General Mathematics Units 1 & 2

Chapter 5 — Matrices

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Theory

Large amounts of information are often easiest to handle when they are set out in a rectangular table of rows and columns. In mathematics such an array is called a **matrix** (plural **matrices**): a rectangular arrangement of numbers, enclosed in brackets, in which each row and each column carries a meaning. Matrices are compact **information stores** — a database of stock levels across a chain of shops, a price list across several stores, or the number of direct links between the towns of a transport network can each be recorded as a matrix. A matrix is named with a capital letter.

For example, a greengrocer sells apples, bananas and grapes at market stalls in Preston and Reservoir. The prices, in dollars per kilogram, are stored in the matrix

$$P = \begin{bmatrix} 4.20 & 4.45 \\ 3.90 & 3.60 \\ 6.80 & 7.10 \end{bmatrix},$$

where the rows are the fruits (apples, bananas, then grapes) and the columns are the stalls (Preston, then Reservoir). Reading down a column gives the full price list at one stall; reading across a row compares the two stalls for one fruit.

Order. The **order** (or size) of a matrix records its number of **rows** and its number of **columns**, always in that sequence: a matrix with m rows and n columns has order $m \times n$, read "m by n". The matrix P above has 3 rows and 2 columns, so its order is 3×2 — not 2×3: the row count always comes first. A matrix of order $m \times n$ contains $m \times n$ elements altogether, so P stores $3 \times 2 = 6$ pieces of information.

Elements. Each individual entry of a matrix is called an **element**. The element in row i and column j of a matrix A is written a_{ij} — a lower-case letter matching the matrix name, with the **row number first** and the **column number second**. In the price matrix P, the element p_{32} sits in row 3, column 2, so $p_{32} = 7.10$: grapes cost $7.10 per kilogram at the Reservoir stall. To locate any element, count down to its row and then across to its column.

Types of matrices. Several shapes of matrix are important enough to have their own names.

- A **row matrix** has a single row, such as $\begin{bmatrix} 5 & -2 & 7 \end{bmatrix}$, of order 1×3.
- A **column matrix** has a single column, such as $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$, of order 2×1.
- A **square matrix** has the same number of rows as columns, such as $\begin{bmatrix} 3 & 1 \\ 2 & 8 \end{bmatrix}$, of order 2×2. The elements running from the top left to the bottom right of a square matrix (here 3 and 8) form its **main diagonal**.
- A **zero matrix**, written O, has every element equal to 0. It may be of any order: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the 2×2 zero matrix.
- An **identity matrix**, written I, is a **square** matrix with 1s on the main diagonal and 0s everywhere else: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the subscript gives the size. The zero and identity matrices play special roles in the matrix arithmetic of the sections that follow.

Equality of matrices. Two matrices are **equal** when two conditions both hold: they have the **same order**, and **every pair of corresponding elements is equal**. Both conditions matter. A 2×2 matrix can never equal a 2×3 matrix, whatever numbers they contain; and two 2×2 matrices are equal only if all four matching entries agree. Equality is often used to find unknown values: when two equal matrices contain pronumerals, equating corresponding elements produces simple equations to solve.

Using CAS. General Mathematics is a technology-active, open-book course, and your CAS calculator has a matrix editor. When you define a matrix on a CAS you must first enter its dimensions as rows × columns — exactly the order convention $m \times n$ of this section — and the calculator then displays an empty array for you to fill in. A stored matrix can be recalled by name and any element read directly from the screen. Entering the dimensions the wrong way around (columns before rows) is a common error, so learn the by-hand conventions here: they are precisely the conventions the calculator uses.

Worked examples

Example 1 — scaffolded

A hardware chain records the numbers of cordless drills, sanders and circular saws in stock at its Ballarat and Bendigo stores in the matrix

$$S = \begin{bmatrix} 12 & 8 & 5 \\ 9 & 14 & 6 \end{bmatrix},$$

where the rows are the stores (Ballarat, then Bendigo) and the columns are the tools (drills, sanders, then saws). State the order of S and its number of elements, write down the values of s_{13} and s_{21}, and explain what s_{22} represents.

Working	Comment
S has 2 rows and 3 columns, so its order is 2×3.	Order is rows × columns, read "2 by 3" — the row count comes first.
Number of elements $= 2 \times 3 = 6$.	An $m \times n$ matrix stores $m \times n$ values.
$s_{13} = 5$.	Row 1, column 3: Ballarat's stock of saws.
$s_{21} = 9$.	Row 2, column 1: Bendigo's stock of drills.
$s_{22} = 14$, the number of sanders in stock at Bendigo.	In s_{ij} the subscripts name the row, then the column; row 2 is Bendigo and column 2 is sanders.

So S is a 2×3 matrix with six elements, $s_{13} = 5$, $s_{21} = 9$, and $s_{22} = 14$ means the Bendigo store holds 14 sanders.

Example 2 — scaffolded

Consider the matrices

$$A = \begin{bmatrix} 7 & -3 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 0 \\ 11 \end{bmatrix}, C = \begin{bmatrix} 2 & 9 \\ 4 & 5 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

State the order of each matrix and classify each as a row, column, square, zero or identity matrix.

Working	Comment
A has order 1×2: a row matrix.	A single row, two columns.
B has order 3×1: a column matrix.	A single column, three rows.
C has order 2×2: a square matrix.	Rows = columns; its main diagonal is 2 and 5.
D has order 2×3: a zero matrix.	Every element is 0. A zero matrix need not be square.
E has order 3×3: the identity matrix I_3.	Square, with 1s on the main diagonal and 0s elsewhere.

So A is a 1×2 row matrix, B a 3×1 column matrix, C a 2×2 square matrix, D the 2×3 zero matrix and E the identity matrix I_3.

Example 3 — unscaffolded

The towns of Anglesea (town 1), Birregurra (town 2) and Colac (town 3) are served by a regional bus company. The matrix

$$M=\begin{bmatrix}0&2&0\\2&0&3\\0&3&0\end{bmatrix}$$

records the morning timetable, where the element m_{ij} is the number of direct bus routes between town i and town j. State the order of M, explain what m_{23} tells you, identify the pair of towns with no direct route, and find the total number of direct routes serving Birregurra.

The matrix M has 3 rows and 3 columns, so its order is 3×3 and M is a square matrix. The element $m_{23}=3$ sits in row 2, column 3: there are three direct bus routes between Birregurra (town 2) and Colac (town 3). Since $m_{13}=0$ (and likewise $m_{31}=0$), there is no direct route between Anglesea and Colac — a passenger must travel via Birregurra. Every element on the main diagonal is 0 because no route runs from a town to itself, and the matrix reads the same across rows as down columns ($m_{ij}=m_{ji}$) because each route can be travelled in either direction. Reading across row 2, Birregurra is served by $2+0+3=5$ direct routes.

∴ M is a 3×3 square matrix; $m_{23}=3$ means three direct routes link Birregurra and Colac; Anglesea and Colac have no direct route; and Birregurra is served by 5 direct routes in total.

Example 4 — unscaffolded

The matrices

$$A=\begin{bmatrix}x+2&7\\5&3y\end{bmatrix}\text{and } B=\begin{bmatrix}9&7\\5&12\end{bmatrix}$$

are equal. Find the values of x and y.

Both matrices have order 2×2, so the first condition for equality is satisfied and it remains to equate the corresponding elements. From the elements in row 1, column 1: $x+2=9$, so $x=7$. The elements 7 and 5 already agree with their partners in B. From the elements in row 2, column 2: $3y=12$, so $y=4$. Substituting back, $A=\begin{bmatrix}9&7\\5&12\end{bmatrix}=B$, as required. (Had A been of order 2×3, no values of x and y could have made the matrices equal, because their orders would differ.)

∴ $x=7$ and $y=4$.

Practice questions

Scaffolded

Q1. A stationery chain records the numbers of pens, notebooks and markers in stock at its Sunshine and Footscray stores in the matrix

$$N=\begin{bmatrix}45&30&24\\38&42&19\end{bmatrix},$$

where the rows are the stores (Sunshine, then Footscray) and the columns are the items (pens, notebooks, then markers). (a) State the order of N. (b) How many elements does N have? (c) Write down the values of n_{12} and n_{23}. (d) What information does n_{21} give the chain?

Q2. On Saturday a juice bar sold 52 orange, 38 apple and 45 mango juices; on Sunday it sold 61 orange, 29 apple and 50 mango juices. (a) Display this information in a matrix J whose rows are the days (Saturday, then Sunday) and whose columns are the flavours (orange, apple, then mango). (b) State the order of J. (c) Write down the value of j_{23}. (d) What does row 2 of J represent?

Q3. Consider the matrices

$$A=\begin{bmatrix}8\\3\end{bmatrix}, B=\begin{bmatrix}6 & 1 & -4 & 9\end{bmatrix}, C=\begin{bmatrix}5 & 2\\7 & 0\end{bmatrix}, D=\begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}, E=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

(a) State the order of each matrix. (b) Classify each matrix as a row, column, square, zero or identity matrix. (c) Which matrix is an identity matrix? Describe the arrangement of its elements.

Q4. Let

$$A=\begin{bmatrix}3 & -1 & 6\\0 & 8 & 2\\5 & 4 & 9\end{bmatrix}.$$

(a) State the order of A and explain why A is a square matrix. (b) Write down the values of a_{12}, a_{31} and a_{23}. (c) Find the values of i and j for which $a_{ij}=8$. (d) Write down the elements on the main diagonal of A.

Q5. Four towns in a farming district — Yandra (town 1), Kellwood (town 2), Mirrup (town 3) and Torrin (town 4) — are linked by a road network recorded in the matrix

$$R=\begin{bmatrix}0 & 1 & 0 & 1\\1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & 1 & 0\end{bmatrix},$$

where $r_{ij}=1$ if a direct road joins town i and town j, and $r_{ij}=0$ otherwise. (a) State the order of R. (b) What does the element r_{23} tell you? (c) Which towns are joined to Yandra by a direct road? (d) Explain why every element on the main diagonal of R is 0.

Q6. (a) The matrices $\begin{bmatrix}a & 5\\b & c\end{bmatrix}$ and $\begin{bmatrix}2 & 5\\-3 & 7\end{bmatrix}$ are equal. Write down the values of a, b and c. (b) The matrices $\begin{bmatrix}x+4 & 9\\6 & 2y\end{bmatrix}$ and $\begin{bmatrix}11 & 9\\6 & 10\end{bmatrix}$ are equal. Find the values of x and y. (c) Explain why the matrices $\begin{bmatrix}3 & 1\\2 & 4\end{bmatrix}$ and $\begin{bmatrix}3 & 1 & 0\\2 & 4 & 0\end{bmatrix}$ are not equal.

Unscaffolded

Q7. State the order of each of the matrices

$$P=\begin{bmatrix}2 & 7 & -1\end{bmatrix}, Q=\begin{bmatrix}4\\0\\8\\3\end{bmatrix}, R=\begin{bmatrix}1 & 5\\9 & 2\\6 & 3\end{bmatrix}, S=\begin{bmatrix}8\end{bmatrix},$$

and state the number of elements in R.

Q8. At a regional swimming carnival the medal tally of three schools is recorded in the matrix

$$M=\begin{bmatrix}5 & 3 & 6\\4 & 7 & 2\\3 & 3 & 5\end{bmatrix},$$

where the rows are the schools (Mildura SC, Swan Hill SC, then Kerang SC) and the columns are the medals (gold, silver, then bronze). State the order of M; write down m_{22} and explain what it represents; state which school won the most gold medals; and find the total number of medals won by Kerang SC.

Q9. Write down the identity matrix I_3, the 2×3 zero matrix, and the column matrix whose elements, from top to bottom, are 2, -1 and 6. State the order of each matrix.

Q10. A matrix B of order 2×3 is defined by the rule $b_{ij}=2i+j$, where i is the row number and j is the column number. Write B in full, and state whether B is a square matrix.

Q11. A small airline operates direct flights between Melbourne (city 1), Devonport (city 2) and King Island (city 3): there are direct flights between Melbourne and Devonport and between Melbourne and King Island, but none between Devonport and King Island. Write down the 3×3 matrix F in which $f_{ij}=1$ if there is a direct flight between city i and city j and $f_{ij}=0$ otherwise, and state which pair of cities has no direct flight.

Q12. The matrices $\begin{bmatrix} 3x & -2 \\ 8 & y+5 \end{bmatrix}$ and $\begin{bmatrix} 15 & -2 \\ 8 & 9 \end{bmatrix}$ are equal. Find the values of x and y.

Q13. Consider the matrices

$$K=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{and } L=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

State the two conditions a matrix must satisfy to be an identity matrix, and explain why neither K nor L is an identity matrix.

Q14. The prices, in dollars, of three grocery items at two supermarkets are recorded in the matrix

$$C=\begin{bmatrix} 2.60 & 2.45 \\ 3.80 & 4.10 \\ 6.20 & 5.90 \end{bmatrix},$$

where the rows are the items (2 L milk, wholemeal loaf, then a dozen eggs) and the columns are the supermarkets (FreshMart, then ValueBuy). State the order of C; explain what c_{31} tells a shopper; state which element gives the price of milk at ValueBuy and write down that price; and decide at which supermarket a wholemeal loaf is cheaper.

Q15. A matrix has 12 elements. (a) List all the possible orders the matrix could have. (b) Explain why a matrix with 12 elements cannot be a square matrix.

Q16. Let

$$D=\begin{bmatrix} 4 & 7 & 1 & 0 \\ 2 & 5 & 8 & 3 \\ 6 & 9 & 2 & 7 \end{bmatrix}.$$

(a) State the order of D. (b) Write down the values of d_{24} and d_{32}. (c) Find all pairs (i,j) for which $d_{ij}=7$. (d) How many elements does D have?

Q17. The matrices $\begin{bmatrix} x+y & 4 \\ 7 & x-y \end{bmatrix}$ and $\begin{bmatrix} 10 & 4 \\ 7 & 2 \end{bmatrix}$ are equal. Find the values of x and y.

Q18. A veterinary clinic in Wodonga records the numbers of dogs and cats treated on each of four days in the matrix

$$V=\begin{bmatrix} 9 & 4 \\ 7 & 6 \\ 10 & 3 \\ 8 & 5 \end{bmatrix},$$

where the rows are the days (Monday to Thursday, in order) and the columns are the animals (dogs, then cats). (a) State the order of V. (b) Write down v_{32} and explain what it represents. (c) On which day were the most dogs treated? (d) How many animals were treated on Tuesday in total?

Answers

Q1. (a) 2×3. (b) 6 elements. (c) $n_{12}=30$, $n_{23}=19$. (d) $n_{21}=38$: the Footscray store has 38 pens in stock. **Q2.** (a) $J=\begin{bmatrix}52 & 38 & 45\\61 & 29 & 50\end{bmatrix}$. (b) 2×3. (c) $j_{23}=50$. (d) Sunday's sales of each flavour. **Q3.** (a) A: 2×1; B: 1×4; C: 2×2; D: 3×2; E: 2×2. (b) A column, B row, C square, D zero, E identity (a special square matrix). (c) $E=I_2$: 1s on the main diagonal, 0s elsewhere. **Q4.** (a) 3×3; rows = columns. (b) $a_{12}=-1$, $a_{31}=5$, $a_{23}=2$. (c) $i=2$, $j=2$. (d) 3, 8, 9. **Q5.** (a) 4×4. (b) $r_{23}=1$: a direct road joins Kellwood and Mirrup. (c) Kellwood and Torrin. (d) No road runs from a town to itself. **Q6.** (a) $a=2$, $b=-3$, $c=7$. (b) $x=7$, $y=5$. (c) Their orders differ (2×2 and 2×3), so they cannot be equal. **Q7.** P: 1×3; Q: 4×1; R: 3×2; S: 1×1; R has 6 elements. **Q8.** Order 3×3; $m_{22}=7$: Swan Hill SC won 7 silver medals; Mildura SC won the most gold (5); Kerang SC won $3+3+5=11$ medals. **Q9.** $I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ (3×3); $O=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ (2×3); $\begin{bmatrix}2\\-1\\6\end{bmatrix}$ (3×1). **Q10.** $B=\begin{bmatrix}3 & 4 & 5\\5 & 6 & 7\end{bmatrix}$; not square ($2\neq3$). **Q11.** $F=\begin{bmatrix}0 & 1 & 1\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix}$; Devonport and King Island have no direct flight. **Q12.** $x=5$, $y=4$. **Q13.** An identity matrix must be square, with 1s on the main diagonal and 0s elsewhere; K is 2×3 (not square), and L has $l_{12}=1\neq0$. **Q14.** Order 3×2; $c_{31}=6.20$: a dozen eggs costs \$6.20 at FreshMart; $c_{12}=2.45$, so milk costs \$2.45 at ValueBuy; the loaf is cheaper at FreshMart (\$3.80 vs \$4.10). **Q15.** (a) 1×12, 2×6, 3×4, 4×3, 6×2, 12×1. (b) A square matrix of order $m\times m$ has m^2 elements, and no whole number squared equals 12. **Q16.** (a) 3×4. (b) $d_{24}=3$, $d_{32}=9$. (c) $(1,2)$ and $(3,4)$. (d) 12 elements. **Q17.** $x=6$, $y=4$. **Q18.** (a) 4×2. (b) $v_{32}=3$: three cats were treated on Wednesday. (c) Wednesday (10 dogs). (d) $7+6=13$ animals.

Fully-worked solutions

Q1. (a) N has 2 rows and 3 columns, so its order is 2×3. (b) An $m\times n$ matrix has $m\times n$ elements, so N has $2\times3=6$ elements. (c) n_{12} is the element in row 1, column 2: $n_{12}=30$. n_{23} is the element in row 2, column 3: $n_{23}=19$. (d) $n_{21}=38$ is in row 2 (Footscray) and column 1 (pens): the Footscray store has 38 pens in stock.

Q2. (a) Row 1 holds Saturday's sales and row 2 Sunday's, with columns for orange, apple and mango in order:

$$J=\begin{bmatrix}52 & 38 & 45\\61 & 29 & 50\end{bmatrix}.$$

(b) J has 2 rows and 3 columns, so its order is 2×3. (c) j_{23} is in row 2, column 3: $j_{23}=50$, the number of mango juices sold on Sunday. (d) Row 2 lists Sunday's sales of each flavour: 61 orange, 29 apple and 50 mango.

Q3. (a) A has 2 rows and 1 column: order 2×1. B has 1 row and 4 columns: order 1×4. C has order 2×2, D has order 3×2 and E has order 2×2. (b) A is a column matrix (a single column); B is a row matrix (a single row); C is a square matrix (rows = columns); D is a zero matrix (every element 0); E is an identity matrix — it is also square, since every identity matrix is square. (c) E is the identity matrix I_2: it is square with 1s on the main diagonal and 0s everywhere else.

Q4. (a) A has 3 rows and 3 columns, so its order is 3×3; because the number of rows equals the number of columns, A is a square matrix. (b) a_{12} (row 1, column 2) $=-1$; a_{31} (row 3, column 1) $=5$; a_{23} (row 2, column 3) $=2$. (c) The element 8 sits in row 2, column 2, so $a_{ij}=8$ when $i=2$ and $j=2$. (d) The main diagonal runs from the top left to the bottom right: $a_{11}=3$, $a_{22}=8$, $a_{33}=9$.

Q5. (a) R has 4 rows and 4 columns, so its order is 4×4. (b) r_{23} is in row 2 (Kellwood) and column 3 (Mirrup), and $r_{23}=1$: a direct road joins Kellwood and Mirrup. (c) Reading across row 1 (Yandra): $r_{12}=1$ and $r_{14}=1$, while $r_{13}=0$. So Yandra is joined by direct road to Kellwood (town 2) and Torrin (town 4), but not to Mirrup. (d) The diagonal elements $r_{11}, r_{22}, r_{33}, r_{44}$ would record a road from a town to itself; no such road exists, so every main-diagonal element is 0.

Q6. (a) The matrices already have the same order (2 × 2), so equate corresponding elements: $a = 2$, $b = -3$ and $c = 7$. (b) Equating the row 1, column 1 elements: $x + 4 = 11$, so $x = 7$. Equating the row 2, column 2 elements: $2y = 10$, so $y = 5$. The remaining elements (9 and 6) already agree. (c) Equal matrices must have the same order. The first matrix has order 2 × 2 and the second has order 2 × 3, so they cannot be equal — even though the four elements they share agree.

Q7. P has 1 row and 3 columns: order 1×3. Q has 4 rows and 1 column: order 4×1. R has 3 rows and 2 columns: order 3×2. S has 1 row and 1 column: order 1×1. The matrix R has $3 \times 2 = 6$ elements.

Q8. M has 3 rows and 3 columns, so its order is 3×3. The element m_{22} is in row 2 (Swan Hill SC) and column 2 (silver): $m_{22} = 7$, so Swan Hill SC won 7 silver medals. The gold medals are column 1, with entries 5 (Mildura), 4 (Swan Hill) and 3 (Kerang), so Mildura SC won the most gold medals. Kerang SC's medals are row 3: $3 + 3 + 5 = 11$ medals in total.

