape. Every length is multiplied by the same k: each side, each height, each diagonal, and therefore the perimeter as well. Once k is known, any unknown length is found by multiplying the matching known length by k.

Conditions for similar triangles. Two triangles are similar if any one of the following holds — there is no need to check every side and every angle:

- **AAA** — two pairs of corresponding angles are equal (the third pair must then also agree, because the angles of a triangle add to 180°);
- **SSS ratio** — the three pairs of corresponding sides are in the same ratio;
- **SAS ratio** — two pairs of corresponding sides are in the same ratio, and the angles included between those sides are equal.

Finding an unknown side. First match up the corresponding sides — use the matching letters in the similarity statement, or pair the sides opposite equal angles. Find k from a pair whose lengths are both known, then multiply. For example, if $ABC \sim PQR$ then

$$\frac{PQ}{AB}=\frac{QR}{BC}=\frac{RP}{CA}=k,$$

so $QR=k\times BC$, and so on for each unknown side.

Area and volume scale factors. When every length is multiplied by k, every **area is multiplied by** k^2 and every **volume is multiplied by** k^3. To see why, scale a rectangle with sides l and w: the sides become kl and kw, so the area becomes $kl\times kw=k^2\times lw$ — both dimensions are scaled, so the factor k acts twice. In a solid all three dimensions are scaled, so the factor acts three times. For **similar solids** the surface area is multiplied by k^2 and the volume — and hence the capacity — by k^3. The rule also runs backwards: if the ratio of two corresponding areas is known then k^2 equals that ratio, so k is its square root; if the ratio of the volumes is known, k is the cube root.

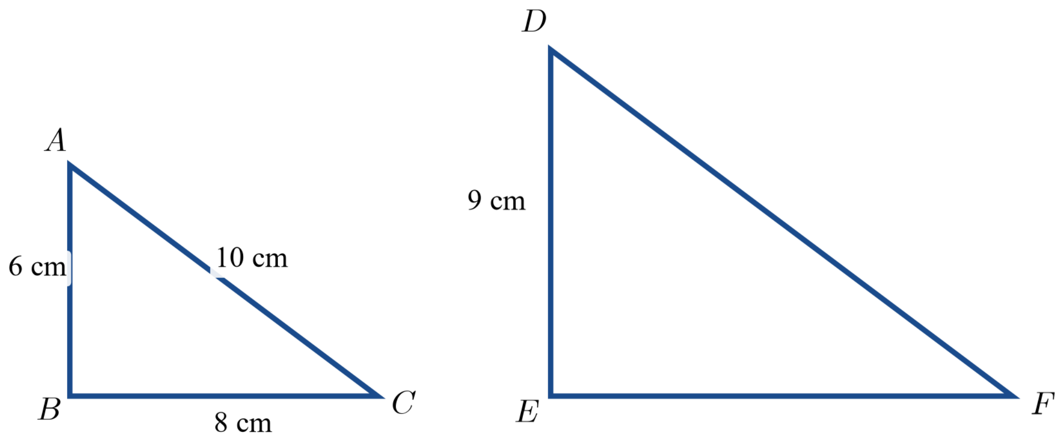
Maps, plans and models. A map scale of 1 : 50 000 means that every 1 cm on the map represents 50 000 cm (that is, 500 m) on the ground: the map and the ground are similar figures, and the scale factor from map to ground is 50 000. Ground distances are map distances multiplied by 50 000, and ground areas are map areas multiplied by $50\,000^2$. A model built to a scale of 1 : 20 works the same way in reverse: real lengths are divided by 20 to give model lengths.

Using CAS. General Mathematics is a technology-active, open-book course. A CAS solves the proportion for an unknown side directly — for example, solving $\frac{x}{8}=\frac{9}{6}$ gives $x=12$ — and evaluates the square and cube roots needed to recover k from an area or a volume ratio. Keep fractional scale factors such as $\frac{4}{3}$ exact on the CAS rather than rounding part-way through a calculation. Always set up the proportion by hand first, so that you know which sides correspond and can tell whether the CAS answer is reasonable.

Worked examples

Example 1 — scaffolded

Triangle ABC has $AB = 6$ cm, $BC = 8$ cm and $CA = 10$ cm. Triangle DEF is similar to triangle ABC, with D corresponding to A, E to B and F to C, and $DE = 9$ cm.