Q9. The identity matrix I_3 is the 3×3 square matrix with 1s on the main diagonal and 0s elsewhere; the 2×3 zero matrix has six 0 elements; and the column matrix is of order 3×1:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

Q10. Apply the rule $b_{ij} = 2i + j$ to each position. Row 1 ($i = 1$): $b_{11} = 2 + 1 = 3$, $b_{12} = 2 + 2 = 4$, $b_{13} = 2 + 3 = 5$. Row 2 ($i = 2$): $b_{21} = 4 + 1 = 5$, $b_{22} = 4 + 2 = 6$, $b_{23} = 4 + 3 = 7$. Hence

$$B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}.$$

B has 2 rows and 3 columns; since $2 \neq 3$, B is not a square matrix.

Q11. Direct flights join Melbourne (1) with Devonport (2) and Melbourne (1) with King Island (3), so $f_{12} = f_{21} = 1$ and $f_{13} = f_{31} = 1$; there is no direct flight between Devonport and King Island, so $f_{23} = f_{32} = 0$; and no city flies to itself, so every main-diagonal element is 0:

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The pair with no direct flight is Devonport and King Island.

Q12. The orders agree (2 × 2), so equate corresponding elements. Row 1, column 1: $3x = 15$, so $x = 5$. Row 2, column 2: $y + 5 = 9$, so $y = 4$. The elements -2 and 8 already agree.

Q13. An identity matrix must (1) be square, and (2) have 1s on the main diagonal and 0s in every other position. K has order 2×3, so it is not square and fails condition (1), even though its 1s and 0s are arranged like part of an identity matrix. L is square, but $l_{12} = 1$ where an identity matrix requires 0, so L fails condition (2). Hence neither K nor L is an identity matrix.

Q14. C has 3 rows and 2 columns, so its order is 3×2. The element c_{31} is in row 3 (a dozen eggs) and column 1 (FreshMart): $c_{31} = 6.20$, so a dozen eggs costs \$6.20 at FreshMart. The price of milk at ValueBuy is row 1, column 2, the element $c_{12} = 2.45$, i.e. \$2.45. For the wholemeal loaf compare row 2: $c_{21} = 3.80$ at FreshMart and $c_{22} = 4.10$ at ValueBuy, so the loaf is cheaper at FreshMart (\$3.80 against \$4.10).

Q15. (a) The order $m \times n$ must satisfy $m \times n = 12$, so list the factor pairs of 12: 1×12, 2×6, 3×4, 4×3, 6×2 and 12×1 — six possible orders. (Note that 3×4 and 4×3 are different orders.) (b) A square matrix has order $m \times m$

and therefore m^2 elements. Since $3^2 = 9$ and $4^2 = 16$, there is no whole number m with $m^2 = 12$, so a matrix with 12 elements cannot be square.

Q16. (a) D has 3 rows and 4 columns, so its order is 3×4. (b) d_{24} (row 2, column 4) $= 3$; d_{32} (row 3, column 2) $= 9$. (c) The element 7 appears in row 1, column 2 and in row 3, column 4, so $d_{ij} = 7$ for $(i, j) = (1, 2)$ and $(i, j) = (3, 4)$. (d) D has $3 \times 4 = 12$ elements.

Q17. The orders agree (2×2), and the elements 4 and 7 already match, so equate the two remaining pairs: $x + y = 10$ and $x - y = 2$. Adding the two equations gives $2x = 12$, so $x = 6$; substituting back, $6 + y = 10$, so $y = 4$. Check: $x + y = 10$ and $x - y = 6 - 4 = 2$, as required.

Q18. (a) V has 4 rows and 2 columns, so its order is 4×2. (b) v_{32} is in row 3 (Wednesday) and column 2 (cats): $v_{32} = 3$, so three cats were treated on Wednesday. (c) The dog numbers are column 1: 9 (Monday), 7 (Tuesday), 10 (Wednesday) and 8 (Thursday). The largest is 10, so the most dogs were treated on Wednesday. (d) Tuesday is row 2: $7 + 6 = 13$ animals were treated in total.

Theory

In section 5A a **matrix** was introduced as a rectangular array of numbers arranged in **rows** and **columns**; a matrix with m rows and n columns has **order** $m \times n$, and each number in it is an **element**. Because so much everyday information — stock lists, price lists, sales figures — is naturally stored in tables, it is natural to ask how the tables themselves can be combined. This section develops three operations: **matrix addition**, **matrix subtraction** and **scalar multiplication**. (Multiplying one matrix by another matrix is a different operation altogether and is taken up in section 5C.)

Matrix addition. Two matrices can be added **only when they have the same order**. Their sum is found **element by element**: each element of the sum is the sum of the two elements in the matching position. For example,

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3+4 & 1+6 \\ 2+1 & 5+0 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 3 & 5 \end{bmatrix}.$$

The sum has the same order as the matrices being added. If two matrices have **different orders**, their sum is **not defined**: corresponding positions do not exist, so there is nothing to pair the elements with. A 2×3 matrix and a 3×2 matrix, for instance, cannot be added even though both contain six elements, because the elements are arranged in different patterns. Since ordinary numbers can be added in either order, matrix addition is **commutative**: $A + B = B + A$.

Matrix subtraction. Subtraction works the same way: the two matrices must have the **same order**, and each element of $A - B$ is the element of A minus the element of B in the same position. Unlike addition, the order of subtraction matters: every element of $B - A$ is the negative of the corresponding element of $A - B$. In context, subtraction measures **change**: subtracting a matrix of items sold from a matrix of opening stock gives the closing stock, element by element.

The zero matrix. The matrix whose elements are all zero is called the **zero matrix**, written O. It behaves in matrix addition exactly as the number 0 behaves in ordinary addition: for any matrix A of the same order, $A + O = A$ and $A - A = O$.

Scalar multiplication. In matrix work an ordinary number is called a **scalar**, to distinguish it from a matrix. To multiply a matrix by a scalar, multiply **every element** of the matrix by that scalar:

$$3\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 0 & 12 \end{bmatrix}.$$

This agrees with repeated addition, since $3A = A + A + A$, but the scalar need not be a whole number: $0.5A$ halves every element, and $1.1A$ increases every element by 10% — which is why scalar multiplication is the natural tool for a uniform price rise. Multiplying by -1 negates every element; the result is written $-A$, and subtraction can then be viewed as $A - B = A + (-B)$.

Combining the operations. Expressions such as $2A + 3B$ or $\frac{1}{2}A - B$ are evaluated by first forming each scalar multiple and then adding or subtracting, element by element as always. Such a combination is defined only when A and B have the same order.

Matrix equations. Two matrices are **equal** only when they have the same order and every pair of corresponding elements is equal. Because addition, subtraction and scalar multiplication all act element by element, a simple matrix equation is solved with the same steps as an ordinary equation. If $X + A = B$, subtracting A from both sides gives $X = B - A$; if $kX = C$ (with $k \neq 0$), multiplying both sides by the scalar $\frac{1}{k}$ gives $X = \frac{1}{k}C$. Equality of corresponding elements also lets us find unknown elements inside a matrix: each position provides its own small equation.

Using CAS. Enter each matrix once in your CAS calculator's matrix editor and give it a name; sums such as $A+B$, differences such as $A-B$ and scalar multiples such as $1.1\,A$ are then computed with the ordinary addition, subtraction and multiplication keys. If you ask the CAS to add matrices of different orders it reports a dimension error — the operation is simply not defined. Work small examples by hand as below so that you understand what the calculator is doing and can recognise when an answer (or an error message) is reasonable.

Worked examples

Example 1 — scaffolded

A bakery runs weekend stalls at the Daylesford and Kyneton farmers' markets. The matrices A and B record the numbers of sourdough loaves and baguettes sold on Saturday and on Sunday respectively, with rows for the two stalls (Daylesford then Kyneton) and columns for the two breads (sourdough then baguettes):

$$A=\begin{bmatrix}24 & 18\\ 31 & 12\end{bmatrix}, B=\begin{bmatrix}19 & 22\\ 27 & 15\end{bmatrix}.$$

Find $A+B$ and $A-B$, and interpret the element in row 1, column 2 of $A-B$.

Working	Comment
Both A and B have order 2×2, so $A+B$ and $A-B$ are defined.	Matrices can be added or subtracted only when their orders are the same.
$A+B=\begin{bmatrix}24+19 & 18+22\\ 31+27 & 12+15\end{bmatrix}=\begin{bmatrix}43 & 40\\ 58 & 27\end{bmatrix}$	Add the elements in matching positions; this gives the weekend totals.
$A-B=\begin{bmatrix}24-19 & 18-22\\ 31-27 & 12-15\end{bmatrix}=\begin{bmatrix}5 & -4\\ 4 & -3\end{bmatrix}$	Subtract element by element: Saturday's sales minus Sunday's.
The element in row 1, column 2 of $A-B$ is -4.	A negative element means Sunday's figure was the larger one.

So over the whole weekend the Daylesford stall sold 43 sourdough loaves and 40 baguettes and the Kyneton stall sold 58 and 27, and the element -4 of $A-B$ shows that the Daylesford stall sold 4 more baguettes on Sunday than on Saturday.

Example 2 — scaffolded

A Torquay surf shop sells two styles of wetsuit, springsuits and steamers, each in junior and adult sizes. Its current prices, in dollars, are stored in the matrix

$$P=\begin{bmatrix}45 & 60\\ 80 & 110\end{bmatrix},$$

with rows for the styles (springsuit then steamer) and columns for the sizes (junior then adult). (a) All prices are to rise by 10%. Write the new price matrix as a scalar multiple of P and evaluate it. (b) Instead, during a clearance sale, every wetsuit is advertised at 25% off. Write the sale price matrix as a scalar multiple of P and evaluate it.

Working	Comment
(a) New prices $=P+0.1\,P=1.1\,P$.	A 10% rise adds 0.1 of each price to itself, so every element is multiplied by 1.1.
$1.1\,P=\begin{bmatrix}1.1\times 45 & 1.1\times 60\\ 1.1\times 80 & 1.1\times 110\end{bmatrix}=\begin{bmatrix}49.50 & 66\\ 88 & 121\end{bmatrix}$	Multiply every element by the scalar 1.1.
(b) Sale prices $=P-0.25\,P=0.75\,P$.	Taking 25% off leaves 75% of each price.
$0.75\,P=\begin{bmatrix}0.75\times 45 & 0.75\times 60\\ 0.75\times 80 & 0.75\times 110\end{bmatrix}=\begin{bmatrix}33.75 & 45\\ 60 & 82.50\end{bmatrix}$	Multiply every element by the scalar 0.75.

So after the rise a junior springsuit costs \$49.50 and an adult steamer \$121, while in the sale those prices would instead be \$33.75 and \$82.50.

Example 3 — unscaffolded

For the matrices

$$A=\begin{bmatrix}4 & -2 & 5\\ 0 & 3 & -1\end{bmatrix} \text{and } B=\begin{bmatrix}1 & 6 & -3\\ 2 & -4 & 7\end{bmatrix},$$

find $2A+3B$.

Both matrices have order 2×3, so the combination is defined and the result also has order 2×3. Form each scalar multiple first:

$$2A=\begin{bmatrix}8 & -4 & 10\\ 0 & 6 & -2\end{bmatrix}, 3B=\begin{bmatrix}3 & 18 & -9\\ 6 & -12 & 21\end{bmatrix}.$$

Then add element by element:

$$2A+3B=\begin{bmatrix}8+3 & -4+18 & 10-9\\ 0+6 & 6-12 & -2+21\end{bmatrix}=\begin{bmatrix}11 & 14 & 1\\ 6 & -6 & 19\end{bmatrix}.$$

$\therefore\ 2A+3B=\begin{bmatrix}11 & 14 & 1\\ 6 & -6 & 19\end{bmatrix}.$

Example 4 — unscaffolded

A garden centre with sites at Ballarat and Bendigo stocks lavender and rosemary seedlings. Its opening stock on Monday, the quantities sold during the week and its end-of-week stocktake are recorded in the matrices

$$S=\begin{bmatrix}38 & 26\\ 44 & 19\end{bmatrix}, Q=\begin{bmatrix}21 & 14\\ 29 & 11\end{bmatrix}, E=\begin{bmatrix}42 & 30\\ 35 & 24\end{bmatrix},$$

with rows for the sites (Ballarat then Bendigo) and columns for the plants (lavender then rosemary). One delivery, described by a matrix D, arrived during the week. Find D.

The stock satisfies the matrix equation $S-Q+D=E$: opening stock, less sales, plus the delivery, gives the end-of-week count. Solving for D exactly as for an ordinary equation gives $D=E-S+Q$. Now

$$E-S=\begin{bmatrix}42-38 & 30-26\\ 35-44 & 24-19\end{bmatrix}=\begin{bmatrix}4 & 4\\ -9 & 5\end{bmatrix},$$

and adding Q element by element gives

$$D=\begin{bmatrix}4+21 & 4+14\\ -9+29 & 5+11\end{bmatrix}=\begin{bmatrix}25 & 18\\ 20 & 16\end{bmatrix}.$$

As a check, $S-Q+D$ has row 1 equal to $38-21+25=42$ and $26-14+18=30$, matching E.

$\therefore$ the delivery brought 25 lavender and 18 rosemary seedlings to Ballarat, and 20 lavender and 16 rosemary seedlings to Bendigo.

Practice questions

Scaffolded

Q1. Let

$$A=\begin{bmatrix}5 & 2\\ -1 & 7\end{bmatrix} \text{ and } B=\begin{bmatrix}3 & -4\\ 6 & 0\end{bmatrix}.$$

(a) Find $A+B$. (b) Find $A-B$. (c) Find $B-A$. (d) Compare your answers to (b) and (c). What do you notice?

Q2. Let

$$C=\begin{bmatrix}4 & -1 & 3\\ 0 & 5 & -2\end{bmatrix}.$$

(a) Find $3C$. (b) Find $-2C$. (c) Find $\frac{1}{2}C$, and state its order.

Q3. A school canteen records its sales of sandwiches and sushi rolls at recess and at lunch. In week 1 and week 2 of term the sales were

$$F=\begin{bmatrix}38 & 52\\ 81 & 94\end{bmatrix} \text{ and } G=\begin{bmatrix}45 & 47\\ 76 & 102\end{bmatrix},$$

with rows for the session (recess then lunch) and columns for the item (sandwiches then sushi rolls). (a) Find $F+G$ and state what its elements represent. (b) Find $G-F$. (c) Interpret the element in row 1, column 2 of $G-F$.

Q4. Let

$$P=\begin{bmatrix}2 & -3\\ 5 & 1\end{bmatrix} \text{ and } Q=\begin{bmatrix}-1 & 4\\ 2 & -2\end{bmatrix}.$$

(a) Find $2P$. (b) Find $3Q$. (c) Find $2P+3Q$. (d) Find $2P-3Q$.

Q5. Solve each matrix equation for the unknown matrix. (a) $X+\begin{bmatrix}3 & 1\\ -2 & 5\end{bmatrix}=\begin{bmatrix}7 & 4\\ 0 & 9\end{bmatrix}$ (b) $4Y=\begin{bmatrix}8 & -12\\ 20 & 0\end{bmatrix}$ (c) $Z-\begin{bmatrix}1 & 6\\ 3 & -2\end{bmatrix}=\begin{bmatrix}4 & -1\\ -3 & 5\end{bmatrix}$

Q6. The matrix A has order 2×3, B has order 3×2, C has order 2×3 and D has order 2×2, where

$$A=\begin{bmatrix}1 & 0 & 4\\ -2 & 3 & 5\end{bmatrix} \text{ and } C=\begin{bmatrix}6 & -2 & 1\\ 3 & 0 & -4\end{bmatrix}.$$

(a) State which of $A+C$, $A+B$, $B+D$ and $5B$ are defined. (b) Explain why $A+B$ is not defined, even though A and B each contain six elements. (c) Find $A+C$. (d) Find $2A-C$.

Unscaffolded

Q7. For

$$A=\begin{bmatrix}6 & -1 & 3\\ 2 & 0 & -5\end{bmatrix} \text{ and } B=\begin{bmatrix}-2 & 4 & 1\\ 7 & -3 & 5\end{bmatrix},$$

find $A+B$ and $A-B$.

Q8. For $M=\begin{bmatrix}-3 & 6\\ 9 & -12\end{bmatrix}$, find $4M$ and $-\frac{1}{3}M$.

Q9. For

$$A=\begin{bmatrix}2 & 5\\ -1 & 3\end{bmatrix} \text{ and } B=\begin{bmatrix}4 & -1\\ 0 & 6\end{bmatrix},$$

find $3A-2B$.

Q10. A bookshop chain has stores in Bendigo and Geelong. At the start of June the stock of paperbacks and hardbacks was $S=\begin{bmatrix}120 & 45\\ 95 & 60\end{bmatrix}$, and during June the stores sold the quantities in $Q=\begin{bmatrix}64 & 18\\ 71 & 25\end{bmatrix}$ (rows: Bendigo then Geelong; columns: paperbacks then hardbacks). Find the matrix of stock remaining at the end of June, and interpret its element in row 2, column 1.

Q11. A Brunswick food truck stores its menu prices, in dollars, in the matrix

$$P=\begin{bmatrix}12.50 & 15.00\\ 9.00 & 11.50\end{bmatrix},$$

with rows for tacos then burritos and columns for the single item then the meal deal. Rising costs force an 8% increase in every price. Find the new price matrix as a scalar multiple of P and evaluate it. Explain why the new matrix is $1.08\,P$ rather than something obtained by "adding 0.08" to P.

Q12. Solve the matrix equation $2X+A=B$ for X, where

$$A=\begin{bmatrix}4 & -2\\ 6 & 0\end{bmatrix} \text{and } B=\begin{bmatrix}10 & 4\\ -2 & 8\end{bmatrix}.$$

Q13. Find the values of x and y given that

$$\begin{bmatrix}x & 3\\ -1 & 2y\end{bmatrix}+\begin{bmatrix}4 & -5\\ 6 & 3\end{bmatrix}=\begin{bmatrix}9 & -2\\ 5 & 11\end{bmatrix}.$$

Q14. Two community pools, at Brunswick and Coburg, record their entries over one long weekend. The matrices for Friday, Saturday and Sunday were

$$A=\begin{bmatrix}42 & 96\\ 30 & 75\end{bmatrix}, B=\begin{bmatrix}51 & 108\\ 27 & 84\end{bmatrix}, C=\begin{bmatrix}60 & 120\\ 36 & 90\end{bmatrix},$$

with rows for the pools (Brunswick then Coburg) and columns for adult then child entries. Find the total-entries matrix $T=A+B+C$, and then find $\frac{1}{3}T$ and state what its elements represent.

Q15. Let $A=\begin{bmatrix}3 & -7\\ 2 & 5\end{bmatrix}$ and let O be the 2×2 zero matrix. Find $A-A$ and $A+O$, and find the matrix B for which $A+B=O$. What role does O play in matrix addition?

Q16. An electronics retailer has stores in Hobart and Launceston. Head office sets a target stock matrix $T=\begin{bmatrix}30 & 25\\ 28 & 22\end{bmatrix}$, while the current stock is $C=\begin{bmatrix}22 & 31\\ 35 & 14\end{bmatrix}$ (rows: Hobart then Launceston; columns: tablets then laptops). Find the adjustment matrix D satisfying $C+D=T$, and interpret the meaning of its negative elements.

Q17. For

$$A=\begin{bmatrix}8 & -4\\ 2 & 6\end{bmatrix} \text{and } B=\begin{bmatrix}-6 & 10\\ 4 & -2\end{bmatrix},$$

find $\frac{1}{2}A+\frac{1}{4}B$.