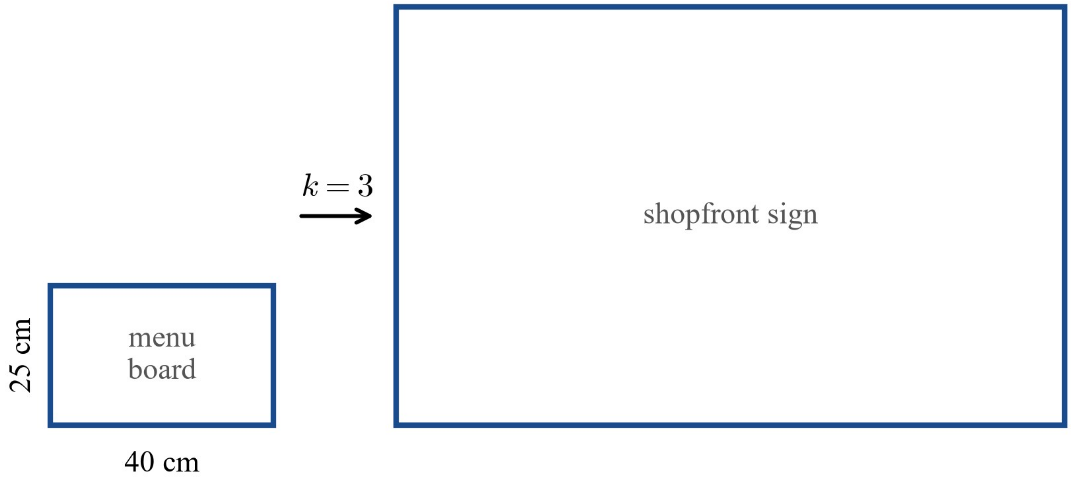
Find the linear scale factor k, and hence the lengths of EF and FD.

Working	Comment
DE corresponds to AB, EF to BC, and FD to CA.	Match the sides using the order of the letters in the correspondence.
$k = \frac{DE}{AB} = \frac{9}{6} = 1.5.$	Use the one pair of corresponding sides whose lengths are both known.
$EF = k \times BC = 1.5 \times 8 = 12$ cm.	Every length in triangle DEF is 1.5 times its match in triangle ABC.
$FD = k \times CA = 1.5 \times 10 = 15$ cm.	The same scale factor applies to every pair of corresponding sides.

So the scale factor is $k = 1.5$, and $EF = 12$ cm and $FD = 15$ cm.

Example 2 — scaffolded

A café’s rectangular menu board is 40 cm wide and 25 cm high. A similar shopfront sign is to be made using a linear scale factor of $k = 3$.



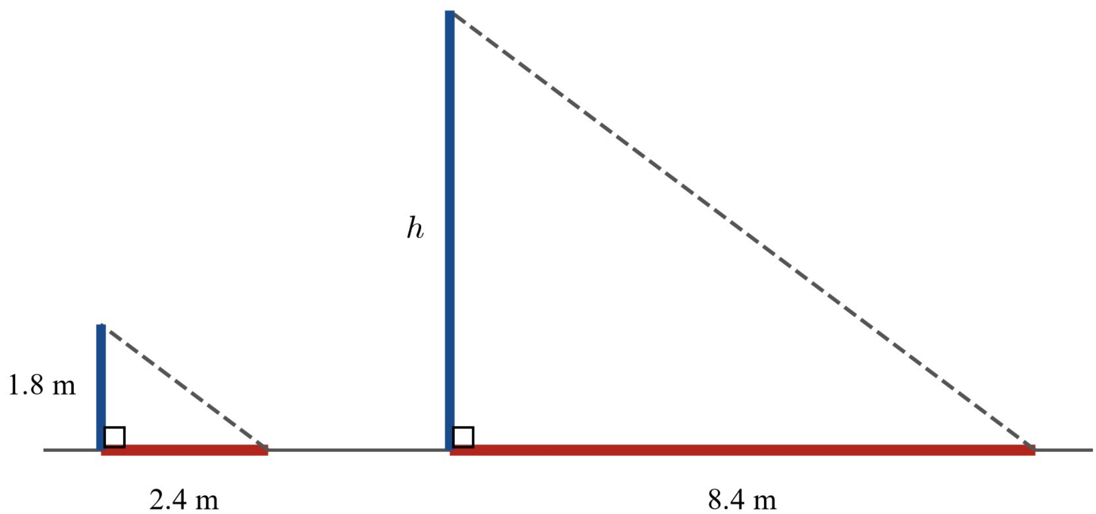
Find the dimensions of the shopfront sign, find the area of each sign, and verify that the ratio of the two areas is k^2.