Q18. Solve the matrix equation $3X-A=2B+X$ for X, where

$$A=\begin{bmatrix}4 & 2\\ -6 & 8\end{bmatrix} \text{and } B=\begin{bmatrix}1 & -3\\ 5 & 0\end{bmatrix}.$$

Answers

Q1. (a) $\begin{bmatrix} 8 & -2 \\ 5 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 6 \\ -7 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -6 \\ 7 & -7 \end{bmatrix}$ (d) Every element of $B-A$ is the negative of the corresponding element of $A-B$; that is, $B-A=-(A-B)$. **Q2.** (a) $\begin{bmatrix} 12 & -3 & 9 \\ 0 & 15 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} -8 & 2 & -6 \\ 0 & -10 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -0.5 & 1.5 \\ 0 & 2.5 & -1 \end{bmatrix}$; order 2×3. **Q3.** (a) $F+G=\begin{bmatrix} 83 & 99 \\ 157 & 196 \end{bmatrix}$; total sales of each item in each session over the fortnight. (b) $G-F=\begin{bmatrix} 7 & -5 \\ -5 & 8 \end{bmatrix}$. (c) In week 2 the canteen sold 5 fewer sushi rolls at recess than in week 1. **Q4.** (a) $\begin{bmatrix} 4 & -6 \\ 10 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 12 \\ 6 & -6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 6 \\ 16 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & -18 \\ 4 & 8 \end{bmatrix}$ **Q5.** (a) $X=\begin{bmatrix} 4 & 3 \\ 2 & 4 \end{bmatrix}$ (b) $Y=\begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}$ (c) $Z=\begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$ **Q6.** (a) $A+C$ and $5B$ are defined; $A+B$ and $B+D$ are not. (b) A and B have different orders (2×3 and 3×2), so corresponding positions do not exist. (c) $\begin{bmatrix} 7 & -2 & 5 \\ 1 & 3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 2 & 7 \\ -7 & 6 & 14 \end{bmatrix}$ **Q7.** $A+B=\begin{bmatrix} 4 & 3 & 4 \\ 9 & -3 & 0 \end{bmatrix}$; $A-B=\begin{bmatrix} 8 & -5 & 2 \\ -5 & 3 & -10 \end{bmatrix}$. **Q8.** $4M=\begin{bmatrix} -12 & 24 \\ 36 & -48 \end{bmatrix}$; $-\frac{1}{3}M=\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$. **Q9.** $3A-2B=\begin{bmatrix} -2 & 17 \\ -3 & -3 \end{bmatrix}$. **Q10.** $S-Q=\begin{bmatrix} 56 & 27 \\ 24 & 35 \end{bmatrix}$; the element 24 means the Geelong store has 24 paperbacks left at the end of June. **Q11.** New prices $=1.08P=\begin{bmatrix} 13.50 & 16.20 \\ 9.72 & 12.42 \end{bmatrix}$; an 8% rise multiplies each price by 1.08 — adding 0.08 would raise every price by only 8 cents, and in any case adding a number to a matrix is not a defined matrix operation. **Q12.** $X=\begin{bmatrix} 3 & 3 \\ -4 & 4 \end{bmatrix}$. **Q13.** $x=5,\ y=4$. **Q14.** $T=\begin{bmatrix} 153 & 324 \\ 93 & 249 \end{bmatrix}$; $\frac{1}{3}T=\begin{bmatrix} 51 & 108 \\ 31 & 83 \end{bmatrix}$, the average number of entries of each type per day at each pool. **Q15.** $A-A=O$; $A+O=A$; $B=-A=\begin{bmatrix} -3 & 7 \\ -2 & -5 \end{bmatrix}$. O plays the role of the number 0 in ordinary addition (the additive identity). **Q16.** $D=T-C=\begin{bmatrix} 8 & -6 \\ -7 & 8 \end{bmatrix}$; a negative element means that store currently holds more than the target, so stock must be moved out (Hobart has 6 laptops too many, Launceston 7 tablets too many). **Q17.** $\frac{1}{2}A+\frac{1}{4}B=\begin{bmatrix} 2.5 & 0.5 \\ 2 & 2.5 \end{bmatrix}$. **Q18.** $X=\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$.

Fully-worked solutions

Q1. (a) Adding element by element, $A+B=\begin{bmatrix} 5+3 & 2-4 \\ -1+6 & 7+0 \end{bmatrix}=\begin{bmatrix} 8 & -2 \\ 5 & 7 \end{bmatrix}$. (b) $A-B=\begin{bmatrix} 5-3 & 2-(-4) \\ -1-6 & 7-0 \end{bmatrix}=\begin{bmatrix} 2 & 6 \\ -7 & 7 \end{bmatrix}$. (c) $B-A=\begin{bmatrix} 3-5 & -4-2 \\ 6-(-1) & 0-7 \end{bmatrix}=\begin{bmatrix} -2 & -6 \\ 7 & -7 \end{bmatrix}$. (d) Each element of $B-A$ is the negative of the corresponding element of $A-B$, so $B-A=-(A-B)$: reversing the order of a subtraction changes the sign of every element.

Q2. (a) Multiplying every element by 3, $3C=\begin{bmatrix} 12 & -3 & 9 \\ 0 & 15 & -6 \end{bmatrix}$. (b) Multiplying every element by -2, $-2C=\begin{bmatrix} -8 & 2 & -6 \\ 0 & -10 & 4 \end{bmatrix}$. (c) Multiplying every element by $\frac{1}{2}$, $\frac{1}{2}C=\begin{bmatrix} 2 & -0.5 & 1.5 \\ 0 & 2.5 & -1 \end{bmatrix}$. Scalar multiplication does not change the shape of a matrix, so the order is still 2×3.

Q3. (a) $F+G=\begin{bmatrix}38+45 & 52+47\\ 81+76 & 94+102\end{bmatrix}=\begin{bmatrix}83 & 99\\ 157 & 196\end{bmatrix}$. Each element is the total number of that item sold in that session over the two weeks — for instance, 157 sandwiches were sold at lunch across the fortnight. (b) $G-F=\begin{bmatrix}45-38 & 47-52\\ 76-81 & 102-94\end{bmatrix}=\begin{bmatrix}7 & -5\\ -5 & 8\end{bmatrix}$. (c) The element in row 1, column 2 is -5: the canteen sold 5 fewer sushi rolls at recess in week 2 than in week 1.

Q4. (a) $2P=\begin{bmatrix}4 & -6\\ 10 & 2\end{bmatrix}$. (b) $3Q=\begin{bmatrix}-3 & 12\\ 6 & -6\end{bmatrix}$. (c) Adding the answers to (a) and (b) element by element, $2P+3Q=\begin{bmatrix}4-3 & -6+12\\ 10+6 & 2-6\end{bmatrix}=\begin{bmatrix}1 & 6\\ 16 & -4\end{bmatrix}$. (d) Subtracting instead, $2P-3Q=\begin{bmatrix}4-(-3) & -6-12\\ 10-6 & 2-(-6)\end{bmatrix}=\begin{bmatrix}7 & -18\\ 4 & 8\end{bmatrix}$.

Q5. (a) Subtract the known matrix from both sides: $X=\begin{bmatrix}7-3 & 4-1\\ 0-(-2) & 9-5\end{bmatrix}=\begin{bmatrix}4 & 3\\ 2 & 4\end{bmatrix}$. (b) Multiply both sides by the scalar $\frac{1}{4}$: $Y=\frac{1}{4}\begin{bmatrix}8 & -12\\ 20 & 0\end{bmatrix}=\begin{bmatrix}2 & -3\\ 5 & 0\end{bmatrix}$. (c) Add the known matrix to both sides: $Z=\begin{bmatrix}4+1 & -1+6\\ -3+3 & 5+(-2)\end{bmatrix}=\begin{bmatrix}5 & 5\\ 0 & 3\end{bmatrix}$.

Q6. (a) Addition requires equal orders, so only $A+C$ (both 2×3) is defined among the sums; $A+B$ (2×3 with 3×2) and $B+D$ (3×2 with 2×2) are not. A scalar multiple is defined for every matrix, so $5B$ is defined. (b) Although A and B both contain six elements, their orders differ: A arranges its elements in 2 rows of 3 and B in 3 rows of 2. Element-by-element addition needs a partner in the **same position**, and (for example) A has no element in row 3 to pair with the row 3 elements of B. (c) $A+C=\begin{bmatrix}1+6 & 0-2 & 4+1\\ -2+3 & 3+0 & 5-4\end{bmatrix}=\begin{bmatrix}7 & -2 & 5\\ 1 & 3 & 1\end{bmatrix}$. (d) $2A=\begin{bmatrix}2 & 0 & 8\\ -4 & 6 & 10\end{bmatrix}$, so $2A-C=\begin{bmatrix}2-6 & 0+2 & 8-1\\ -4-3 & 6-0 & 10+4\end{bmatrix}=\begin{bmatrix}-4 & 2 & 7\\ -7 & 6 & 14\end{bmatrix}$.

Q7. Both matrices have order 2×3, so both results are defined. $A+B=\begin{bmatrix}6-2 & -1+4 & 3+1\\ 2+7 & 0-3 & -5+5\end{bmatrix}=\begin{bmatrix}4 & 3 & 4\\ 9 & -3 & 0\end{bmatrix}$ and $A-B=\begin{bmatrix}6-(-2) & -1-4 & 3-1\\ 2-7 & 0-(-3) & -5-5\end{bmatrix}=\begin{bmatrix}8 & -5 & 2\\ -5 & 3 & -10\end{bmatrix}$.

Q8. Multiplying every element by 4 gives $4M=\begin{bmatrix}-12 & 24\\ 36 & -48\end{bmatrix}$; multiplying every element by $-\frac{1}{3}$ gives $-\frac{1}{3}M=\begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}$.

Q9. $3A=\begin{bmatrix}6 & 15\\ -3 & 9\end{bmatrix}$ and $2B=\begin{bmatrix}8 & -2\\ 0 & 12\end{bmatrix}$, so $3A-2B=\begin{bmatrix}6-8 & 15-(-2)\\ -3-0 & 9-12\end{bmatrix}=\begin{bmatrix}-2 & 17\\ -3 & -3\end{bmatrix}$.

Q10. The remaining stock is the opening stock less the sales, element by element: $S-Q=\begin{bmatrix}120-64 & 45-18\\ 95-71 & 60-25\end{bmatrix}=\begin{bmatrix}56 & 27\\ 24 & 35\end{bmatrix}$. The element in row 2, column 1 is 24: the Geelong store has 24 paperbacks in stock at the end of June.

Q11. An 8% rise means each new price is $100\%+8\%=108\%$ of the old price, so every element of P is multiplied by the scalar 1.08:

$$1.08P=\begin{bmatrix}1.08\times12.50 & 1.08\times15.00\\ 1.08\times9.00 & 1.08\times11.50\end{bmatrix}=\begin{bmatrix}13.50 & 16.20\\ 9.72 & 12.42\end{bmatrix}.$$

“Adding 0.08” is wrong on two counts: adding a plain number to a matrix is not a defined matrix operation, and even adding 0.08 to every element would raise each price by only 8 cents, not by 8% of that price. A percentage change is a **multiplicative** change, so scalar multiplication is the correct tool.

Q12. Subtract A from both sides: $2X=B-A=\begin{bmatrix}10-4 & 4-(-2)\\ -2-6 & 8-0\end{bmatrix}=\begin{bmatrix}6 & 6\\ -8 & 8\end{bmatrix}$. Multiplying both sides by $\frac{1}{2}$ gives $X=\begin{bmatrix}3 & 3\\ -4 & 4\end{bmatrix}$.

Q13. Adding the left-hand side element by element gives $\begin{bmatrix}x+4 & -2\\ 5 & 2y+3\end{bmatrix}$. Two matrices are equal only when all corresponding elements are equal, so $x+4=9$, giving $x=5$, and $2y+3=11$, giving $y=4$. The remaining two positions already agree ($3+(-5)=-2$ and $-1+6=5$), so the equation is consistent.

Q14. Adding all three matrices element by element,

$$T=A+B+C=\begin{bmatrix}42+51+60 & 96+108+120\\ 30+27+36 & 75+84+90\end{bmatrix}=\begin{bmatrix}153 & 324\\ 93 & 249\end{bmatrix}.$$

Multiplying by the scalar $\frac{1}{3}$, $\frac{1}{3}T=\begin{bmatrix}51 & 108\\ 31 & 83\end{bmatrix}$. Its elements are the average daily entries over the three days — for instance, Brunswick averaged 51 adult and 108 child entries per day.

Q15. Subtracting each element from itself gives $A-A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$, and adding zero to each element leaves it unchanged, so $A+O=A$. For $A+B=O$ every element of B must cancel the corresponding element of A, so $B=-A=\begin{bmatrix}-3 & 7\\ -2 & -5\end{bmatrix}$. Thus O is the **additive identity** of matrix addition: it plays exactly the role that the number 0 plays in ordinary addition.

Q16. Solving $C+D=T$ gives $D=T-C=\begin{bmatrix}30-22 & 25-31\\ 28-35 & 22-14\end{bmatrix}=\begin{bmatrix}8 & -6\\ -7 & 8\end{bmatrix}$. A positive element is stock that must be brought in; a negative element means the store already holds more than the target, so stock must be moved out: Hobart must send away 6 laptops and Launceston 7 tablets, while Hobart needs 8 more tablets and Launceston 8 more laptops.

Q17. $\frac{1}{2}A=\begin{bmatrix}4 & -2\\ 1 & 3\end{bmatrix}$ and $\frac{1}{4}B=\begin{bmatrix}-1.5 & 2.5\\ 1 & -0.5\end{bmatrix}$, so adding element by element, $\frac{1}{2}A+\frac{1}{4}B=\begin{bmatrix}4-1.5 & -2+2.5\\ 1+1 & 3-0.5\end{bmatrix}=\begin{bmatrix}2.5 & 0.5\\ 2 & 2.5\end{bmatrix}$.

Q18. Collect the X terms exactly as in an ordinary equation. Subtracting X from both sides of $3X-A=2B+X$ gives $2X-A=2B$, and adding A to both sides gives $2X=2B+A$. Multiplying by $\frac{1}{2}$, $X=B+\frac{1}{2}A$. Now $\frac{1}{2}A=\begin{bmatrix}2 & 1\\ -3 & 4\end{bmatrix}$, so $X=\begin{bmatrix}1+2 & -3+1\\ 5-3 & 0+4\end{bmatrix}=\begin{bmatrix}3 & -2\\ 2 & 4\end{bmatrix}$. As a check, $3X-A=\begin{bmatrix}9-4 & -6-2\\ 6+6 & 12-8\end{bmatrix}=\begin{bmatrix}5 & -8\\ 12 & 4\end{bmatrix}$ and $2B+X=\begin{bmatrix}2+3 & -6-2\\ 10+2 & 0+4\end{bmatrix}=\begin{bmatrix}5 & -8\\ 12 & 4\end{bmatrix}$, which agree.

Theory

Earlier in this chapter matrices were used to store information, and matrices of the same order were added, subtracted and multiplied by a number (a scalar). This section defines the **product** of two matrices. Matrix multiplication is not done element-by-element — it has its own rule, chosen because it does exactly the bookkeeping needed in practical problems such as costing an order or counting the connections in a network. Recall that a matrix with m rows and n columns has **order** $m \times n$.

The row-by-column rule. The building block of every matrix product is a **row** multiplied by a **column** with the same number of elements: multiply the elements in matching positions, then add.

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a \times p + b \times q \end{bmatrix}.$$

For example, $\begin{bmatrix} 2 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$, a 1×1 matrix.

When does a product exist? The product AB exists only when the number of **columns of** A equals the number of **rows of** B — otherwise the rows of A and the columns of B do not have matching lengths and the rule cannot be applied. If A has order $m \times n$ and B has order $n \times p$, then AB exists and has order $m \times p$. A quick way to check is to write the two orders side by side:

$$(m \times n)(n \times p) \to m \times p.$$

The **inside** numbers must match for the product to exist, and the **outside** numbers give the order of the product.

Multiplying larger matrices. The element in **row** i**, column** j of the product AB is row i of A multiplied by column j of B (work **across** the row and **down** the column). For example,

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 4 + 1 \times 5 \\ 0 \times 4 + 3 \times 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}.$$

The order of multiplication matters. For numbers, $3 \times 5 = 5 \times 3$; for matrices, $AB \neq BA$ **in general**. One of the two products may exist while the other does not; when both exist they may have different orders; and even when both exist and have the same order, the elements are usually different. Always multiply in the order the problem requires, and never swap the order of a matrix product.

Practical products. Matrix multiplication automatically pairs, multiplies and adds — exactly what costing an order requires. If each row of a **quantities matrix** Q lists the amounts of each item bought (one row per buyer or per day), and a **price column** P lists the cost of one unit of each item in the same item order, then $C = QP$ is a column of total costs. The rows of the product keep the meaning of the rows of the first matrix, and the columns keep the meaning of the columns of the second — a useful check that the product has been formed the right way around.

Powers of a square matrix. Just as $5^2 = 5 \times 5$, the **square** of a matrix is $A^2 = A \times A$, and $A^3 = A \times A \times A$, which can be found as $A^2 A$ (or $A A^2$ — the result is the same). Writing the orders side by side, $(m \times n)(m \times n)$ requires $n = m$, so **only square matrices** can be raised to powers.

Connection matrices and two-step links. A network of direct connections (friends who have each other's phone numbers, towns joined by direct routes) can be stored in a square **connection matrix** (also called an adjacency matrix) M: the element in row i, column j is 1 if i and j are directly connected and 0 if they are not, with 0s on the diagonal. The power M^2 then counts the **two-step connections**: its element in row i, column j is the number of ways to get from i to j in exactly two steps, through some third member k. This works because the row-by-column rule forms the sum of the products $m_{ik} \times m_{kj}$, and each product equals 1 exactly when both the link i–k and the link k–j exist. In an undirected network each diagonal element of M^2 counts the "there-and-back" trips *i*–*k*–*i*, which is simply the number of direct connections *i* has.

Using CAS. General Mathematics is a technology-active, open-book course. Your CAS has a matrix editor: enter each matrix, then use the ordinary multiplication key for products and the power key for A^2 and A^3. If the orders do not match, the calculator reports a dimension error — the same condition you check by hand. Powers of larger connection matrices are best left to the CAS, but you must be able to form small products by hand, predict the order of a product before computing it, and recognise when a product does not exist.

Worked examples

Example 1 — scaffolded

Consider the matrices

$$A=\begin{bmatrix}3 & 1 & 2\end{bmatrix}, B=\begin{bmatrix}5\\4\\6\end{bmatrix}.$$

Explain why the product AB exists and state its order, then find AB. Then decide whether BA exists and, if it does, find it.

Working	Comment
A is 1×3 and B is 3×1: $(1\times3)(3\times1)$, and the inside numbers agree ($3=3$), so AB exists.	The number of columns of A must equal the number of rows of B.
The outside numbers give the order of AB: 1×1.	Write the orders side by side; outside numbers give the order of the product.
$AB=\begin{bmatrix}3\times5+1\times4+2\times6\end{bmatrix}=\begin{bmatrix}31\end{bmatrix}$.	Row-by-column: multiply matching elements, then add.
For BA: $(3\times1)(1\times3)$ — the inside numbers agree ($1=1$), so BA also exists, with order 3×3.	The same check, with the matrices in the other order.
$BA=\begin{bmatrix}5\times3 & 5\times1 & 5\times2\\4\times3 & 4\times1 & 4\times2\\6\times3 & 6\times1 & 6\times2\end{bmatrix}=\begin{bmatrix}15 & 5 & 10\\12 & 4 & 8\\18 & 6 & 12\end{bmatrix}$.	Each row of B has one element, so each row-by-column product is a single multiplication.

So $AB=\begin{bmatrix}31\end{bmatrix}$ while BA is a 3×3 matrix — the two products do not even have the same order, so certainly $AB\neq BA$.

Example 2 — scaffolded

For the matrices

$$A=\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}, B=\begin{bmatrix}4 & 1\\2 & 5\end{bmatrix},$$

find AB and BA, and hence show that $AB\neq BA$.

Working	Comment
Both products exist: $(2\times2)(2\times2)$ gives a 2×2 product each way.	The inside numbers match in both orders.
Row 1 of A times the columns of B: $1\times4+2\times2=8$ and $1\times1+2\times5=11$.	Work across row 1, down each column of B in turn.
Row 2 of A times the columns of B: $3\times4+0\times2=12$ and $3\times1+0\times5=3$.	The element in row i, column j uses row i of A and column j of B.

Working	Comment
$AB=\begin{bmatrix}8 & 11\\12 & 3\end{bmatrix}$.	Assemble the four elements.
$BA=\begin{bmatrix}4\times1+1\times3 & 4\times2+1\times0\\2\times1+5\times3 & 2\times2+5\times0\end{bmatrix}=\begin{bmatrix}7 & 8\\17 & 4\end{bmatrix}$.	Now the rows of B multiply the columns of A.

So $AB=\begin{bmatrix}8 & 11\\12 & 3\end{bmatrix}$ and $BA=\begin{bmatrix}7 & 8\\17 & 4\end{bmatrix}$: the products have the same order but different elements, so $AB\neq BA$.

Example 3 — unscaffolded

A school canteen sells sandwiches for \$6.50, sushi rolls for \$7.00 and juice boxes for \$3.50. On Monday it sold 30 sandwiches, 25 sushi rolls and 40 juice boxes; on Tuesday it sold 24 sandwiches, 30 sushi rolls and 36 juice boxes. Use a matrix product to find the canteen's takings on each day.

Store the sales in a quantities matrix Q (one row per day, one column per item) and the prices in a column matrix P, listing the items in the same order:

$$Q=\begin{bmatrix}30 & 25 & 40\\24 & 30 & 36\end{bmatrix}, P=\begin{bmatrix}6.50\\7.00\\3.50\end{bmatrix}.$$

The product QP exists because $(2\times3)(3\times1)$ has matching inside numbers, and its order is 2×1 — one total per day. (The reverse product PQ does not exist, since $(3\times1)(2\times3)$ has inside numbers $1\neq2$: the quantities matrix must come first.)