Working	Comment
Width $= 3 \times 40 = 120$ cm; height $= 3 \times 25 = 75$ cm.	Every length is multiplied by $k = 3$.
Menu board area $= 40 \times 25 = 1000$ cm².	Area of a rectangle = width × height.
Sign area $= 120 \times 75 = 9000$ cm².	Use the enlarged dimensions just found.
$\frac{9000}{1000} = 9 = 3^2 = k^2$.	The areas are in the ratio k^2, not k: the width and the height have each been multiplied by 3.

So the shopfront sign is 120 cm wide and 75 cm high, and its area of 9000 cm² is $k^2 = 9$ times the 1000 cm² area of the menu board.

Example 3 — unscaffolded

A student who is 1.8 m tall casts a shadow 2.4 m long. At the same moment, the school flagpole casts a shadow 8.4 m long. Find the height of the flagpole.



The student and the flagpole both stand at right angles to the level ground, and the sun's rays are parallel, so the rays meet the ground at the same angle. Each triangle therefore has a right angle and an equal angle at the tip of its shadow, so the two triangles are similar (AAA). Matching the two shadows, $k = \frac{8.4}{2.4} = 3.5$, so every length in the flagpole's triangle is 3.5 times the matching length in the student's triangle. The flagpole's height matches the student's height, so its height is $3.5 \times 1.8 = 6.3$ m.

∴ the flagpole is 6.3 m tall.

Example 4 — unscaffolded

Two similar bronze trophies are made by the same foundry. The smaller trophy has surface area 240 cm² and volume 320 cm³; the larger trophy has surface area 540 cm². Find the linear scale factor, and hence the volume of the larger trophy.

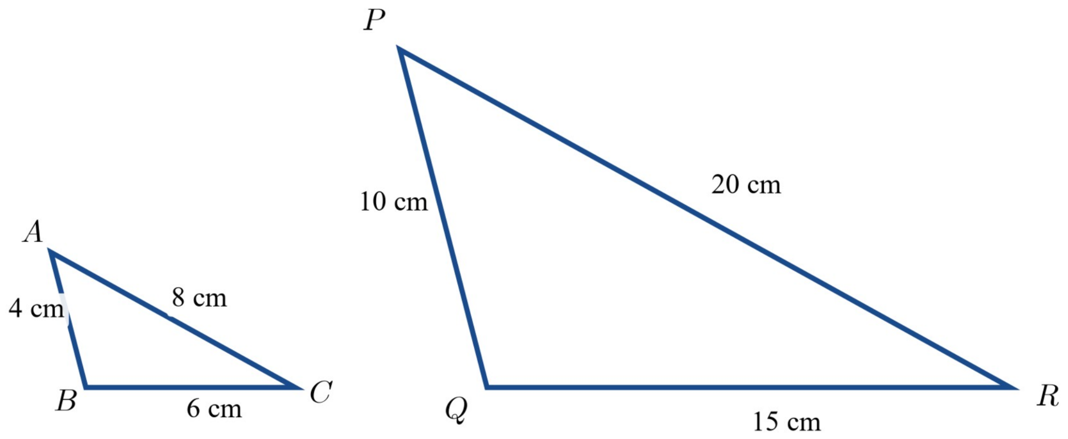
The surface areas of similar solids are in the ratio k^2, so $k^2 = \frac{540}{240} = 2.25$, giving $k = \sqrt{2.25} = 1.5$. The volumes of similar solids are in the ratio k^3, and $k^3 = 1.5^3 = 3.375$, so the larger trophy has volume $320 \times 3.375 = 1080$ cm³.

∴ the linear scale factor is $k = 1.5$ and the larger trophy has volume 1080 cm³.