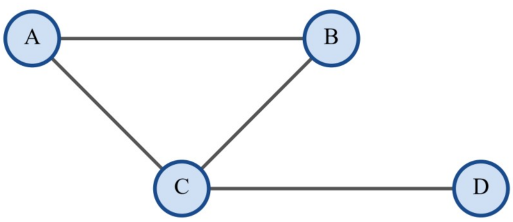
$$QP=\begin{bmatrix}30\times6.50+25\times7.00+40\times3.50\\24\times6.50+30\times7.00+36\times3.50\end{bmatrix}=\begin{bmatrix}195+175+140\\156+210+126\end{bmatrix}=\begin{bmatrix}510\\492\end{bmatrix}.$$

Each row of the product keeps the meaning of the corresponding row of Q: row 1 is Monday, row 2 is Tuesday.

$\therefore$ the canteen took \$510 on Monday and \$492 on Tuesday.

Example 4 — unscaffolded

Four members of a community garden — Ana, Ben, Cara and Dev — swap seedlings by phone. Ana and Ben have each other's numbers, and so do Ana and Cara, Ben and Cara, and Cara and Dev. The network is shown below, with each member represented by the first letter of their name.



Write down the connection matrix M (rows and columns in the order A, B, C, D), find M^2, and interpret the element in row A, column D of M^2, the element in row C, column D of M^2, and the diagonal element in row C.

Each element of M is 1 for a pair with each other's numbers and 0 otherwise:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Each element of M^2 is a row of M times a column of M. For instance, the element in row A, column D is $0 \times 0 + 1 \times 0 + 1 \times 1 + 0 \times 0 = 1$, and the diagonal element in row C is $1 \times 1 + 1 \times 1 + 0 \times 0 + 1 \times 1 = 3$. Computing all sixteen elements the same way (a CAS check is quick):

$$M^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

The element 1 in row A, column D means there is exactly **one two-step link** from Ana to Dev — a message passed through one other member, here Ana–Cara–Dev. The element 0 in row C, column D means Cara and Dev have **no** two-step link (their only link is the direct one). The diagonal element 3 in row C counts the three there-and-back trips Cara–Ana–Cara, Cara–Ben–Cara and Cara–Dev–Cara — one through each of Cara's three direct contacts.

$\therefore$ M^2 counts the two-step links: Ana can reach Dev through one intermediary (Cara), Cara and Dev have no two-step link, and Cara's diagonal element 3 equals her number of direct contacts.

Practice questions

Scaffolded

Q1. Matrix A has order 2×3, matrix B has order 3×2 and matrix C has order 2×2. (a) Explain why the product AB exists and state its order. (b) Does BA exist? If so, state its order. (c) Does AC exist? Explain. (d) Does CA exist? If so, state its order. (e) One of A^2 and C^2 exists. Which one, and why?

Q2. Consider $R = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$. (a) Find RC and state its order. (b) State the order of CR without calculating it. (c) Now calculate CR. (d) Is $RC = CR$? What does this tell you about matrix multiplication?

Q3. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. (a) Find AB. (b) Find AC. (c) Find CA. (d) Is $AC = CA$?

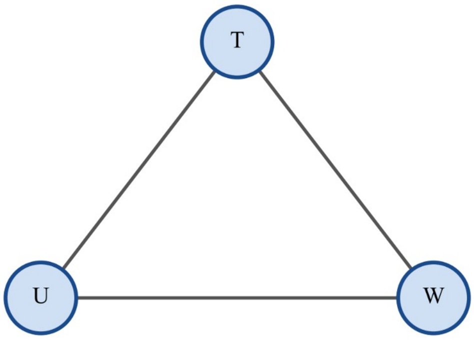
Q4. At a hardware store, a box of screws costs \$7, a length of treated pine costs \$12 and a tin of paint costs \$25. Jack buys 4 boxes of screws, 6 lengths of pine and 2 tins of paint; Maya buys 3 boxes of screws, 8 lengths of pine and 5 tins of paint. Let

$$Q = \begin{bmatrix} 4 & 6 & 2 \\ 3 & 8 & 5 \end{bmatrix}, P = \begin{bmatrix} 7 \\ 12 \\ 25 \end{bmatrix},$$

where row 1 of Q is Jack's order and row 2 is Maya's. (a) Write the orders of Q and P side by side and hence state the order of QP. (b) Calculate QP. (c) Interpret each element of QP in context.

Q5. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. (a) Find A^2. (b) Use $A^3 = A^2 A$ to find A^3. (c) Check that $A A^2$ gives the same answer as part (b). (d) Explain why a 2×3 matrix cannot be squared.

Q6. The towns of Tarra (T), Ulan (U) and Wirri (W) are joined by direct shuttle-bus routes: there is a direct route between each pair of towns, as shown.



(a) Write down the connection matrix M, with rows and columns in the order T, U, W.

(b) Calculate M^2.

(c) Interpret the element in row T, column U of M^2 in terms of bus trips.

(d) Interpret the diagonal element in row T of M^2.

Unscaffolded

Q7. Matrix P has order 3×2, matrix Q has order 2×4 and matrix R has order 4×1. State whether each of the products PQ, QP, QR, RP and $(PQ)R$ exists, and give the order of each product that does exist.

Q8. For $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}$, find AB and BA, and hence show that $AB \neq BA$.

Q9. For $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 1 & 3 \end{bmatrix}$, find AB, and state (without calculating it) the order of BA.

Q10. A farm-gate stall near Woodend sells small boxes of apples for \$15 and large boxes for \$25. On Saturday it sold 12 small and 8 large boxes; on Sunday it sold 15 small and 10 large boxes. Write a quantities matrix S (rows Saturday and Sunday) and a price column P, and use the product SP to find the stall's takings on each day.

Q11. An electrician restocking her van buys 8 power points at \$14 each, 5 light switches at \$9 each and 20 metres of cable at \$3.50 per metre. Express the total cost as the product of a 1×3 quantities matrix and a 3×1 price matrix, calculate it, and state the order of the resulting matrix.

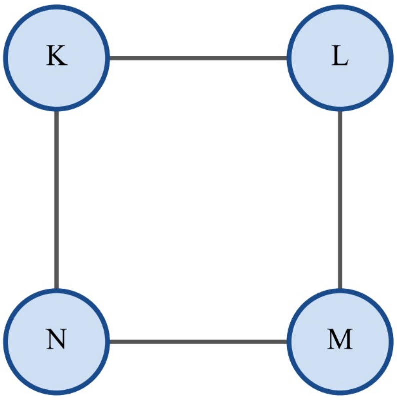
Q12. Let $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find B^2 and B^3.

Q13. A Bendigo café sells flat whites for \$4.50, pots of tea for \$4.00 and hot chocolates for \$5.00. Its weekend sales are recorded in the quantities matrix

$$Q = \begin{bmatrix} 85 & 40 & 25 \\ 96 & 44 & 30 \end{bmatrix},$$

where the rows are Saturday and Sunday and the columns are flat whites, teas and hot chocolates, and the prices are stored in the column matrix P (same item order). Exactly one of the products QP and PQ exists. State which one, explain why the other does not exist, and calculate the one that does to find the café's takings on each day.

Q14. Four campsites — Karri (K), Lowan (L), Mallee (M) and Nardoo (N) — are joined by walking tracks that form a loop: K–L, L–M, M–N and N–K, as shown.



Write down the connection matrix C (rows and columns in the order K, L, M, N) and calculate C^2. Interpret the element in row K, column M of C^2, and explain why the element in row K, column L of C^2 is zero.

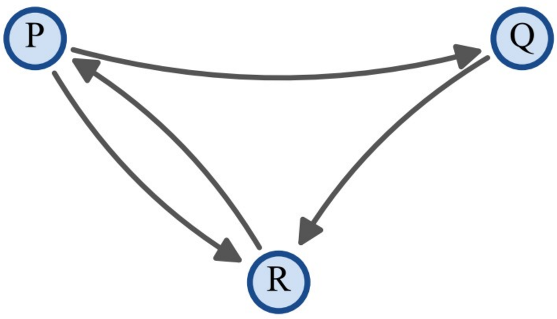
Q15. For two school camps, the quantities of food ordered are stored in

$$Q = \begin{bmatrix} 18 & 12 & 6 \\ 24 & 15 & 8 \end{bmatrix},$$

where row 1 is Camp Coolamon, row 2 is Camp Barrabool, and the columns are loaves of bread, cartons of milk and dozens of eggs. Bread costs \$4 a loaf, milk \$3 a carton and eggs \$8 a dozen. Form a suitable price matrix P, calculate QP, and interpret the element in row 2 of the product in context.

Q16. Let $T = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Find T^2 and T^3.

Q17. In a small business, tasks are emailed one way only: Priya (P) can email Quinn (Q) and Rosa (R), Quinn can email Rosa, and Rosa can email Priya, as shown (arrows point from sender to receiver).



The communication matrix M has a 1 in row i, column j if the person in row i can email the person in column j directly. Write down M (rows and columns in the order P, Q, R) and calculate M^2. Interpret the element in row Q, column P of M^2, and explain what the zero in row P, column Q of M^2 tells you.

Q18. Three alpine huts — Falls (F), Gorge (G) and Heath (H) — have two-way radios, but only Falls–Gorge and Gorge–Heath are within range of each other. The connection matrix is

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Calculate M^2 and $M + M^2$. Explain what the elements of $M + M^2$ count, and hence explain how the matrix $M + M^2$ shows that every hut can contact every other hut using at most one relay.

Answers

Q1. (a) Columns of A = rows of $B = 3$; order 2×2. (b) Yes; 3×3. (c) No — A has 3 columns but C has 2 rows. (d) Yes; 2×3. (e) C^2 (C is square); A^2 does not exist since $(2 \times 3)(2 \times 3)$ has inside numbers $3 \neq 2$. **Q2.** (a) $RC = [40]$, order 1×1. (b) 3×3. (c) $CR = \begin{bmatrix} 20 & 8 & 12 \\ 35 & 14 & 21 \\ 10 & 4 & 6 \end{bmatrix}$. (d) No — matrix multiplication is not commutative; here RC and CR do not even have the same order. **Q3.** (a) $AB = \begin{bmatrix} 16 \\ 13 \end{bmatrix}$. (b) $AC = \begin{bmatrix} 2 & 13 \\ 1 & 14 \end{bmatrix}$. (c) $CA = \begin{bmatrix} 4 & 11 \\ 3 & 12 \end{bmatrix}$. (d) No, $AC \neq CA$. **Q4.** (a) $(2 \times 3)(3 \times 1) \to 2 \times 1$. (b) $QP = \begin{bmatrix} 150 \\ 242 \end{bmatrix}$. (c) Jack's order costs \$150; Maya's order costs \$242. **Q5.** (a) $A^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$. (b) $A^3 = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$. (c) $AA^2 = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ — the same. (d) $(2 \times 3)(2 \times 3)$ has inside numbers $3 \neq 2$, so the product does not exist. **Q6.** (a) $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. (b) $M^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. (c) Exactly one two-bus trip from Tarra to Ulan (via Wirri). (d) Two there-and-back trips from Tarra (via Ulan or via Wirri) — Tarra's number of direct routes. **Q7.** PQ: 3×4. QP: does not exist ($4 \neq 3$). QR: 2×1. RP: does not exist ($1 \neq 3$). $(PQ)R$: 3×1. **Q8.** $AB = \begin{bmatrix} 13 & 1 \\ 19 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 2 \\ 22 & 8 \end{bmatrix}$; the elements differ, so $AB \neq BA$. **Q9.** $AB = \begin{bmatrix} 4 & 7 \\ 11 & 6 \end{bmatrix}$; BA has order 3×3. **Q10.** $SP = \begin{bmatrix} 380 \\ 475 \end{bmatrix}$: takings \$380 on Saturday and \$475 on Sunday. **Q11.** $\begin{bmatrix} 8 & 5 & 20 \end{bmatrix} \begin{bmatrix} 14 \\ 9 \\ 3.50 \end{bmatrix} = [227]$; total \$227; order 1×1. **Q12.** $B^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $B^3 = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$. **Q13.** QP exists: $(2 \times 3)(3 \times 1) \to 2 \times 1$; PQ does not, since $(3 \times 1)(2 \times 3)$ has inside numbers $1 \neq 2$. $QP = \begin{bmatrix} 667.50 \\ 758.00 \end{bmatrix}$: \$667.50 on Saturday, \$758.00 on Sunday. **Q14.** $C^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$; two two-step walks from Karri to Mallee (via Lowan or via Nardoo); zero because no two-step walk from Karri ends at Lowan (each such walk ends back at Karri or at Mallee). **Q15.** $P = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}$; $QP = \begin{bmatrix} 156 \\ 205 \end{bmatrix}$; row 2: the food order for Camp Barrabool costs \$205 in total. **Q16.** $T^2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, $T^3 = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$. **Q17.** $M = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $M^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; row Q, column P: one two-step route Quinn–Rosa–Priya; the zero means Priya cannot reach Quinn in two steps (only directly). **Q18.** $M^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $M + M^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; elements count routes using one or two steps; every off-diagonal element is positive, so every pair of huts can communicate directly or through one relay.

Fully-worked solutions

Q1. (a) Write the orders side by side: $(2 \times 3)(3 \times 2)$. The inside numbers are both 3 (columns of A = rows of B), so AB exists; the outside numbers give its order, 2×2. (b) $(3 \times 2)(2 \times 3)$: inside numbers both 2, so BA exists, with order 3×3. (c) $(2 \times 3)(2 \times 2)$: the inside numbers are 3 and 2, which do not match, so AC does not exist. (d) $(2 \times 2)(2 \times 3)$: inside numbers both 2, so CA exists, with order 2×3. (e) A^2 would be $(2 \times 3)(2 \times 3)$, inside numbers $3 \neq 2$, so it does not exist. C^2 is $(2 \times 2)(2 \times 2)$, which exists — only **square** matrices can be raised to powers.

Q2. (a) $(1 \times 3)(3 \times 1) \rightarrow 1 \times 1$: $RC = [5 \times 4 + 2 \times 7 + 3 \times 2] = [20 + 14 + 6] = [40]$. (b) $(3 \times 1)(1 \times 3) \rightarrow 3 \times 3$. (c) Each element is one element of C times one element of R:

$$CR = \begin{bmatrix} 4 \times 5 & 4 \times 2 & 4 \times 3 \\ 7 \times 5 & 7 \times 2 & 7 \times 3 \\ 2 \times 5 & 2 \times 2 & 2 \times 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 12 \\ 35 & 14 & 21 \\ 10 & 4 & 6 \end{bmatrix}.$$

(d) No: RC is 1×1 but CR is 3×3. Matrix multiplication is not commutative — reversing the order can change the answer completely, even its order.

Q3. (a) $(2 \times 2)(2 \times 1) \rightarrow 2 \times 1$: $AB = \begin{bmatrix} 2 \times 5 + 3 \times 2 \\ 1 \times 5 + 4 \times 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \end{bmatrix}$. (b) $AC = \begin{bmatrix} 2 \times 1 + 3 \times 0 & 2 \times 2 + 3 \times 3 \\ 1 \times 1 + 4 \times 0 & 1 \times 2 + 4 \times 3 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ 1 & 14 \end{bmatrix}$. (c) $CA = \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times 3 + 2 \times 4 \\ 0 \times 2 + 3 \times 1 & 0 \times 3 + 3 \times 4 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 3 & 12 \end{bmatrix}$. (d) No: the corresponding elements of AC and CA differ, so $AC \neq CA$.

Q4. (a) $(2 \times 3)(3 \times 1)$: the inside numbers match ($3 = 3$), so QP exists and its order is given by the outside numbers, 2×1. (b) $QP = \begin{bmatrix} 4 \times 7 + 6 \times 12 + 2 \times 25 \\ 3 \times 7 + 8 \times 12 + 5 \times 25 \end{bmatrix} = \begin{bmatrix} 28 + 72 + 50 \\ 21 + 96 + 125 \end{bmatrix} = \begin{bmatrix} 150 \\ 242 \end{bmatrix}$. (c) The rows of the product keep the meaning of the rows of Q: Jack's order costs \$150 and Maya's order costs \$242.

Q5. (a) $A^2 = \begin{bmatrix} 1 \times 1 + 2 \times 0 & 1 \times 2 + 2 \times 3 \\ 0 \times 1 + 3 \times 0 & 0 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$. (b) $A^3 = A^2 A = \begin{bmatrix} 1 \times 1 + 8 \times 0 & 1 \times 2 + 8 \times 3 \\ 0 \times 1 + 9 \times 0 & 0 \times 2 + 9 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$. (c) $AA^2 = \begin{bmatrix} 1 \times 1 + 2 \times 0 & 1 \times 8 + 2 \times 9 \\ 0 \times 1 + 3 \times 0 & 0 \times 8 + 3 \times 9 \end{bmatrix} = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$, the same as part (b). (d) Squaring a 2×3 matrix would require the product $(2 \times 3)(2 \times 3)$, whose inside numbers are 3 and 2. They do not match, so the product does not exist: only square matrices have powers.

Q6. (a) Every pair of towns is directly joined, so every off-diagonal element is 1: $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. (b) For example, the element in row T, column U of M^2 is $0 \times 1 + 1 \times 0 + 1 \times 1 = 1$, and the diagonal element in row T is $0 \times 0 + 1 \times 1 + 1 \times 1 = 2$. Completing all nine elements:

$$M^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(c) The 1 in row T, column U means there is exactly one two-bus trip from Tarra to Ulan — travel via the third town, Tarra–Wirri–Ulan. (d) The diagonal element 2 counts the two-step trips from Tarra back to Tarra: out and back via Ulan, or out and back via Wirri. It equals the number of direct routes from Tarra.

Q7. Write the orders side by side in each case. PQ: $(3 \times 2)(2 \times 4)$ — inside numbers match, order 3×4. QP: $(2 \times 4)(3 \times 2)$ — inside numbers $4 \neq 3$, does not exist. QR: $(2 \times 4)(4 \times 1)$ — inside numbers match, order 2×1. RP: $(4 \times 1)(3 \times 2)$ — inside numbers $1 \neq 3$, does not exist. $(PQ)R$: PQ is 3×4, so $(3 \times 4)(4 \times 1)$ — inside numbers match, order 3×1.

Q8. $AB = \begin{bmatrix} 4 \times 2 + 1 \times 5 & 4 \times 0 + 1 \times 1 \\ 2 \times 2 + 3 \times 5 & 2 \times 0 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} 13 & 1 \\ 19 & 3 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 \times 4 + 0 \times 2 & 2 \times 1 + 0 \times 3 \\ 5 \times 4 + 1 \times 2 & 5 \times 1 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 22 & 8 \end{bmatrix}$. The two products have the same order but different elements (for example $13 \neq 8$ in row 1, column 1), so $AB \neq BA$.

Q9. $(2 \times 3)(3 \times 2) \rightarrow 2 \times 2$:

$$AB=\begin{bmatrix}1\times2+0\times4+2\times1 & 1\times1+0\times0+2\times3\\3\times2+1\times4+1\times1 & 3\times1+1\times0+1\times3\end{bmatrix}=\begin{bmatrix}4 & 7\\11 & 6\end{bmatrix}.$$

For BA: $(3\times2)(2\times3)$ — the inside numbers match, and the outside numbers show that BA has order 3×3.

Q10. With rows Saturday and Sunday and columns small and large boxes, $S=\begin{bmatrix}12 & 8\\15 & 10\end{bmatrix}$ and $P=\begin{bmatrix}15\\25\end{bmatrix}$. Then

$$SP=\begin{bmatrix}12\times15+8\times25\\15\times15+10\times25\end{bmatrix}=\begin{bmatrix}180+200\\225+250\end{bmatrix}=\begin{bmatrix}380\\475\end{bmatrix},$$

so the stall took \$380 on Saturday and \$475 on Sunday.

Q11. The quantities matrix is $\begin{bmatrix}8 & 5 & 20\end{bmatrix}$ and the price matrix is $\begin{bmatrix}14\\9\\3.50\end{bmatrix}$ (items in the same order). The product is

$$\begin{bmatrix}8 & 5 & 20\end{bmatrix}\begin{bmatrix}14\\9\\3.50\end{bmatrix}=\begin{bmatrix}8\times14+5\times9+20\times3.50\end{bmatrix}=\begin{bmatrix}112+45+70\end{bmatrix}=\begin{bmatrix}227\end{bmatrix},$$

a 1×1 matrix: the total cost is \$227.

Q12. $B^2=\begin{bmatrix}2\times2+1\times1 & 2\times1+1\times2\\1\times2+2\times1 & 1\times1+2\times2\end{bmatrix}=\begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}$. Then $B^3=B^2B=\begin{bmatrix}5\times2+4\times1 & 5\times1+4\times2\\4\times2+5\times1 & 4\times1+5\times2\end{bmatrix}=\begin{bmatrix}14 & 13\\13 & 14\end{bmatrix}$. (On a CAS, entering B and pressing the power key gives the same results.)

Q13. Here Q is 2×3 and $P=\begin{bmatrix}4.50\\4.00\\5.00\end{bmatrix}$ is 3×1. For QP: $(2\times3)(3\times1)$ — the inside numbers match, so QP exists with order 2×1. For PQ: $(3\times1)(2\times3)$ — the inside numbers are 1 and 2, which do not match, so PQ does not exist. Multiplying,

$$QP=\begin{bmatrix}85\times4.50+40\times4.00+25\times5.00\\96\times4.50+44\times4.00+30\times5.00\end{bmatrix}=\begin{bmatrix}382.50+160+125\\432+176+150\end{bmatrix}=\begin{bmatrix}667.50\\758.00\end{bmatrix},$$

so the café took \$667.50 on Saturday and \$758.00 on Sunday.

Q14. Each campsite is directly joined to its two neighbours on the loop:

$$C=\begin{bmatrix}0 & 1 & 0 & 1\\1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & 1 & 0\end{bmatrix}.$$

For example, the element of C^2 in row K, column M is $0\times0+1\times1+0\times0+1\times1=2$, and in row K, column L it is $0\times1+1\times0+0\times1+1\times0=0$. Completing the sixteen elements (or squaring C on a CAS):

$$C^2=\begin{bmatrix}2 & 0 & 2 & 0\\0 & 2 & 0 & 2\\2 & 0 & 2 & 0\\0 & 2 & 0 & 2\end{bmatrix}.$$

The 2 in row K, column M means there are exactly two two-step walks from Karri to Mallee: via Lowan (K–L–M) or via Nardoo (K–N–M). The element in row K, column L is zero because a two-step walk from Karri goes to a neighbour (Lowan or Nardoo) and then to one of that campsite's neighbours, which is either back to Karri or across to Mallee — no two-step walk can end at Lowan.

Q15. The prices, in the same item order as the columns of Q, form $P=\begin{bmatrix}4\\3\\8\end{bmatrix}$. Then $(2\times3)(3\times1)\to2\times1$:

$$QP=\begin{bmatrix}18\times4+12\times3+6\times8\\24\times4+15\times3+8\times8\end{bmatrix}=\begin{bmatrix}72+36+48\\96+45+64\end{bmatrix}=\begin{bmatrix}156\\205\end{bmatrix}.$$

Row 2 of the product corresponds to row 2 of Q: the total cost of the food order for Camp Barrabool is \$205 (and Camp Coolamon's order costs \$156).

Q16. $T^2=\begin{bmatrix}0\times0+2\times1 & 0\times2+2\times1\\1\times0+1\times1 & 1\times2+1\times1\end{bmatrix}=\begin{bmatrix}2&2\\1&3\end{bmatrix}$. Then $T^3=T^2T=\begin{bmatrix}2\times0+2\times1 & 2\times2+2\times1\\1\times0+3\times1 & 1\times2+3\times1\end{bmatrix}=\begin{bmatrix}2&6\\3&5\end{bmatrix}$.

Q17. Reading the arrows (sender in the row, receiver in the column):

$$M=\begin{bmatrix}0&1&1\\0&0&1\\1&0&0\end{bmatrix}.$$

For example, the element of M^2 in row Q, column P is $0\times0+0\times0+1\times1=1$. Completing all nine elements:

$$M^2=\begin{bmatrix}1&0&1\\1&0&0\\0&1&1\end{bmatrix}.$$

The 1 in row Q, column P means Quinn's task can reach Priya in exactly one two-step route: Quinn emails Rosa, who emails Priya. The zero in row P, column Q means there is **no** two-step route from Priya to Quinn — the only way Priya can send a task to Quinn is her direct email, since neither Quinn nor Rosa can email Quinn.

Q18. Squaring M:

$$M^2=\begin{bmatrix}0\times0+1\times1+0\times0 & 0\times1+1\times0+0\times1 & 0\times0+1\times1+0\times0\\1\times0+0\times1+1\times0 & 1\times1+0\times0+1\times1 & 1\times0+0\times1+1\times0\\0\times0+1\times1+0\times0 & 0\times1+1\times0+0\times1 & 0\times0+1\times1+0\times0\end{bmatrix}=\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}.$$

Adding the matrices element by element,

$$M+M^2=\begin{bmatrix}1&1&1\\1&2&1\\1&1&1\end{bmatrix}.$$

An element of M counts the direct (one-step) links between two huts and the matching element of M^2 counts the two-step links, so each element of $M+M^2$ counts the routes between two huts that use **at most two steps** — that is, at most one relay through the middle hut. Every off-diagonal element of $M+M^2$ is positive: Falls–Gorge and Gorge–Heath can talk directly, and Falls can reach Heath by the two-step route Falls–Gorge–Heath. So every hut can contact every other hut using at most one relay.

Theory

Earlier in this chapter we learned to multiply matrices. This section asks the reverse question: can matrix multiplication be **undone**? In ordinary arithmetic, multiplying by 5 is undone by multiplying by its reciprocal $\frac{1}{5}$, because $5\times\frac{1}{5}=1$. The matrix version of the number 1 is the identity matrix, and the matrix version of the reciprocal is the inverse matrix. Together they give a powerful method for solving simultaneous linear equations.

The identity matrix. The 2×2 **identity matrix** is

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

with 1s on the main diagonal (top left to bottom right) and 0s elsewhere. It behaves like the number 1 in multiplication: for every 2×2 matrix A,

$$A\,I=I\,A=A.$$

There is an identity matrix of every square size — the 3×3 identity has three 1s down its main diagonal — and multiplying by it always leaves a matrix unchanged.

The determinant. For a 2×2 matrix $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, the **determinant** is the number

$$det(A)=a\,d-b\,c,$$

found by multiplying the two main-diagonal elements and subtracting the product of the other two elements. The determinant is a single number, not a matrix, and it decides whether or not A can be inverted.

The inverse matrix. The **inverse** of A, written A^{-1}, is the matrix that multiplies with A to give the identity:

$$A\,A^{-1}=A^{-1}A=I.$$

For a 2×2 matrix the inverse is found with the rule

$$A^{-1}=\frac{1}{a\,d-b\,c}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}.$$

In words: **swap** the two main-diagonal elements, **change the sign** of the other two elements, and multiply the result by $\frac{1}{det(A)}$. If $det(A)=0$ this fraction is undefined, so **no inverse exists**; a matrix whose determinant is zero is called **singular**. After finding an inverse by hand, always multiply it back with A and confirm that the product is I — this simple check catches almost every slip.

Writing simultaneous equations as a matrix equation. A pair of linear equations in two unknowns, written in the standard form $a\,x+b\,y=e$ and $c\,x+d\,y=f$, can be captured in a single matrix equation:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}e\\ f\end{bmatrix},\text{ that is } A\,x=b,$$

where A is the **coefficient matrix**, x is the column of unknowns and b is the column of constants. Multiplying out the left-hand side row by column gives back $a\,x+b\,y$ and $c\,x+d\,y$, so the one matrix equation says exactly the same thing as the two original equations. Line up the x terms, the y terms and the constants before reading off the coefficients.

Solving with the inverse. To solve $Ax=b$, multiply both sides **on the left** by A^{-1}:

$$A^{-1}Ax=A^{-1}b\text{, so }Ix=A^{-1}b\text{, giving }x=A^{-1}b.$$

The order matters — matrix multiplication is not commutative, so A^{-1} must be written in front of b. If $det(A)=0$ the coefficient matrix has no inverse and the method fails: the two lines are parallel (no solution) or are the same line (infinitely many solutions), so there is no unique solution to find.

Using CAS. General Mathematics is a technology-active, open-book course. Define the coefficient matrix A and the constant column b in your CAS calculator's matrix editor; the calculator then returns $det(A)$, A^{-1} (raise the matrix to the power -1) and the product $A^{-1}b$ directly. For systems of three or more equations the same method $x=A^{-1}b$ applies and the CAS handles the larger matrices — by hand you are only expected to invert 2×2 matrices. Learn the by-hand method so you understand what the calculator reports and can spot an unreasonable answer, and remember that a CAS error such as "singular matrix" is telling you that $det(A)=0$.

Worked examples

Example 1 — scaffolded

For the matrix $A=\begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}$, find $det(A)$, find A^{-1}, and check the inverse by computing AA^{-1}.

Working	Comment
$det(A)=ad-bc=5\times 3-2\times 7=15-14=1.$	Main-diagonal product minus the product of the other two elements.
$A^{-1}=\frac{1}{1}\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}=\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}.$	Swap the 5 and the 3, change the signs of the 2 and the 7, and multiply by $\frac{1}{det(A)}$.
$AA^{-1}=\begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}=\begin{bmatrix}15-14 & -10+10\\21-21 & -14+15\end{bmatrix}=\left[\begin{matrix}1\\0\end{matrix}\right.$.	Each element is a row of A multiplied by a column of A^{-1}; the product is the identity, which confirms the inverse.

So $det(A)=1$ and $A^{-1}=\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}$.

Example 2 — scaffolded

Use an inverse matrix to solve the simultaneous equations $3x+2y=8$ and $4x+3y=11$.

Working	Comment
$\begin{bmatrix}3 & 2\\4 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\11\end{bmatrix}$, that is $Ax=b$.	The coefficients of x and y fill the coefficient matrix; the constants form the right-hand column.
$det(A)=3\times 3-2\times 4=9-8=1$, so $A^{-1}=\begin{bmatrix}3 & -2\\-4 & 3\end{bmatrix}$.	The determinant is not zero, so a unique solution exists. Swap, change signs, divide by 1.
$x=A^{-1}b=\begin{bmatrix}3 & -2\\-4 & 3\end{bmatrix}\begin{bmatrix}8\\11\end{bmatrix}=\begin{bmatrix}24-22\\-32+33\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}.$	Multiply the constant column by A^{-1}, with A^{-1} written in front.
Check: $3\times 2+2\times 1=8$ and $4\times 2+3\times 1=11$.	Substituting into both original equations confirms the solution.

So $x=2$ and $y=1$.

Example 3 — unscaffolded

At a Bendigo café, three coffees and four muffins cost $34, while two coffees and three muffins cost $24. Write a matrix equation for this information and use an inverse matrix to find the price of a coffee and the price of a muffin.

Let c and m be the prices, in dollars, of a coffee and a muffin. The two purchases give the equations $3c+4m=34$ and $2c+3m=24$, which as a matrix equation are

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c \\ m \end{bmatrix}=\begin{bmatrix} 34 \\ 24 \end{bmatrix}.$$

The determinant of the coefficient matrix is $3\times3-4\times2=9-8=1$, so $A^{-1}=\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Then

$$\begin{bmatrix} c \\ m \end{bmatrix}=A^{-1}b=\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 34 \\ 24 \end{bmatrix}=\begin{bmatrix} 102-96 \\ -68+72 \end{bmatrix}=\begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

As a check, $3\times6+4\times4=34$ and $2\times6+3\times4=24$, matching both purchases.

∴ a coffee costs $6 and a muffin costs $4.

Example 4 — unscaffolded

Find the value of k for which the matrix $\begin{bmatrix} 2 & k \\ 4 & 6 \end{bmatrix}$ has no inverse, and explain what happens, for that value of k, to the pair of simultaneous equations $2x+ky=8$ and $4x+6y=20$.

A matrix has no inverse exactly when its determinant is zero. Here $det=2\times6-k\times4=12-4k$, and $12-4k=0$ gives $k=3$. When $k=3$ the equations are $2x+3y=8$ and $4x+6y=20$. Doubling the first equation gives $4x+6y=16$, so the two equations have the same left-hand side but different right-hand sides: they describe parallel lines that never meet, and no pair (x, y) can satisfy both at once. The matrix method cannot be used because the coefficient matrix is singular, and indeed no solution exists.

∴ $k=3$; the equations then represent parallel lines, so the system has no solution.

Practice questions

Scaffolded

Q1. Let $A=\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. (a) Write down the 2×2 identity matrix I. (b) Compute the products AI and IA, and confirm that both equal A. (c) Find $det(A)$. (d) Find A^{-1}.

Q2. Find the determinant of each matrix. (a) $\begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ (e) One of these four matrices is singular. Which one, and how do you know?

Q3. Let $A=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 8 & 6 \\ 2 & 2 \end{bmatrix}$. (a) Show that $det(A)=1$ and find A^{-1}. (b) Check your answer to part (a) by computing AA^{-1}. (c) Find $det(B)$ and B^{-1}.

Q4. Consider the simultaneous equations $3x+4y=18$ and $x+2y=8$. (a) Write the pair as a matrix equation $Ax=b$. (b) Find $det(A)$ and A^{-1}. (c) Solve the system using $x=A^{-1}b$. (d) Check your solution by substituting into both original equations.

Q5. At a school fete in Geelong, one family buys four sausages in bread and two bottled drinks for $22, and another family buys three sausages in bread and two bottled drinks for $19. Let s be the price of a sausage in bread and d the price of a bottled drink, both in dollars. (a) Write two equations for the two purchases. (b) Write the pair of equations as a matrix equation. (c) Solve the matrix equation using an inverse matrix. (d) Interpret your solution in context: what does each item cost?

Q6. Let $M=\begin{bmatrix} k & 10 \\ 2 & 5 \end{bmatrix}$. (a) Find $det(M)$ in terms of k. (b) For what value of k does M have no inverse? (c) When $k=6$, find M^{-1}.

Unscaffolded

Q7. Find the determinant and the inverse of $\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$.

Q8. Find the determinant and the inverse of $\begin{bmatrix} 6 & 4 \\ 1 & 1 \end{bmatrix}$.

Q9. Show that $P=\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$ is the inverse of $A=\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$ by computing suitable matrix products.

Q10. Two of the matrices $P=\begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}$, $Q=\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$ and $R=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ are singular. Identify them, and find the inverse of the matrix that is not singular.

Q11. Use an inverse matrix to solve the simultaneous equations $5x+2y=24$ and $2x+y=10$.

Q12. Use an inverse matrix to solve the simultaneous equations $3x+4y=10$ and $2x+2y=6$.

Q13. An electrician in Ballarat charges a fixed call-out fee plus an hourly rate. A job that took 2 hours cost $190 in total, and a job that took 5 hours cost $340 in total. Write a pair of equations for the call-out fee f and the hourly rate r (both in dollars), express the pair as a matrix equation, and solve it with an inverse matrix. Interpret your answer in context.

Q14. At a suburban football match, two adults and three children pay $90 altogether for entry, while three adults and five children pay $140 altogether. Use the inverse matrix method to find the adult entry price and the child entry price.

Q15. At a CWA bake sale, two pies and four lamingtons cost $28, and three pies and six lamingtons cost $42. A student tries to find the price of each item by the inverse matrix method. Explain why the method fails here and what that says about the information given. What kind of extra information would allow the two prices to be found?

Q16. Find the value of k for which $\begin{bmatrix} 5 & k \\ 2 & 6 \end{bmatrix}$ has no inverse.

Q17. A stall at the Queen Victoria Market sells potted herbs in small, medium and large pots. Let x, y and z be the prices, in dollars, of a small, a medium and a large pot. Two small pots, one medium pot and one large pot cost $25; one small, two medium and one large cost $28; and one small, one medium and two large cost $31. Write this information as a matrix equation and solve it using CAS.

Q18. Solve the matrix equation $\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 26 \\ 19 \end{bmatrix}$.

Answers

Q1. (a) $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. (b) $A\,I=I\,A=A$. (c) $det(A)=2$. (d) $A^{-1}=\frac{1}{2}\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$. **Q2.** (a) 4. (b) 1. (c) 0. (d) 1. (e) The matrix in (c): its determinant is 0. **Q3.** (a) $A^{-1}=\begin{bmatrix}2&-5\\-1&3\end{bmatrix}$. (b) $A\,A^{-1}=I$. (c) $det(B)=4$, $B^{-1}=\frac{1}{4}\begin{bmatrix}2&-6\\-2&8\end{bmatrix}$. **Q4.** (a) $\begin{bmatrix}3&4\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}18\\8\end{bmatrix}$. (b) $det(A)=2$, $A^{-1}=\frac{1}{2}\begin{bmatrix}2&-4\\-1&3\end{bmatrix}$. (c) $x=2$, $y=3$. (d) Both equations are satisfied. **Q5.** (a) $4s+2d=22$, $3s+2d=19$. (b) $\begin{bmatrix}4&2\\3&2\end{bmatrix}\begin{bmatrix}s\\d\end{bmatrix}=\begin{bmatrix}22\\19\end{bmatrix}$. (c) $s=3$, $d=5$. (d) A sausage in bread costs \$3 and a bottled drink costs \$5. **Q6.** (a) $det(M)=5k-20$. (b) $k=4$. (c) $M^{-1}=\frac{1}{10}\begin{bmatrix}5&-10\\-2&6\end{bmatrix}$. **Q7.** $det=1$; inverse $=\begin{bmatrix}1&-4\\-2&9\end{bmatrix}$. **Q8.** $det=2$; inverse $=\frac{1}{2}\begin{bmatrix}1&-4\\-1&6\end{bmatrix}$. **Q9.** $A\,P=P\,A=I$, so $P=A^{-1}$. **Q10.** P and Q are singular (each has determinant 0); $det(R)=-2$ and $R^{-1}=-\frac{1}{2}\begin{bmatrix}5&-3\\-4&2\end{bmatrix}$. **Q11.** $x=4$, $y=2$. **Q12.** $x=2$, $y=1$. **Q13.** $f=90$, $r=50$: the call-out fee is \$90 and the hourly rate is \$50 per hour. **Q14.** Adult \$30, child \$10. **Q15.** The determinant is 0, so the coefficient matrix is singular and has no inverse; the second purchase is exactly 1.5 times the first, so it adds no new information. A purchase that is not a multiple of the first (for example, the price of a single pie) would be needed. **Q16.** $k=15$. **Q17.** $x=4$, $y=7$, $z=10$: small \$4, medium \$7, large \$10. **Q18.** $x=5$, $y=2$.

Fully-worked solutions

Q1. (a) $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. (b) $A\,I=\begin{bmatrix}3\times1+1\times0&3\times0+1\times1\\4\times1+2\times0&4\times0+2\times1\end{bmatrix}=\begin{bmatrix}3&1\\4&2\end{bmatrix}=A$, and similarly $I\,A=\begin{bmatrix}1\times3+0\times4&1\times1+0\times2\\0\times3+1\times4&0\times1+1\times2\end{bmatrix}=A$. Multiplying by the identity leaves A unchanged. (c) $det(A)=3\times2-1\times4=6-4=2$. (d) $A^{-1}=\frac{1}{2}\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$ (swap the 3 and the 2, change the signs of the 1 and the 4, divide by the determinant).

Q2. (a) $5\times2-3\times2=10-6=4$. (b) $7\times1-2\times3=7-6=1$. (c) $4\times3-6\times2=12-12=0$. (d) $(-2)\times(-3)-5\times1=6-5=1$. (e) The matrix in (c) is singular, because its determinant is 0; a matrix has an inverse only when its determinant is non-zero.

Q3. (a) $det(A)=3\times2-5\times1=6-5=1$, so $A^{-1}=\frac{1}{1}\begin{bmatrix}2&-5\\-1&3\end{bmatrix}=\begin{bmatrix}2&-5\\-1&3\end{bmatrix}$. (b) $A\,A^{-1}=\begin{bmatrix}3&5\\1&2\end{bmatrix}\begin{bmatrix}2&-5\\-1&3\end{bmatrix}=\begin{bmatrix}6-5&-15+15\\2-2&-5+6\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$, which confirms the inverse. (c) $det(B)=8\times2-6\times2=16-12=4$, so $B^{-1}=\frac{1}{4}\begin{bmatrix}2&-6\\-2&8\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{3}{2}\\-\frac{1}{2}&2\end{bmatrix}$.

Q4. (a) $\begin{bmatrix}3&4\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}18\\8\end{bmatrix}$. (b) $det(A)=3\times2-4\times1=6-4=2$, so $A^{-1}=\frac{1}{2}\begin{bmatrix}2&-4\\-1&3\end{bmatrix}$. (c) $x=A^{-1}b=\frac{1}{2}\begin{bmatrix}2&-4\\-1&3\end{bmatrix}\begin{bmatrix}18\\8\end{bmatrix}=\frac{1}{2}\begin{bmatrix}36-32\\-18+24\end{bmatrix}=\frac{1}{2}\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$, so $x=2$ and $y=3$. (d) $3\times2+4\times3=18$ and $2+2\times3=8$: both equations are satisfied.

Q5. (a) The purchases give $4s+2d=22$ and $3s+2d=19$. (b) $\begin{bmatrix}4 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}s\\d\end{bmatrix}=\begin{bmatrix}22\\19\end{bmatrix}$. (c) The determinant is $4\times2-2\times3=8-6=2$, so $A^{-1}=\frac{1}{2}\begin{bmatrix}2 & -2\\-3 & 4\end{bmatrix}$. Then $\begin{bmatrix}s\\d\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & -2\\-3 & 4\end{bmatrix}\begin{bmatrix}22\\19\end{bmatrix}=\frac{1}{2}\begin{bmatrix}44-38\\-66+76\end{bmatrix}=\frac{1}{2}\begin{bmatrix}6\\10\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$. (d) A sausage in bread costs \$3 and a bottled drink costs \$5. As a check, $4\times3+2\times5=22$ and $3\times3+2\times5=19$, matching both purchases.