Practice questions

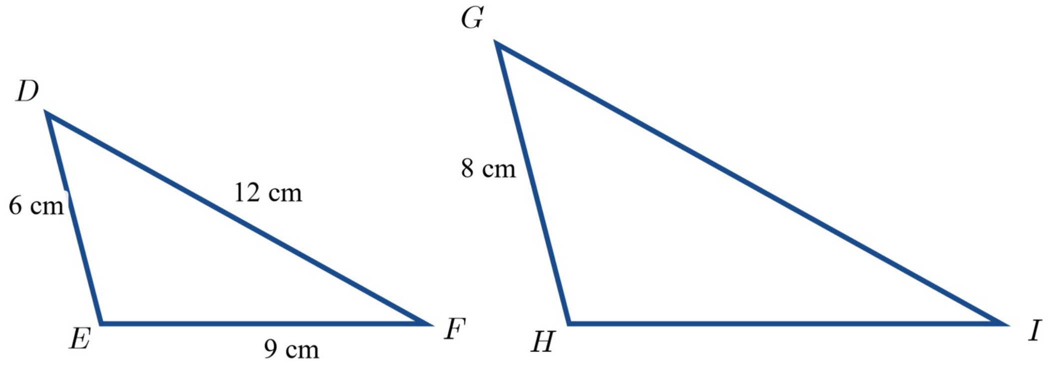
Scaffolded

Q1. Triangle ABC has sides $AB = 4$ cm, $BC = 6$ cm and $CA = 8$ cm; triangle PQR has sides $PQ = 10$ cm, $QR = 15$ cm and $RP = 20$ cm.



(a) Show that the two triangles are similar, naming the condition you used.
(b) State the linear scale factor k that takes triangle ABC to triangle PQR.
(c) Find the perimeter of each triangle, and show that the perimeters are also in the ratio k.

Q2. Triangle DEF is similar to triangle GHI, with G corresponding to D, H to E and I to F. $DE = 6$ cm, $EF = 9$ cm, $DF = 12$ cm and $GH = 8$ cm.



(a) Find the linear scale factor k that takes triangle DEF to triangle GHI.
(b) Find HI.
(c) Find GI.

Q3. A bushwalking map is drawn to a scale of 1 : 50 000. (a) Two campsites are 3 cm apart on the map. Find the real distance between them, in kilometres. (b) A straight firebreak is 4 km long. How long is it on the map, in centimetres? (c) A rectangular nature reserve measures 2 cm by 1.5 cm on the map. Find its real dimensions, in kilometres, and hence its real area, in square kilometres.

Q4. A graphic designer's logo is 6 cm wide and has area 24 cm². An enlarged, similar version of the logo for a banner is 15 cm wide. (a) Find the linear scale factor k. (b) State the factor by which the area is multiplied. (c) Find the area of the enlarged logo.

Q5. Two storage tubs are similar in shape. The small tub is 20 cm long, has surface area 880 cm² and holds 1600 cm³; the large tub is 30 cm long. (a) Find the linear scale factor k. (b) Find the capacity of the large tub. (c) Find the surface area of the large tub.

Q6. For each pair of triangles, decide whether the two triangles must be similar. If they are, name the condition you used (AAA, SSS ratio or SAS ratio); if they are not, explain why not. (a) One triangle has angles of 40° and 60°; another has angles of 60° and 80°. (b) One triangle has sides 4 cm, 6 cm and 8 cm; another has sides 6 cm, 9 cm and 12 cm. (c) One triangle has sides of 5 cm and 7 cm meeting at 50°; another has sides of 10 cm and 15 cm meeting at 50°.

Unscaffolded

Q7. Triangle LMN is similar to triangle XYZ, with X corresponding to L, Y to M and Z to N. Given $LM=8$ cm, $MN=10$ cm, $NL=14$ cm and $XY=12$ cm, find YZ and ZX.

Q8. A surveying pole 1.5 m tall casts a shadow 2 m long. At the same time, a gum tree casts a shadow 18 m long. Explain why the two triangles formed are similar, and find the height of the tree.

Q9. A 10 cm by 15 cm photograph is enlarged so that its longer side becomes 36 cm. Find the length of the shorter side of the enlargement, and show that the area of the enlargement is k^2 times the area of the original.

Q10. A model yacht is built to a scale of 1 : 20. The model's hull is 45 cm long; find the length of the real hull, in metres. The real mast is 5 m tall; find the height of the model's mast, in centimetres. The model's deck has area 600 cm²; find the area of the real deck, in square metres.