Q6. (a) $det(M)=k\times5-10\times2=5k-20$. (b) M has no inverse when $det(M)=0$: $5k-20=0$ gives $k=4$. (c) When $k=6$, $det(M)=30-20=10$, so $M^{-1}=\frac{1}{10}\begin{bmatrix}5 & -10\\-2 & 6\end{bmatrix}$.

Q7. $det=9\times1-4\times2=9-8=1$, so the inverse is $\frac{1}{1}\begin{bmatrix}1 & -4\\-2 & 9\end{bmatrix}=\begin{bmatrix}1 & -4\\-2 & 9\end{bmatrix}$.

Q8. $det=6\times1-4\times1=6-4=2$, so the inverse is $\frac{1}{2}\begin{bmatrix}1 & -4\\-1 & 6\end{bmatrix}$.

Q9. $AP=\begin{bmatrix}2 & 7\\1 & 4\end{bmatrix}\begin{bmatrix}4 & -7\\-1 & 2\end{bmatrix}=\begin{bmatrix}8-7 & -14+14\\4-4 & -7+8\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$, and $PA=\begin{bmatrix}4 & -7\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & 7\\1 & 4\end{bmatrix}=\begin{bmatrix}8-7 & 28-28\\-2+2 & -7+8\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$. Since the product each way is the identity, $P=A^{-1}$.

Q10. $det(P)=3\times3-9\times1=0$ and $det(Q)=5\times4-2\times10=0$, so P and Q are singular. $det(R)=2\times5-3\times4=10-12=-2$, so R has an inverse: $R^{-1}=\frac{1}{-2}\begin{bmatrix}5 & -3\\-4 & 2\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}5 & -3\\-4 & 2\end{bmatrix}=\begin{bmatrix}-\frac{5}{2} & \frac{3}{2}\\2 & -1\end{bmatrix}$.

Q11. As a matrix equation, $\begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}24\\10\end{bmatrix}$. The determinant is $5\times1-2\times2=1$, so $A^{-1}=\begin{bmatrix}1 & -2\\-2 & 5\end{bmatrix}$ and $x=A^{-1}b=\begin{bmatrix}1 & -2\\-2 & 5\end{bmatrix}\begin{bmatrix}24\\10\end{bmatrix}=\begin{bmatrix}24-20\\-48+50\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$. So $x=4$ and $y=2$; checking, $5\times4+2\times2=24$ and $2\times4+2=10$.

Q12. As a matrix equation, $\begin{bmatrix}3 & 4\\2 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10\\6\end{bmatrix}$. The determinant is $3\times2-4\times2=6-8=-2$ (a determinant may be negative — the inverse still exists because it is non-zero), so $A^{-1}=\frac{1}{-2}\begin{bmatrix}2 & -4\\-2 & 3\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}2 & -4\\-2 & 3\end{bmatrix}$. Then $x=-\frac{1}{2}\begin{bmatrix}2 & -4\\-2 & 3\end{bmatrix}\begin{bmatrix}10\\6\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}20-24\\-20+18\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-4\\-2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$. So $x=2$ and $y=1$; checking, $3\times2+4\times1=10$ and $2\times2+2\times1=6$.

Q13. The two jobs give $f+2r=190$ and $f+5r=340$, or $\begin{bmatrix}1 & 2\\1 & 5\end{bmatrix}\begin{bmatrix}f\\r\end{bmatrix}=\begin{bmatrix}190\\340\end{bmatrix}$. The determinant is $1\times5-2\times1=3$, so $A^{-1}=\frac{1}{3}\begin{bmatrix}5 & -2\\-1 & 1\end{bmatrix}$ and $\begin{bmatrix}f\\r\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5 & -2\\-1 & 1\end{bmatrix}\begin{bmatrix}190\\340\end{bmatrix}=\frac{1}{3}\begin{bmatrix}950-680\\-190+340\end{bmatrix}=\frac{1}{3}\begin{bmatrix}270\\150\end{bmatrix}=\begin{bmatrix}90\\50\end{bmatrix}$. In context: the electrician charges a call-out fee of \$90 plus \$50 per hour. As a check, $90+2\times50=190$ and $90+5\times50=340$, matching both job costs.

Q14. With a and c the adult and child prices in dollars, $2a+3c=90$ and $3a+5c=140$, so $\begin{bmatrix}2 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}90\\140\end{bmatrix}$. The determinant is $2\times5-3\times3=10-9=1$, so $A^{-1}=\begin{bmatrix}5 & -3\\-3 & 2\end{bmatrix}$ and $\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}5 & -3\\-3 & 2\end{bmatrix}\begin{bmatrix}90\\140\end{bmatrix}=\begin{bmatrix}450-420\\-270+280\end{bmatrix}=\begin{bmatrix}30\\10\end{bmatrix}$. So adult entry costs \$30 and child entry costs \$10; checking, $2\times30+3\times10=90$ and $3\times30+5\times10=140$.

Q15. With p and l the prices of a pie and a lamington, the purchases give $2p+4l=28$ and $3p+6l=42$, with coefficient matrix $\begin{bmatrix}2&4\\3&6\end{bmatrix}$. Its determinant is $2\times6-4\times3=12-12=0$, so the matrix is singular and has no inverse: the method $x=A^{-1}b$ cannot be used. The reason is that the second purchase is exactly 1.5 times the first ($3=1.5\times2$, $6=1.5\times4$ and $42=1.5\times28$), so the second equation carries no new information — both reduce to $p+2l=14$, which infinitely many price pairs satisfy. To pin down the two prices we would need a purchase that is **not** a multiple of the first — for example, knowing the price of a single pie, or a purchase such as one pie and one lamington.

Q16. The matrix has no inverse when its determinant is zero: $det=5\times6-k\times2=30-2k$, and $30-2k=0$ gives $k=15$.

Q17. The three bundles give $2x+y+z=25$, $x+2y+z=28$ and $x+y+2z=31$, or

$$\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}25\\28\\31\end{bmatrix}.$$

This is a 3×3 system, so we solve it with CAS. The calculator gives $det(A)=4$ (so a unique solution exists) and

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}b=\frac{1}{4}\begin{bmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{bmatrix}\begin{bmatrix}25\\28\\31\end{bmatrix}=\frac{1}{4}\begin{bmatrix}16\\28\\40\end{bmatrix}=\begin{bmatrix}4\\7\\10\end{bmatrix}.$$

So a small pot costs \$4, a medium pot \$7 and a large pot \$10. As a check, $2\times4+7+10=25$, $4+2\times7+10=28$ and $4+7+2\times10=31$, matching all three bundles.

Q18. The coefficient matrix has determinant $4\times2-3\times3=8-9=-1$, so $A^{-1}=\frac{1}{-1}\begin{bmatrix}2&-3\\-3&4\end{bmatrix}=\begin{bmatrix}-2&3\\3&-4\end{bmatrix}$. Then $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2&3\\3&-4\end{bmatrix}\begin{bmatrix}26\\19\end{bmatrix}=\begin{bmatrix}-52+57\\78-76\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}$, so $x=5$ and $y=2$. Checking, $4\times5+3\times2=26$ and $3\times5+2\times2=19$.

Theory

Many everyday situations involve a system that, at each stage, is in exactly one of a small number of **states**, and at regular time steps its members move between those states. If the **proportion** that moves from one state to another over a step stays the same from step to step, and depends only on the state a member is in **now** (not on how it got there), the whole process can be described with a matrix. This section introduces that model, using it for population movement, market share and land use.

States and the transition matrix. Suppose a system has two states, labelled A and B. Over one step a fixed proportion of those in each state stays, and the rest move to the other state. We collect these proportions in a **transition matrix** T. The element in **row** i**, column** j of T is the proportion of those currently in **state** j who are in **state** i after the next step — so each column describes where the members of one state go. For example,

$$T=\begin{bmatrix}0.85 & 0.10\\ 0.15 & 0.90\end{bmatrix},$$

where column 1 (‘from A’) says that each step 0.85 of state A stays in A and 0.15 moves to B, and column 2 (‘from B’) says that 0.10 of state B moves to A and 0.90 stays in B.

Columns add to 1. Everyone in a state must go somewhere — either stay or move — so the proportions leaving a state account for the whole of it. Hence **every column of a transition matrix adds to** 1. (The rows need not add to 1 .) Checking the column sums is the quickest way to test whether a matrix could be a transition matrix.

The state vector. The **state vector** S_0 records the starting distribution across the states, written as a column in the same order as the matrix — either as actual amounts (people, hectares, vehicles) or as proportions of the whole. For two states,

$$S_0=\begin{bmatrix}a\\ b\end{bmatrix}.$$

Stepping forward. To find the distribution after one step, multiply the state vector by the transition matrix:

$$S_1=T\,S_0.$$

The row-by-column rule then does exactly the right bookkeeping: each new state total is built from the correct fraction of every old state total. For instance, with $S_0=\begin{bmatrix}4000\\ 2000\end{bmatrix}$,

$$S_1=T\,S_0=\begin{bmatrix}0.85\times 4000+0.10\times 2000\\ 0.15\times 4000+0.90\times 2000\end{bmatrix}=\begin{bmatrix}3600\\ 2400\end{bmatrix}.$$

Repeating the step,

$$S_2=T\,S_1, S_3=T\,S_2, \ldots, S_{n+1}=T\,S_n.$$

Because $S_2=T\,S_1=T\left(T\,S_0\right)=T^2\,S_0$, and likewise for later steps, the distribution after n steps is also given directly by a power of T:

$$S_n=T^n\,S_0.$$

So you can step forward one stage at a time, or jump straight to stage n using T^n. A useful check is that the state totals are **conserved**: because each column of T adds to 1, the elements of every S_n add to the same total as S_0 (above, each vector totals 6000).

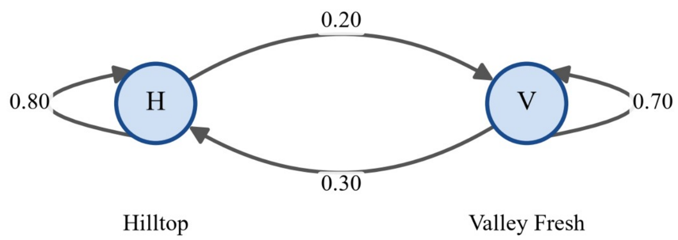
Long-run behaviour. If we keep iterating $S_{n+1} = T\,S_n$ for many steps, the state vector often **settles down**: successive vectors change less and less and approach a fixed vector that stays (almost) unchanged under a further step. This limiting distribution is called the **steady state**, or long-run state. A transition matrix is called **regular** if some power of it has every element greater than zero; a **regular** transition matrix always has a single steady state that the system settles into whatever the starting vector — which is why the examples here approach the same long-run distribution no matter what S_0 is. In this course we find the steady state **informally** — by iterating many steps and observing the value the state vector approaches — with no special algebra required.

Using CAS. General Mathematics is a technology-active, open-book course. Enter T and S_0 into the matrix editor; multiply T by S_0 to get the next state, then multiply the running vector by T again for each further step. To reach stage n in one move, raise T to the power n and multiply by S_0. To study the long run, compute S_n for a large n (or a high power of T) and watch the elements stabilise. Keep the by-hand row-by-column method in mind so that you can check the first few steps and interpret each number in context.

Worked examples

Example 1 — scaffolded

Two supermarkets in a country town, **Hilltop Grocers** (H) and **Valley Fresh** (V), compete for the town's 1000 regular weekly shoppers. Valley Fresh has just opened, so at the start all 1000 shoppers use Hilltop. Each week 80 % of Hilltop's shoppers return to Hilltop and 20 % switch to Valley Fresh, while 30 % of Valley Fresh's shoppers switch to Hilltop and 70 % stay. The movement is shown below.



Write down the transition matrix T and the initial state vector S_0, then find the number of shoppers at each supermarket after one week (S_1) and after two weeks (S_2).

Working	Comment
Order the states H then V; the columns are 'from H' and 'from V': $T = \begin{bmatrix} 0.80 & 0.30 \\ 0.20 & 0.70 \end{bmatrix}$.	Column 1: of Hilltop's shoppers, 0.80 stay and 0.20 move to Valley. Column 2: of Valley's, 0.30 move to Hilltop and 0.70 stay.
Each column adds to 1: $0.80 + 0.20 = 1$ and $0.30 + 0.70 = 1$.	A valid transition matrix — every shopper is accounted for.
$S_0 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$.	All 1000 shoppers start at Hilltop; none at Valley yet.
$S_1 = T\,S_0 = \begin{bmatrix} 0.80 \times 1000 + 0.30 \times 0 \\ 0.20 \times 1000 + 0.70 \times 0 \end{bmatrix} = \begin{bmatrix} 800 \\ 200 \end{bmatrix}$.	Row-by-column: work across each row of T and down S_0.
$S_2 = T\,S_1 = \begin{bmatrix} 0.80 \times 800 + 0.30 \times 200 \\ 0.20 \times 800 + 0.70 \times 200 \end{bmatrix} = \begin{bmatrix} 700 \\ 300 \end{bmatrix}$.	Multiply the new state vector by T again.

So after one week 800 shoppers use Hilltop and 200 use Valley Fresh; after two weeks the split is 700 and 300. (Each vector still totals 1000, as it must.)

Example 2 — scaffolded

In a large workplace, staff commute either by **car** (C) or by **public transport** (P). From one week to the next, 70 % of those who drive keep driving and 30 % switch to public transport, while 20 % of public-transport users switch to driving and 80 % keep using public transport. At present 80 % of staff drive and 20 % use public transport. Taking the state vector as proportions, find the proportions after one, two and three weeks, and describe what happens in the long run.

Working	Comment
$T=\begin{bmatrix}0.70 & 0.20\\0.30 & 0.80\end{bmatrix}, S_0=\begin{bmatrix}0.80\\0.20\end{bmatrix}.$	States ordered C, P; columns 'from C' and 'from P'. The proportions add to 1.
$S_1=T\,S_0=\begin{bmatrix}0.70\times0.80+0.20\times0.20\\0.30\times0.80+0.80\times0.20\end{bmatrix}=\begin{bmatrix}0.60\\0.40\end{bmatrix}.$	$0.56+0.04=0.60$ and $0.24+0.16=0.40$.
$S_2=T\,S_1=\begin{bmatrix}0.70\times0.60+0.20\times0.40\\0.30\times0.60+0.80\times0.40\end{bmatrix}=\begin{bmatrix}0.50\\0.50\end{bmatrix}.$	Multiply the running vector by T again.
$S_3=T\,S_2=\begin{bmatrix}0.70\times0.50+0.20\times0.50\\0.30\times0.50+0.80\times0.50\end{bmatrix}=\begin{bmatrix}0.45\\0.55\end{bmatrix}.$	The car proportion keeps falling towards a limit.
Continuing to iterate, the vectors approach $\begin{bmatrix}0.40\\0.60\end{bmatrix}$ and then barely change.	Many steps reveal the steady state, found here by observation.

So the proportion driving falls week by week — $0.80, 0.60, 0.50, 0.45, \ldots$ — and in the long run about 40 % drive while 60 % use public transport.

Example 3 — unscaffolded

A regional council models the land in a growth corridor as being in one of three uses each year: **residential** (R), **commercial** (C) and **open space** (O). Each year 90 % of residential land stays residential and 10 % becomes commercial; 95 % of commercial land stays commercial and 5 % becomes open space; and 80 % of open space stays open space while 20 % becomes residential. At present the corridor has 200 hectares residential, 100 hectares commercial and 300 hectares open space. Find the areas in each use after one year and after two years.

Ordering the states R, C, O, each column of the transition matrix lists where one use goes:

$$T=\begin{bmatrix}0.90 & 0 & 0.20\\0.10 & 0.95 & 0\\0 & 0.05 & 0.80\end{bmatrix}, S_0=\begin{bmatrix}200\\100\\300\end{bmatrix}.$$

Each column adds to 1. After one year,

$$S_1=T\,S_0=\begin{bmatrix}0.90\times200+0\times100+0.20\times300\\0.10\times200+0.95\times100+0\times300\\0\times200+0.05\times100+0.80\times300\end{bmatrix}=\begin{bmatrix}240\\115\\245\end{bmatrix}.$$

After two years, multiply again by T:

$$S_2=T\,S_1=\begin{bmatrix}0.90\times240+0\times115+0.20\times245\\0.10\times240+0.95\times115+0\times245\\0\times240+0.05\times115+0.80\times245\end{bmatrix}=\begin{bmatrix}265\\133.25\\201.75\end{bmatrix}.$$

Both vectors still total 600 hectares.

$\therefore$ after one year the corridor has 240 ha residential, 115 ha commercial and 245 ha open space; after two years, 265 ha residential, 133.25 ha commercial and 201.75 ha open space.

Example 4 — unscaffolded

Two rideshare apps, **RideOn** (R) and **CityCab** (C), share a city's users. Each month 91% of RideOn's users stay with RideOn and 9% switch to CityCab, while 21% of CityCab's users switch to RideOn and 79% stay. The two apps currently hold half the market each. Taking the state vector as market-share proportions, find the shares after one and two months, and investigate the long-run market share.

The states are ordered R, C, so

$$T=\begin{bmatrix}0.91 & 0.21\\ 0.09 & 0.79\end{bmatrix}, S_0=\begin{bmatrix}0.5\\ 0.5\end{bmatrix}.$$

Then

$$S_1=T\,S_0=\begin{bmatrix}0.91\times 0.5+0.21\times 0.5\\ 0.09\times 0.5+0.79\times 0.5\end{bmatrix}=\begin{bmatrix}0.56\\ 0.44\end{bmatrix},$$

and

$$S_2=T\,S_1=\begin{bmatrix}0.91\times 0.56+0.21\times 0.44\\ 0.09\times 0.56+0.79\times 0.44\end{bmatrix}=\begin{bmatrix}0.602\\ 0.398\end{bmatrix}.$$

RideOn's share is creeping upward. Continuing to iterate $S_{n+1}=T\,S_n$ for many months, the vector settles at $\begin{bmatrix}0.7\\ 0.3\end{bmatrix}$ and then stops changing. Repeating the iteration from a very different start, such as $\begin{bmatrix}0.9\\ 0.1\end{bmatrix}$, leads to the **same** limit, so the long-run share does not depend on today's split.

$\therefore$ in the long run RideOn holds about 70% of the market and CityCab about 30%, whatever the current shares.

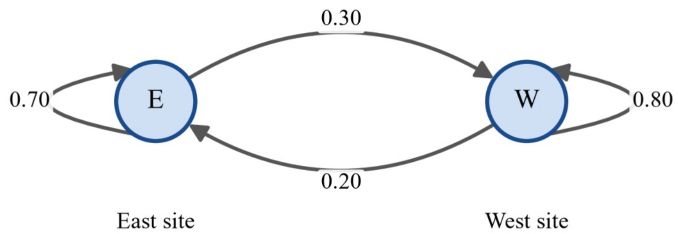
Practice questions

Scaffolded

Q1. A town has two cinemas, the **Star** (S) and the **Vista** (V). The Vista has just opened, so all 500 regular movie-goers currently attend the Star. Each week 90% of the Star's patrons return to the Star and 10% switch to the Vista, while 20% of the Vista's patrons switch to the Star and 80% stay. (a) Write down the transition matrix T, with states in the order S, V. (b) Show that each column of T adds to 1. (c) Write down S_0 and find S_1, the numbers at each cinema after one week. (d) Find S_2, the numbers after two weeks.

Q2. At a school canteen, students each week either **buy** their lunch (B) or **bring** it from home (L). Each week 70% of buyers buy again and 30% switch to bringing, while 10% of bringers buy and 90% bring again. This week the students are split half and half, so $S_0=\begin{bmatrix}0.5\\ 0.5\end{bmatrix}$ (states in the order B, L). (a) Write down the transition matrix T. (b) Find S_1 and S_2. (c) Find S_3. (d) By continuing the iteration, state the long-run proportions who buy and who bring, and interpret them in context.

Q3. A wildlife sanctuary has two feeding sites, **East** (E) and **West** (W). The west site has just been set up, so all 600 waterbirds currently feed at the east site. Each week the birds move between the sites as shown below.



(a) Write down the transition matrix T (states in the order E, W) and the initial state vector S_0.

(b) Find S_1, the number of birds at each site after one week.

(c) Find S_2.

(d) Interpret the first element of S_2 in context.

Q4. Three mobile-phone networks — **Alpha** (A), **Beta** (B) and **Cee** (C) — share a region's customers. Each month the customers move according to

$$T = \begin{bmatrix} 0.80 & 0.15 & 0.10 \\ 0.10 & 0.80 & 0.10 \\ 0.10 & 0.05 & 0.80 \end{bmatrix},$$

with states in the order A, B, C. At present (in thousands of customers) $S_0 = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$. (a) Verify that each column of T adds to 1. (b) Find S_1, the numbers (in thousands) on each network after one month. (c) Find S_2. (d) Interpret the first element of S_1 in context.

Q5. A paddock is either **cropped** (C) or left **fallow** (F) each season. The transition matrix (states in the order C, F) is $T = \begin{bmatrix} 0.30 & 0.60 \\ 0.70 & 0.40 \end{bmatrix}$, and a district has $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ paddocks in each state. (a) Find T^2. (b) Show that each column of T^2 also adds to 1. (c) Use $S_2 = T^2 S_0$ to find the number of cropped and fallow paddocks after two seasons. (d) Find $S_1 = T\,S_0$ and then $T\,S_1$, and check that this agrees with part (c).

Q6. Two meal-kit delivery services, **FreshBox** (F) and **QuickPlate** (Q), share a city's subscribers equally at present, so $S_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ (states in the order F, Q). Each month $85\,\%$ of FreshBox subscribers stay and $15\,\%$ switch to QuickPlate, while $10\,\%$ of QuickPlate subscribers switch to FreshBox and $90\,\%$ stay. (a) Write down T and explain why each of its columns must add to 1. (b) Find S_1. (c) Find S_2. (d) By iterating many months, the market share settles down. State the long-run share of each service and interpret it.

Unscaffolded

Q7. Motorists in a suburb buy petrol from either **BowserCo** (B) or **FuelFast** (F). Each week $70\,\%$ of BowserCo's customers return and $30\,\%$ switch to FuelFast, while $25\,\%$ of FuelFast's customers switch to BowserCo and $75\,\%$ stay. This week 1000 motorists use BowserCo and 400 use FuelFast. Taking the states in the order B, F, find the numbers at each station after one week and after two weeks.

Q8. At a school, students choose between two sports drinks, **Zap** (Z) and **Volt** (V). Each week $80\,\%$ of Zap drinkers stay with Zap and $20\,\%$ switch to Volt, while $5\,\%$ of Volt drinkers switch to Zap and $95\,\%$ stay. The drinks currently have equal shares. Using proportions and states in the order Z, V, find the shares after one and two weeks, and by iterating many weeks state the long-run share of each drink.

Q9. In a catchment, land is classed as **forest** (F), **grassland** (G) or **cleared** (C). Each decade the areas change according to

$$T=\begin{bmatrix}0.90 & 0.05 & 0.10\\ 0.10 & 0.85 & 0.20\\ 0 & 0.10 & 0.70\end{bmatrix},$$

with states in the order F, G, C. The catchment currently has 400 hectares of forest, 300 of grassland and 300 cleared. Find the areas after one decade and after two decades.

Q10. Two airlines, **SkyLink** (S) and **AeroJet** (A), compete on a route. Each quarter $90\,\%$ of SkyLink's passengers rebook with SkyLink and $10\,\%$ switch to AeroJet, while $20\,\%$ of AeroJet's passengers switch to SkyLink and $80\,\%$ stay. SkyLink currently holds $60\,\%$ of the market, and the transition matrix has states in the order S, A. (a) The transition matrix contains the element 0.20. Explain what this number means in context. (b) Using proportions, find the market shares after one and two quarters. (c) After many quarters the shares settle near $\begin{bmatrix}0.667\\ 0.333\end{bmatrix}$. Interpret this long-run result in context.

Q11. A city is divided into an **inner** zone (I) and an **outer** zone (O). Each year $80\,\%$ of inner-zone residents stay and $20\,\%$ move to the outer zone, while $10\,\%$ of outer-zone residents move to the inner zone and $90\,\%$ stay. The transition matrix has states in the order I, O, and currently each zone has 5000 residents. (a) Find T^2 and T^3. (b) Use $S_3=T^3 S_0$ to find the number of residents in each zone after three years.

Q12. Two cafes, **Bean Scene** (B) and **Grind House** (G), compete for regular customers, with transition matrix $T=\begin{bmatrix}0.70 & 0.20\\ 0.30 & 0.80\end{bmatrix}$ (states in the order B, G). (a) Starting from $S_0=\begin{bmatrix}1\\ 0\end{bmatrix}$ (everyone at Bean Scene), find S_1 and S_2. (b) Starting instead from $S_0=\begin{bmatrix}0\\ 1\end{bmatrix}$ (everyone at Grind House), find S_1 and S_2. (c) Both starting vectors lead to the same long-run share $\begin{bmatrix}0.4\\ 0.6\end{bmatrix}$. What does this tell you about the effect of the starting distribution on the long-run state?

Q13. Three supermarkets — **ThriftMart** (T), **GreenGrocer** (G) and **MegaFoods** (M) — share a suburb's shoppers. Each month the shoppers move according to

$$P=\begin{bmatrix}0.70 & 0.10 & 0.20\\ 0.20 & 0.80 & 0.20\\ 0.10 & 0.10 & 0.60\end{bmatrix},$$

with states in the order T, G, M. (Here T is a store name; the transition matrix is called P.) At present, in thousands of shoppers, $S_0=\begin{bmatrix}40\\ 40\\ 20\end{bmatrix}$. Find the numbers (in thousands) at each supermarket after one month and after two months, and comment on how MegaFoods' total changes.

Q14. A student writes down two matrices,

$$P=\begin{bmatrix}0.6 & 0.3\\ 0.4 & 0.7\end{bmatrix}, Q=\begin{bmatrix}0.6 & 0.3\\ 0.3 & 0.7\end{bmatrix},$$

and claims that both could be transition matrices. Decide which of P and Q can be a transition matrix, giving a reason. Then, using the valid matrix with $S_0=\begin{bmatrix}200\\ 300\end{bmatrix}$, find S_1.

Q15. Two brands of coffee pod, **PodPerfect** (P) and **BrewBox** (B), share a market equally at present. Each month $90\,\%$ of PodPerfect's customers stay and $10\,\%$ switch to BrewBox, while $30\,\%$ of BrewBox's customers switch to

PodPerfect and $70\,\%$ stay. Using proportions and states in the order P, B, find the shares after one and two months, and by iterating state the long-run share of each brand. Which brand dominates in the long run?

Q16. A national park classifies its land as **native vegetation** (N), **pasture** (P) or **crop** (C). Each decade the areas change according to

$$T=\begin{bmatrix} 0.90 & 0.05 & 0 \\ 0.10 & 0.80 & 0.10 \\ 0 & 0.15 & 0.90 \end{bmatrix},$$

with states in the order N, P, C. The park currently has 500 hectares of native vegetation, 300 of pasture and 200 of crop. Find the areas after one decade (S_1) and after two decades ($S_2=T^2\,S_0$, or equivalently $T\,S_1$).

Q17. A coastal shire has residents in two areas, **Seaside** (S) and **Hinterland** (H). Each year $95\,\%$ of Seaside residents stay and $5\,\%$ move to the Hinterland, while $10\,\%$ of Hinterland residents move to Seaside and $90\,\%$ stay. At present Seaside has 6000 residents and the Hinterland has $12\,000$. Taking states in the order S, H, find the populations after one year and after two years, and, given that the populations settle in the long run, state the long-run distribution.

Q18. A council runs a kerbside recycling campaign. Each household either **participates** (P) or does **not** (N). From one year to the next, $90\,\%$ of participating households keep participating and $10\,\%$ stop, while $40\,\%$ of non-participating households start participating and $60\,\%$ do not. At the start of the campaign $30\,\%$ of households participate, so $S_0=\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$ (states in the order P, N). (a) Write down the transition matrix and find the participation proportions after one and two years. (b) By iterating many years, state the long-run proportion of households that participate. (c) Explain what the steady state means for the council, and why the campaign can be judged worthwhile.

Answers

Q1. (a) $T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$. (b) $0.9+0.1=1$ and $0.2+0.8=1$. (c) $S_0=\begin{bmatrix}500\\0\end{bmatrix}$, $S_1=\begin{bmatrix}450\\50\end{bmatrix}$. (d) $S_2=\begin{bmatrix}415\\85\end{bmatrix}$. **Q2.** (a) $T=\begin{bmatrix}0.7 & 0.1\\0.3 & 0.9\end{bmatrix}$. (b) $S_1=\begin{bmatrix}0.4\\0.6\end{bmatrix}$, $S_2=\begin{bmatrix}0.34\\0.66\end{bmatrix}$. (c) $S_3=\begin{bmatrix}0.304\\0.696\end{bmatrix}$. (d) Long-run $\begin{bmatrix}0.25\\0.75\end{bmatrix}$: about $25\,\%$ buy and $75\,\%$ bring each week. **Q3.** (a) $T=\begin{bmatrix}0.7 & 0.2\\0.3 & 0.8\end{bmatrix}$, $S_0=\begin{bmatrix}600\\0\end{bmatrix}$. (b) $S_1=\begin{bmatrix}420\\180\end{bmatrix}$. (c) $S_2=\begin{bmatrix}330\\270\end{bmatrix}$. (d) After two weeks about 330 birds feed at the east site. **Q4.** (a) Columns: $0.8+0.1+0.1=1$, $0.15+0.8+0.05=1$, $0.1+0.1+0.8=1$. (b) $S_1=\begin{bmatrix}46.5\\31\\22.5\end{bmatrix}$. (c) $S_2=\begin{bmatrix}44.1\\31.7\\24.2\end{bmatrix}$. (d) About 46 500 customers are with Alpha after one month. **Q5.** (a) $T^2=\begin{bmatrix}0.51 & 0.42\\0.49 & 0.58\end{bmatrix}$. (b) $0.51+0.49=1$ and $0.42+0.58=1$. (c) $S_2=\begin{bmatrix}93\\107\end{bmatrix}$. (d) $S_1=\begin{bmatrix}90\\110\end{bmatrix}$, then $T\,S_1=\begin{bmatrix}93\\107\end{bmatrix}$ — the same. **Q6.** (a) $T=\begin{bmatrix}0.85 & 0.1\\0.15 & 0.9\end{bmatrix}$; each column adds to 1 because every subscriber either stays or switches, so the proportions leaving a service total the whole of it. (b) $S_1=\begin{bmatrix}0.475\\0.525\end{bmatrix}$. (c) $S_2=\begin{bmatrix}0.45625\\0.54375\end{bmatrix}$. (d) Long-run $\begin{bmatrix}0.4\\0.6\end{bmatrix}$: FreshBox settles near $40\,\%$ and QuickPlate near $60\,\%$. **Q7.** $S_1=\begin{bmatrix}800\\600\end{bmatrix}$, $S_2=\begin{bmatrix}710\\690\end{bmatrix}$. **Q8.** $S_1=\begin{bmatrix}0.425\\0.575\end{bmatrix}$, $S_2=\begin{bmatrix}0.36875\\0.63125\end{bmatrix}$; long-run $\begin{bmatrix}0.2\\0.8\end{bmatrix}$ ($20\,\%$ Zap, $80\,\%$ Volt). **Q9.** $S_1=\begin{bmatrix}405\\355\\240\end{bmatrix}$, $S_2=\begin{bmatrix}406.25\\390.25\\203.5\end{bmatrix}$ hectares. **Q10.** (a) Each quarter $20\,\%$ of AeroJet's passengers switch to SkyLink. (b) $S_1=\begin{bmatrix}0.62\\0.38\end{bmatrix}$, $S_2=\begin{bmatrix}0.634\\0.366\end{bmatrix}$. (c) In the long run SkyLink keeps about two-thirds ($66.7\,\%$) of the market and AeroJet about one-third ($33.3\,\%$), and this split then stays roughly constant. **Q11.** (a) $T^2=\begin{bmatrix}0.66 & 0.17\\0.34 & 0.83\end{bmatrix}$, $T^3=\begin{bmatrix}0.562 & 0.219\\0.438 & 0.781\end{bmatrix}$. (b) $S_3=\begin{bmatrix}3905\\6095\end{bmatrix}$. **Q12.** (a) $S_1=\begin{bmatrix}0.7\\0.3\end{bmatrix}$, $S_2=\begin{bmatrix}0.55\\0.45\end{bmatrix}$. (b) $S_1=\begin{bmatrix}0.2\\0.8\end{bmatrix}$, $S_2=\begin{bmatrix}0.3\\0.7\end{bmatrix}$. (c) The long-run state is the same for both starts, so the starting distribution does not affect the long-run share. **Q13.** $S_1=\begin{bmatrix}36\\44\\20\end{bmatrix}$, $S_2=\begin{bmatrix}33.6\\46.4\\20\end{bmatrix}$; MegaFoods stays at 20 thousand both months (its gains balance its losses). **Q14.** P can be a transition matrix (columns $0.6+0.4=1$ and $0.3+0.7=1$); Q cannot, since its first column $0.6+0.3=0.9\neq 1$. Using P: $S_1=\begin{bmatrix}210\\290\end{bmatrix}$. **Q15.** $S_1=\begin{bmatrix}0.6\\0.4\end{bmatrix}$, $S_2=\begin{bmatrix}0.66\\0.34\end{bmatrix}$; long-run $\begin{bmatrix}0.75\\0.25\end{bmatrix}$ — PodPerfect dominates with about $75\,\%$. **Q16.** $S_1=\begin{bmatrix}465\\310\\225\end{bmatrix}$, $S_2=\begin{bmatrix}434\\317\\249\end{bmatrix}$ hectares. **Q17.** $S_1=\begin{bmatrix}6900\\11100\end{bmatrix}$, $S_2=\begin{bmatrix}7665\\10335\end{bmatrix}$; long-run $\begin{bmatrix}12000\\6000\end{bmatrix}$. **Q18.** (a) $T=\begin{bmatrix}0.9 & 0.4\\0.1 & 0.6\end{bmatrix}$, $S_1=\begin{bmatrix}0.55\\0.45\end{bmatrix}$, $S_2=\begin{bmatrix}0.675\\0.325\end{bmatrix}$. (b) Long-run $\begin{bmatrix}0.8\\0.2\end{bmatrix}$. (c) Participation levels off near $80\,\%$ and stays there; the campaign lifts participation from $30\,\%$ to a stable $80\,\%$, so it is worthwhile.

Fully-worked solutions

Q1. With states S, V, column 1 is 'from Star' and column 2 'from Vista', so $T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$. Each column adds to 1: $0.9+0.1=1$ and $0.2+0.8=1$. Starting from $S_0=\begin{bmatrix}500\\0\end{bmatrix}$,

$$S_1=T\,S_0=\begin{bmatrix}0.9\times 500+0.2\times 0\\0.1\times 500+0.8\times 0\end{bmatrix}=\begin{bmatrix}450\\50\end{bmatrix},\ S_2=T\,S_1=\begin{bmatrix}0.9\times 450+0.2\times 50\\0.1\times 450+0.8\times 50\end{bmatrix}=\begin{bmatrix}415\\85\end{bmatrix}.$$

Q2. $T=\begin{bmatrix}0.7 & 0.1\\0.3 & 0.9\end{bmatrix}$. Then

$$S_1=\begin{bmatrix}0.7\times 0.5+0.1\times 0.5\\0.3\times 0.5+0.9\times 0.5\end{bmatrix}=\begin{bmatrix}0.4\\0.6\end{bmatrix},\ S_2=\begin{bmatrix}0.7\times 0.4+0.1\times 0.6\\0.3\times 0.4+0.9\times 0.6\end{bmatrix}=\begin{bmatrix}0.34\\0.66\end{bmatrix},$$

$$S_3=\begin{bmatrix}0.7\times 0.34+0.1\times 0.66\\0.3\times 0.34+0.9\times 0.66\end{bmatrix}=\begin{bmatrix}0.304\\0.696\end{bmatrix}.$$

Iterating further, the vector approaches $\begin{bmatrix}0.25\\0.75\end{bmatrix}$: in the long run about a quarter of the students buy their lunch and three-quarters bring it from home.

Q3. From the diagram, $T=\begin{bmatrix}0.7 & 0.2\\0.3 & 0.8\end{bmatrix}$ and $S_0=\begin{bmatrix}600\\0\end{bmatrix}$. Then

$$S_1=\begin{bmatrix}0.7\times 600+0.2\times 0\\0.3\times 600+0.8\times 0\end{bmatrix}=\begin{bmatrix}420\\180\end{bmatrix},\ S_2=\begin{bmatrix}0.7\times 420+0.2\times 180\\0.3\times 420+0.8\times 180\end{bmatrix}=\begin{bmatrix}330\\270\end{bmatrix}.$$

The first element of S_2 is 330: after two weeks about 330 of the 600 birds feed at the east site.

Q4. The columns add to $0.8+0.1+0.1=1$, $0.15+0.8+0.05=1$ and $0.1+0.1+0.8=1$, so T is a valid transition matrix. Then

$$S_1=T\,S_0=\begin{bmatrix}0.8\times 50+0.15\times 30+0.1\times 20\\0.1\times 50+0.8\times 30+0.1\times 20\\0.1\times 50+0.05\times 30+0.8\times 20\end{bmatrix}=\begin{bmatrix}46.5\\31\\22.5\end{bmatrix},$$

$$S_2=T\,S_1=\begin{bmatrix}0.8\times 46.5+0.15\times 31+0.1\times 22.5\\0.1\times 46.5+0.8\times 31+0.1\times 22.5\\0.1\times 46.5+0.05\times 31+0.8\times 22.5\end{bmatrix}=\begin{bmatrix}44.1\\31.7\\24.2\end{bmatrix}.$$

The first element of S_1 is 46.5 thousand, so after one month about 46 500 customers are with Alpha.

Q5. (a) $T^2=T\times T=\begin{bmatrix}0.3 & 0.6\\0.7 & 0.4\end{bmatrix}\begin{bmatrix}0.3 & 0.6\\0.7 & 0.4\end{bmatrix}=\begin{bmatrix}0.3\times 0.3+0.6\times 0.7 & 0.3\times 0.6+0.6\times 0.4\\0.7\times 0.3+0.4\times 0.7 & 0.7\times 0.6+0.4\times 0.4\end{bmatrix}=\begin{bmatrix}0.51 & 0.42\\0.49 & 0.58\end{bmatrix}$. (b) The columns add to $0.51+0.49=1$ and $0.42+0.58=1$. (c) $S_2=T^2S_0=\begin{bmatrix}0.51\times 100+0.42\times 100\\0.49\times 100+0.58\times 100\end{bmatrix}=\begin{bmatrix}93\\107\end{bmatrix}$. (d) $S_1=T\,S_0=\begin{bmatrix}0.3\times 100+0.6\times 100\\0.7\times 100+0.4\times 100\end{bmatrix}=\begin{bmatrix}90\\110\end{bmatrix}$, and then $T\,S_1=\begin{bmatrix}0.3\times 90+0.6\times 110\\0.7\times 90+0.4\times 110\end{bmatrix}=\begin{bmatrix}93\\107\end{bmatrix}$, which agrees with T^2S_0.

Q6. (a) $T=\begin{bmatrix}0.85 & 0.1\\0.15 & 0.9\end{bmatrix}$. Each column adds to 1 because every subscriber either stays with their service or switches to the other, so the two proportions leaving a service must account for all of it. (b)

$S_1=\begin{bmatrix}0.85\times0.5+0.1\times0.5\\0.15\times0.5+0.9\times0.5\end{bmatrix}=\begin{bmatrix}0.475\\0.525\end{bmatrix}$. (c) $S_2=\begin{bmatrix}0.85\times0.475+0.1\times0.525\\0.15\times0.475+0.9\times0.525\end{bmatrix}=\begin{bmatrix}0.45625\\0.54375\end{bmatrix}$. (d) Iterating many months, the shares approach $\begin{bmatrix}0.4\\0.6\end{bmatrix}$: FreshBox settles near $40\,\%$ of the market and QuickPlate near $60\,\%$.

Q7. $T=\begin{bmatrix}0.7 & 0.25\\0.3 & 0.75\end{bmatrix}$, $S_0=\begin{bmatrix}1000\\400\end{bmatrix}$.

$$S_1=\begin{bmatrix}0.7\times1000+0.25\times400\\0.3\times1000+0.75\times400\end{bmatrix}=\begin{bmatrix}800\\600\end{bmatrix}, S_2=\begin{bmatrix}0.7\times800+0.25\times600\\0.3\times800+0.75\times600\end{bmatrix}=\begin{bmatrix}710\\690\end{bmatrix}.$$

After one week 800 motorists use BowserCo and 600 FuelFast; after two weeks, 710 and 690.

Q8. $T=\begin{bmatrix}0.8 & 0.05\\0.2 & 0.95\end{bmatrix}$, $S_0=\begin{bmatrix}0.5\\0.5\end{bmatrix}$.

$$S_1=\begin{bmatrix}0.8\times0.5+0.05\times0.5\\0.2\times0.5+0.95\times0.5\end{bmatrix}=\begin{bmatrix}0.425\\0.575\end{bmatrix}, S_2=\begin{bmatrix}0.8\times0.425+0.05\times0.575\\0.2\times0.425+0.95\times0.575\end{bmatrix}=\begin{bmatrix}0.36875\\0.63125\end{bmatrix}.$$

Iterating many weeks, the shares approach $\begin{bmatrix}0.2\\0.8\end{bmatrix}$: about $20\,\%$ of students choose Zap and $80\,\%$ Volt.