Q11. One triangular sail has sides of 3 m and 4.5 m meeting at 35°; a larger sail has sides of 4 m and 6 m meeting at 35°. Show that the two sails are similar, stating the condition you used, and find the linear scale factor. The smaller sail uses 5.4 m² of cloth; how much cloth does the larger sail use?

Q12. A road map of country Victoria is drawn to a scale of 1 : 250 000. Two towns are 6.4 cm apart on the map; find the real distance between them, in kilometres. A straight stretch of highway is 20 km long; how long is it on the map, in centimetres?

Q13. Two similar triangles have areas of 45 cm² and 125 cm². The base of the smaller triangle is 9 cm. Find the linear scale factor, and hence the base of the larger triangle.

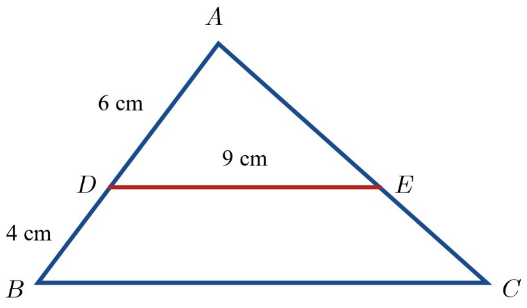
Q14. Two soft-drink bottles are similar in shape. The small bottle holds 250 mL and the large bottle holds 2 L. Show that the linear scale factor is $k=2$. The small bottle is 12 cm tall and its label has area 30 cm²; find the height of the large bottle and the area of its label.

Q15. A garden centre sells two similar statues of a kangaroo, one 0.8 m tall and one 2 m tall. Painting the small statue uses 250 mL of paint. Assuming the amount of paint needed is proportional to the surface area, how much paint does the large statue need? Give your answer in millilitres, and also in litres correct to two decimal places.

Q16. A photocopier is set to $150\,\%$, so every length on the copy is 1.5 times the matching length on the original. A student copies an 8 cm by 6 cm diagram and claims that the copy's area will be $50\,\%$ larger than the original's. Find the dimensions and the area of the copy, and explain why the student's claim is wrong. What is the actual percentage increase in area?

Q17. Two cylindrical rainwater tanks are similar in shape. The small tank is 1.2 m tall, has base area 1.6 m² and holds 1920 L. The large tank is 1.8 m tall. Find the linear scale factor, the base area of the large tank, and the capacity of the large tank in litres.

Q18. In triangle ABC, the point D lies on AB with $AD=6$ cm and $DB=4$ cm, and the point E lies on AC so that DE is parallel to BC, with $DE=9$ cm.



Explain why triangle ADE is similar to triangle ABC, and find the length of BC. Given that the area of triangle ADE is 27 cm², find the area of triangle ABC.

Answers

Q1. (a) $\frac{10}{4}=\frac{15}{6}=\frac{20}{8}=2.5$; similar by SSS ratio. (b) $k=2.5$. (c) Perimeters 18 cm and 45 cm; $45\div18=2.5=k$. **Q2.** (a) $k=\frac{8}{6}=\frac{4}{3}$. (b) $HI=12$ cm. (c) $GI=16$ cm. **Q3.** (a) 1.5 km. (b) 8 cm. (c) 1 km by 0.75 km; area $=0.75$ km². **Q4.** (a) $k=2.5$. (b) $k^2=6.25$. (c) 150 cm². **Q5.** (a) $k=1.5$. (b) 5400 cm³. (c) 1980 cm². **Q6.** (a) Similar (AAA); both have angles 40°, 60° and 80°. (b) Similar (SSS ratio); common ratio 1.5. (c) Not similar: $\frac{10}{5}=2$ but $\frac{15}{7}\neq2$, so the SAS ratio condition fails. **Q7.** $k=1.5$; $YZ=15$ cm, $ZX=21$ cm. **Q8.** Similar by AAA (right angles and a common sun angle); height $=13.5$ m. **Q9.** Shorter side $=24$ cm; areas 150 cm² and 864 cm², and $\frac{864}{150}=5.76=2.4^2=k^2$. **Q10.** Hull 9 m; mast 25 cm; deck 24 m². **Q11.** Similar by SAS ratio ($\frac{4}{3}=\frac{6}{4.5}$, included 35° angles equal); $k=\frac{4}{3}$; cloth $=9.6$ m². **Q12.** 16 km; 8 cm. **Q13.** $k=\frac{5}{3}$; base $=15$ cm. **Q14.** $k^3=\frac{2000}{250}=8$, so $k=2$; height $=24$ cm; label area $=120$ cm². **Q15.** $250\times6.25=1562.5$ mL ≈1.56 L. **Q16.** Copy is 12 cm by 9 cm; areas 48 cm² and 108 cm²; area factor $=1.5^2=2.25$, an increase of 125%, not 50%. **Q17.** $k=1.5$; base area $=3.6$ m²; capacity $=6480$ L. **Q18.** AAA (common angle at A; equal corresponding angles from the parallel lines); $AB=10$ cm, $k=\frac{5}{3}$, $BC=15$ cm; area $=75$ cm².

Fully-worked solutions

Q1. (a) The ratios of corresponding sides are $\frac{PQ}{AB}=\frac{10}{4}=2.5$, $\frac{QR}{BC}=\frac{15}{6}=2.5$ and $\frac{RP}{CA}=\frac{20}{8}=2.5$. All three pairs of corresponding sides are in the same ratio, so the triangles are similar by the SSS ratio condition. (b) The common ratio is the scale factor: $k=2.5$. (c) Perimeter of triangle $ABC=4+6+8=18$ cm; perimeter of triangle $PQR=10+15+20=45$ cm; and $\frac{45}{18}=2.5=k$. A perimeter is a length, so it scales by k like every other length.

Q2. (a) GH corresponds to DE, so $k=\frac{GH}{DE}=\frac{8}{6}=\frac{4}{3}$. (b) HI corresponds to EF: $HI=\frac{4}{3}\times9=12$ cm. (c) GI corresponds to DF: $GI=\frac{4}{3}\times12=16$ cm.

Q3. (a) The real distance is $3\times50\,000=150\,000$ cm $=1500$ m $=1.5$ km. (b) 4 km $=400\,000$ cm on the ground, and $400\,000\div50\,000=8$ cm on the map. (c) The real sides are $2\times50\,000=100\,000$ cm $=1$ km and $1.5\times50\,000=75\,000$ cm $=0.75$ km, so the real area is $1\times0.75=0.75$ km².

Q4. (a) The widths correspond, so $k=\frac{15}{6}=2.5$. (b) Areas are multiplied by $k^2=2.5^2=6.25$. (c) Area of the enlarged logo $=24\times6.25=150$ cm².

Q5. (a) $k=\frac{30}{20}=1.5$. (b) Volumes scale by $k^3=1.5^3=3.375$, so the large tub holds $1600\times3.375=5400$ cm³. (c) Surface areas scale by $k^2=1.5^2=2.25$, so the large tub's surface area is $880\times2.25=1980$ cm². (As a check: a 20 cm by 10 cm by 8 cm tub has volume 1600 cm³ and surface area $2(200+160+80)=880$ cm²; multiplying each side by 1.5 gives a 30 cm by 15 cm by 12 cm tub, with volume $30\times15\times12=5400$ cm³ and surface area $2(450+360+180)=1980$ cm².)

Q6. (a) The third angle of the first triangle is $180-40-60=80°$, so its angles are 40°, 60° and 80°. The third angle of the second triangle is $180-60-80=40°$, so it has the same three angles. The corresponding angles are equal, so the triangles are similar (AAA). (b) $\frac{6}{4}=1.5$, $\frac{9}{6}=1.5$ and $\frac{12}{8}=1.5$: all three pairs of corresponding sides are in the same ratio, so the triangles are similar (SSS ratio). (c) The pairs of sides about the equal 50° angles have ratios $\frac{10}{5}=2$ and $\frac{15}{7}\neq 2$. The two ratios are different, so the SAS ratio condition fails and the triangles are not similar.

Q7. XY corresponds to LM, so $k=\frac{XY}{LM}=\frac{12}{8}=1.5$. Then $YZ=1.5\times MN=1.5\times 10=15$ cm and $ZX=1.5\times NL=1.5\times 14=21$ cm.

Q8. The pole and the tree both stand at right angles to the ground, and the sun's rays are parallel, so each triangle has a right angle and an equal angle at the tip of its shadow: the triangles are similar (AAA). Matching the shadows, $k=\frac{18}{2}=9$, so the tree's height is $9\times 1.5=13.5$ m.

Q9. The longer sides correspond, so $k=\frac{36}{15}=2.4$, and the shorter side of the enlargement is $2.4\times 10=24$ cm. The original's area is $10\times 15=150$ cm² and the enlargement's area is $24\times 36=864$ cm²; and $\frac{864}{150}=5.76=2.4^2=k^2$, as the area rule predicts.

Q10. From model to yacht every length is multiplied by 20. Hull: $45\times 20=900$ cm $=9$ m. Mast: 5 m $=500$ cm, and $500\div 20=25$ cm on the model. Deck: areas are multiplied by $20^2=400$, so the real deck has area $600\times 400=240\,000$ cm²; since 1 m² $=10\,000$ cm², this is $240\,000\div 10\,000=24$ m².

Q11. The pairs of sides about the 35° angles have ratios $\frac{4}{3}$ and $\frac{6}{4.5}=\frac{4}{3}$: two pairs of corresponding sides are in the same ratio and the included angles are equal, so the sails are similar (SAS ratio) with $k=\frac{4}{3}$. Cloth is an area, which scales by $k^2=\frac{16}{9}$, so the larger sail uses $5.4\times\frac{16}{9}=9.6$ m² of cloth.

Q12. The real distance is $6.4\times 250\,000=1\,600\,000$ cm $=16$ km. In reverse, 20 km $=2\,000\,000$ cm on the ground, and $2\,000\,000\div 250\,000=8$ cm on the map.

Q13. Areas of similar figures are in the ratio k^2, so $k^2=\frac{125}{45}=\frac{25}{9}$ and $k=\sqrt{\frac{25}{9}}=\frac{5}{3}$. The base of the larger triangle is $\frac{5}{3}\times 9=15$ cm.

Q14. 2 L $=2000$ mL, so the volumes are in the ratio $\frac{2000}{250}=8$. Volumes of similar solids are in the ratio k^3, so $k^3=8$ and $k=\sqrt[3]{8}=2$. The height is a length: $12\times 2=24$ cm. The label is an area, which scales by $k^2=4$: $30\times 4=120$ cm².

Q15. $k=\frac{2}{0.8}=2.5$. Paint covers surface area, which scales by $k^2=2.5^2=6.25$, so the large statue needs $250\times 6.25=1562.5$ mL of paint, which is $1.5625\approx 1.56$ L.

Q16. Every length is multiplied by $k=1.5$, so the copy is $1.5\times 8=12$ cm by $1.5\times 6=9$ cm. The original's area is $8\times 6=48$ cm² and the copy's area is $12\times 9=108$ cm². The area factor is $\frac{108}{48}=2.25=1.5^2$, not 1.5: the student multiplied the area by k instead of k^2. Both the length and the width grow by $50\,\%$, so the area grows by the factor 1.5 twice over. The actual increase is $\frac{108-48}{48}=1.25$, that is $125\,\%$.

Q17. The heights correspond, so $k = \frac{1.8}{1.2} = 1.5$. The base is an area, which scales by $k^2 = 2.25$: base area $= 1.6 \times 2.25 = 3.6$ m². Capacity is a volume, which scales by $k^3 = 3.375$: capacity $= 1920 \times 3.375 = 6480$ L. (Check: base area × height gives $3.6 \times 1.8 = 6.48$ m³ $= 6480$ L, which agrees.)

Q18. Triangles ADE and ABC share the angle at A, and because DE is parallel to BC the angles ADE and ABC are equal corresponding angles. Two pairs of corresponding angles are equal, so triangle ADE is similar to triangle ABC (AAA). Now $AB = AD + DB = 6 + 4 = 10$ cm, so the scale factor from triangle ADE to triangle ABC is $k = \frac{AB}{AD} = \frac{10}{6} = \frac{5}{3}$. Since BC corresponds to DE, $BC = \frac{5}{3} \times 9 = 15$ cm. Areas scale by $k^2 = \frac{25}{9}$, so the area of triangle ABC is $27 \times \frac{25}{9} = 75$ cm².