Q9.

$$S_1=T\,S_0=\begin{bmatrix}0.9\times400+0.05\times300+0.1\times300\\0.1\times400+0.85\times300+0.2\times300\\0\times400+0.1\times300+0.7\times300\end{bmatrix}=\begin{bmatrix}405\\355\\240\end{bmatrix},$$

$$S_2=T\,S_1=\begin{bmatrix}0.9\times405+0.05\times355+0.1\times240\\0.1\times405+0.85\times355+0.2\times240\\0\times405+0.1\times355+0.7\times240\end{bmatrix}=\begin{bmatrix}406.25\\390.25\\203.5\end{bmatrix}\text{hectares.}$$

Q10. (a) The element 0.20 sits in row S, column A, so it is the proportion of AeroJet's passengers who switch to SkyLink each quarter — that is, $20\,\%$ of AeroJet's passengers rebook with SkyLink the next quarter. (b) $T=\begin{bmatrix}0.9 & 0.2\\0.1 & 0.8\end{bmatrix}$, $S_0=\begin{bmatrix}0.6\\0.4\end{bmatrix}$:

$$S_1=\begin{bmatrix}0.9\times0.6+0.2\times0.4\\0.1\times0.6+0.8\times0.4\end{bmatrix}=\begin{bmatrix}0.62\\0.38\end{bmatrix}, S_2=\begin{bmatrix}0.9\times0.62+0.2\times0.38\\0.1\times0.62+0.8\times0.38\end{bmatrix}=\begin{bmatrix}0.634\\0.366\end{bmatrix}.$$

(c) The shares settle near $\begin{bmatrix}0.667\\0.333\end{bmatrix}$: in the long run SkyLink holds about two-thirds of the market and AeroJet about one-third, and this split then stays roughly constant.

Q11. (a) $T=\begin{bmatrix}0.8 & 0.1\\0.2 & 0.9\end{bmatrix}$, so

$$T^2=\begin{bmatrix}0.8\times0.8+0.1\times0.2 & 0.8\times0.1+0.1\times0.9\\0.2\times0.8+0.9\times0.2 & 0.2\times0.1+0.9\times0.9\end{bmatrix}=\begin{bmatrix}0.66 & 0.17\\0.34 & 0.83\end{bmatrix},$$

$$T^3=T^2T=\begin{bmatrix}0.66\times0.8+0.17\times0.2 & 0.66\times0.1+0.17\times0.9\\0.34\times0.8+0.83\times0.2 & 0.34\times0.1+0.83\times0.9\end{bmatrix}=\begin{bmatrix}0.562 & 0.219\\0.438 & 0.781\end{bmatrix}.$$

(b) $S_3=T^3\,S_0=\begin{bmatrix}0.562\times5000+0.219\times5000\\0.438\times5000+0.781\times5000\end{bmatrix}=\begin{bmatrix}3905\\6095\end{bmatrix}$.

Q12. (a) From $S_0=\begin{bmatrix}1\\0\end{bmatrix}$: $S_1=\begin{bmatrix}0.7\times1+0.2\times0\\0.3\times1+0.8\times0\end{bmatrix}=\begin{bmatrix}0.7\\0.3\end{bmatrix}$, and $S_2=\begin{bmatrix}0.7\times0.7+0.2\times0.3\\0.3\times0.7+0.8\times0.3\end{bmatrix}=\begin{bmatrix}0.55\\0.45\end{bmatrix}$. (b) From $S_0=\begin{bmatrix}0\\1\end{bmatrix}$: $S_1=\begin{bmatrix}0.2\\0.8\end{bmatrix}$, and $S_2=\begin{bmatrix}0.7\times0.2+0.2\times0.8\\0.3\times0.2+0.8\times0.8\end{bmatrix}=\begin{bmatrix}0.3\\0.7\end{bmatrix}$. (c) Both starting vectors head towards the same long-run state $\begin{bmatrix}0.4\\0.6\end{bmatrix}$, so the long-run share is determined by the transition matrix, not by the starting distribution.

Q13.

$$S_1=P\,S_0=\begin{bmatrix}0.7\times40+0.1\times40+0.2\times20\\0.2\times40+0.8\times40+0.2\times20\\0.1\times40+0.1\times40+0.6\times20\end{bmatrix}=\begin{bmatrix}36\\44\\20\end{bmatrix},$$

$$S_2=P\,S_1=\begin{bmatrix}0.7\times36+0.1\times44+0.2\times20\\0.2\times36+0.8\times44+0.2\times20\\0.1\times36+0.1\times44+0.6\times20\end{bmatrix}=\begin{bmatrix}33.6\\46.4\\20\end{bmatrix}.$$

MegaFoods' total stays at 20 thousand in both months: the shoppers it loses each month are exactly balanced by those it gains.

Q14. P can be a transition matrix, because its columns add to $0.6+0.4=1$ and $0.3+0.7=1$. Q cannot, because its first column adds to $0.6+0.3=0.9\neq1$, so those proportions do not account for everyone leaving the first state. Using P with $S_0=\begin{bmatrix}200\\300\end{bmatrix}$,

$$S_1=P\,S_0=\begin{bmatrix}0.6\times200+0.3\times300\\0.4\times200+0.7\times300\end{bmatrix}=\begin{bmatrix}210\\290\end{bmatrix}.$$

Q15. $T=\begin{bmatrix}0.9&0.3\\0.1&0.7\end{bmatrix}$, $S_0=\begin{bmatrix}0.5\\0.5\end{bmatrix}$.

$$S_1=\begin{bmatrix}0.9\times0.5+0.3\times0.5\\0.1\times0.5+0.7\times0.5\end{bmatrix}=\begin{bmatrix}0.6\\0.4\end{bmatrix},\ S_2=\begin{bmatrix}0.9\times0.6+0.3\times0.4\\0.1\times0.6+0.7\times0.4\end{bmatrix}=\begin{bmatrix}0.66\\0.34\end{bmatrix}.$$

Iterating many months, the shares approach $\begin{bmatrix}0.75\\0.25\end{bmatrix}$. In the long run PodPerfect dominates, holding about $75\,\%$ of the market to BrewBox's $25\,\%$.

Q16.

$$S_1=T\,S_0=\begin{bmatrix}0.9\times500+0.05\times300+0\times200\\0.1\times500+0.8\times300+0.1\times200\\0\times500+0.15\times300+0.9\times200\end{bmatrix}=\begin{bmatrix}465\\310\\225\end{bmatrix},$$

$$S_2=T\,S_1=\begin{bmatrix}0.9\times465+0.05\times310+0\times225\\0.1\times465+0.8\times310+0.1\times225\\0\times465+0.15\times310+0.9\times225\end{bmatrix}=\begin{bmatrix}434\\317\\249\end{bmatrix}\text{hectares.}$$

Q17. $T=\begin{bmatrix}0.95&0.1\\0.05&0.9\end{bmatrix}$, $S_0=\begin{bmatrix}6000\\12000\end{bmatrix}$.

$$S_1=\begin{bmatrix}0.95\times6000+0.1\times12000\\0.05\times6000+0.9\times12000\end{bmatrix}=\begin{bmatrix}6900\\11100\end{bmatrix},\ S_2=\begin{bmatrix}0.95\times6900+0.1\times11100\\0.05\times6900+0.9\times11100\end{bmatrix}=\begin{bmatrix}7665\\10335\end{bmatrix}.$$

Continuing the iteration, the populations settle at $\begin{bmatrix} 12000 \\ 6000 \end{bmatrix}$ — in the long run about $12\,000$ residents live in Seaside and 6000 in the Hinterland.

Q18. (a) $T = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$, $S_0 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$.

$$S_1 = \begin{bmatrix} 0.9 \times 0.3 + 0.4 \times 0.7 \\ 0.1 \times 0.3 + 0.6 \times 0.7 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}, S_2 = \begin{bmatrix} 0.9 \times 0.55 + 0.4 \times 0.45 \\ 0.1 \times 0.55 + 0.6 \times 0.45 \end{bmatrix} = \begin{bmatrix} 0.675 \\ 0.325 \end{bmatrix}.$$

(b) Iterating many years, the proportions approach $\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$, so about $80\,\%$ of households participate in the long run. (c) The steady state means participation levels off near $80\,\%$ and then holds there year after year, even though individual households keep joining and leaving. Because the campaign lifts participation from its starting $30\,\%$ up to a stable $80\,\%$, and keeps it there, it can be judged worthwhile.

Topic 1 Test A — Multiple-choice

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 3 minutes. Writing time: 20 minutes. 12 multiple-choice questions, 12 marks. Choose the response that is ***correct*** *for the question.*

Question 1

A real-estate agency records four variables for each rental property it lists: the suburb, the number of bedrooms, the weekly rent (in dollars), and the distance to the city centre (in kilometres). How many of these four variables are numerical?

A. 1

B. 2

C. 3

D. 4

Question 2

For a data set, the first quartile is $Q_1 = 15$ and the third quartile is $Q_3 = 27$. Using the $1.5 \times \mathrm{IQR}$ rule, the upper fence for identifying outliers is

A. 45

B. 39

C. 51

D. 33

Question 3

The heights of a large group of Year 11 students are approximately normally distributed with a mean of 168 cm and a standard deviation of 6 cm. Using the 68–95–99.7% rule, the percentage of students with heights between 156 cm and 174 cm is closest to

A. 68%

B. 95%

C. 99.7%

D. 81.5%

Question 4

A sequence is generated by the recurrence relation

$$t_{n+1} = t_n - 6, t_0 = 40.$$

The value of t_4 is

A. 28

B. 16

C. 22

D. 34

Question 5

A geometric sequence is generated by the recurrence relation

$$t_{n+1} = 1.5t_n, t_0 = 8.$$

The value of t_3 is

A. 18

B. 24

C. 27

D. 40.5

Question 6

A sum of $5000 is invested at 6% per annum simple interest. The total interest earned after 3 years is

A. $300

B. $900

C. $954

D. $1800

Question 7

A machine bought for $20 000 depreciates by 10% of its value each year, using reducing-balance depreciation. Its value after 2 years is

A. $16 200

B. $16 000

C. $18 000

D. $4000

Question 8

A plumber quotes $440 for a job, and this price includes 10% GST. The amount of GST included in the $440 price is

A. $44

B. $400

C. $4

D. $40

Question 9

The cost $C of hiring a marquee for d days is modelled by

$$C = 120 + 85d.$$

In this model, the number 85 represents

A. the fixed cost of hiring the marquee

B. the increase in cost for each additional day of hire

C. the cost of hiring the marquee for one day

D. the total cost of hiring the marquee

Question 10

A car park charges $4 for the first hour and $3 for each additional hour or part thereof. The cost of parking for 4 hours is

A. $12

B. $15

C. $13

D. $16

Question 11

Matrix A has order 2×3 and matrix B has order 3×4. The order of the product AB is

A. 2×3

B. 3×4

C. AB is not defined

D. 2×4

Question 12

The product

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

is equal to

A. $\begin{bmatrix} 18 \\ 19 \end{bmatrix}$

B. $\begin{bmatrix} 19 \\ 18 \end{bmatrix}$

C. $\begin{bmatrix} 17 \\ 23 \end{bmatrix}$

D. $\begin{bmatrix} 12 \\ 15 \end{bmatrix}$

End of Topic 1 Test A

Test A — answers and solutions

Q	1	2	3	4	5	6	7	8	9	10	11	12
Answer	C	A	D	B	C	B	A	D	B	C	D	A

Q1. C. The number of bedrooms, the weekly rent and the distance to the city centre are numerical; the suburb is categorical (nominal), so 3 variables are numerical.

Q2. A. $\mathrm{IQR}=27-15=12$, so the upper fence is $Q_3+1.5\times\mathrm{IQR}=27+1.5\times12=45$.

Q3. D. $156=168-2\,\mathrm{sd}$ and $174=168+1\,\mathrm{sd}$; the percentage between is $95/2+68/2=47.5+34=81.5\,\%$.

Q4. B. Subtracting 6 four times: $40, 34, 28, 22, 16$, so $t_4=16$.

Q5. C. Multiplying by 1.5 three times: $8, 12, 18, 27$, so $t_3=27$.

Q6. B. Simple interest $=P\times r\times t=5000\times0.06\times3=\$900$.

Q7. A. Reducing-balance: value $=20\,000\times0.90^2=20\,000\times0.81=\$16\,200$. (The \$16 000 option is the flat-rate answer.)

Q8. D. The \$440 is $110\,\%$ of the pre-GST price, so the pre-GST price is $440\div1.1=\$400$ and the GST is $440-400=\$40$.

Q9. B. In $C=120+85\,d$ the coefficient of d is the slope, the constant rate of change: the cost rises by \$85 for each additional day.

Q10. C. First hour \$4, then 3 more hours at \$3 each: $4+3\times3=\$13$.

Q11. D. $A\,B$ is defined because the columns of A (3) equal the rows of B (3); its order is rows of A by columns of B, i.e. 2×4.

Q12. A. Row by column: $\begin{bmatrix}3\times4+2\times3\\1\times4+5\times3\end{bmatrix}=\begin{bmatrix}18\\19\end{bmatrix}$.

Topic 1 Test B — Short-answer

One bound reference and one approved technology (CAS calculator or software) are permitted; a scientific calculator may also be used. Reading time: 5 minutes. Writing time: 40 minutes. Total 40 marks. In all questions where a numerical answer is required, an exact value must be given unless otherwise specified. Where more than one mark is available, appropriate working must be shown.

Question 1 (8 marks)

A conservation group counts the number of eastern grey kangaroos grazing at a nature reserve at dawn on 15 mornings. The counts, arranged in order, are

$$8, 11, 12, 14, 15, 16, 18, 19, 21, 22, 24, 25, 27, 29, 46.$$

a. Determine the five-number summary for this data set. 2 marks

b. Using the $1.5 \times$ IQR rule, show that 46 is an outlier. 2 marks

c. Construct a boxplot for the data, displaying the outlier separately. 2 marks

d. Describe the shape of the distribution, referring to your boxplot. 1 mark

e. State whether the median and IQR or the mean and standard deviation better summarise these counts, and give a reason. 1 mark

Question 2 (8 marks)

A regional library launches a "Friends of the Library" membership scheme. At the end of the first month there are 34 members. The number of members M_n at the end of month n is modelled by the recurrence relation

$$M_1 = 34, M_{n+1} = M_n + 12.$$

a. Find the number of members at the end of months 2 and 3. 2 marks

b. Deduce an explicit rule for M_n in terms of n. 2 marks

c. Use your rule to find the number of members at the end of month 24. 1 mark

d. During which month does the membership first exceed 250? 2 marks

e. State whether the sequence is increasing, decreasing or constant, and explain why. 1 mark

Question 3 (8 marks)

Marcus has $8000 to invest.

a. If he invests the $8000 at 4% per annum simple interest, find the interest earned and the value of the investment after 5 years. 2 marks

b. If instead he invests the $8000 at 4% per annum compound interest, compounding annually, find the value of the investment after 5 years, correct to the nearest cent. 2 marks

c. State which investment is worth more after 5 years and by how much, and explain briefly why. 1 mark

d. Marcus also buys a $3000 laptop for his business. It depreciates by a flat rate of 15% of its purchase price each year. Find the value of the laptop after 4 years. 2 marks

e. Write down a recurrence relation for the value V_n of the laptop after n years. 1 mark

Question 4 (8 marks)

A landscaping company hires out a mini-excavator. Company A charges a $90 call-out fee plus $45 for each hour of hire. Company B charges a $150 call-out fee plus $35 for each hour of hire. Let h be the number of hours the excavator is hired.

a. Write down a linear model for the total cost \$C in terms of h for each company. 2 marks

b. Interpret the slope of Company A's model in terms of the cost of hire. 1 mark

c. Find the number of hours for which the two companies charge the same amount, and state that amount. 2 marks

d. A customer needs the excavator for 9 hours. Which company is cheaper, and by how much? 2 marks

e. A hire is booked for a whole number of hours, up to a maximum of a 10-hour day. State the domain of h appropriate for these models. 1 mark

Question 5 (8 marks)

A sports store sells caps, shirts and balls at two branches. Over one weekend the numbers sold are given by matrix S, and the selling prices (in dollars) by matrix P:

$$S = \begin{bmatrix} 20 & 35 & 12 \\ 18 & 28 & 15 \end{bmatrix}, P = \begin{bmatrix} 25 \\ 40 \\ 18 \end{bmatrix},$$

where the rows of S are the two branches and the columns are caps, shirts and balls respectively.

a. State the order of matrix S. 1 mark

b. Explain why the product SP is defined but the product PS is not. 1 mark

c. Evaluate SP and interpret its entries in this context. 2 marks

A car-share scheme keeps its cars at two depots, the Airport (A) and the City (C). Each day, 85% of the cars at the Airport stay there and the rest move to the City, while 25% of the cars at the City move to the Airport and the rest stay.

d. Write down the transition matrix T for this daily movement of cars. 2 marks

e. One morning there are 60 cars at the Airport and 40 cars at the City. Find the number of cars at each depot the next morning. 2 marks

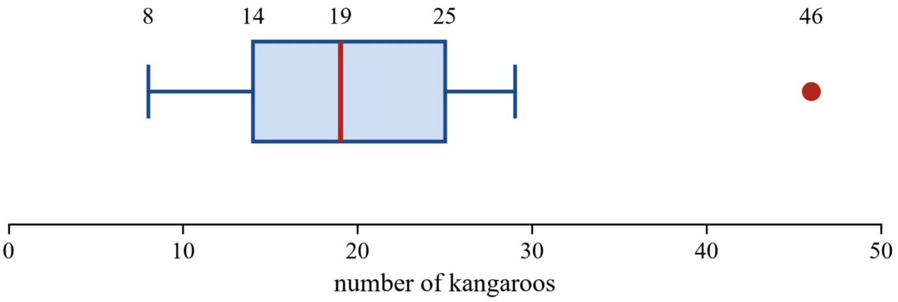
End of Topic 1 Test B

Test B — answers and solutions

Q1. (a) With $n=15$ the median is the 8th value, $M=19$. Excluding the median, the lower half is the first 7 values $8, 11, 12, 14, 15, 16, 18$, giving $Q_1=14$; the upper half is the last 7 values $21, 22, 24, 25, 27, 29, 46$, giving $Q_3=25$. With $min=8$ and $max=46$, the five-number summary is

$$8, 14, 19, 25, 46.$$

(b) $\text{IQR}=Q_3-Q_1=25-14=11$. The upper fence is $Q_3+1.5\times\text{IQR}=25+1.5\times 11=41.5$. Since $46>41.5$, the count 46 is an outlier. (The lower fence is $14-16.5=-2.5$, and $8>-2.5$, so there is no low outlier.) (c) The box spans $Q_1=14$ to $Q_3=25$ with the median at 19; the outlier 46 is plotted separately and the upper whisker stops at the largest non-outlier, 29.



(d) The distribution is positively skewed: the single high outlier (46) stretches the upper tail, and the mean (≈ 20.5) is pulled above the median (19).

(e) The **median and IQR** are the better summary, because they are resistant to the outlier of 46, whereas the mean and standard deviation are inflated by that one large count.

Q2. (a) Adding 12 each month: $M_2=34+12=46$ and $M_3=46+12=58$ members. (b) The first term is $M_1=34$ and the common difference is $d=12$, so $M_n=34+12(n-1)=22+12n$. (c) $M_{24}=22+12\times 24=22+288=310$ members. (d) Solve $22+12n>250$: $12n>228$, so $n>19$. Since $M_{19}=22+228=250$ (exactly 250, not more) and $M_{20}=22+240=262$, the membership first exceeds 250 during **month 20**. (e) The sequence is **increasing**, because the common difference $d=12$ is positive, so each month adds 12 members.

Q3. (a) Simple interest $=P\times r\times t=8000\times 0.04\times 5=\$1600$. The value after 5 years is $8000+1600=\$9600$. (b) Compound interest: value $=8000\times 1.04^5=8000\times 1.216652\ldots=\$9733.22$ (to the nearest cent). (c) The **compound** investment is worth more, by $9733.22-9600=\$133.22$. This is because compound interest earns interest on the interest already added, whereas simple interest is paid only on the original \$8000. (d) Flat-rate depreciation of 15% of the \$3000 purchase price is $0.15\times 3000=\$450$ each year. After 4 years the value is $3000-4\times 450=3000-1800=\$1200$. (e) $V_0=3000, V_{n+1}=V_n-450.$

Q4. (a) Company A: $C=90+45h$. Company B: $C=150+35h$. (b) The slope of Company A's model is 45, so the cost increases by \$45 for each additional hour the excavator is hired. (c) Setting the costs equal: $90+45h=150+35h$, so $10h=60$ and $h=6$ hours. The common cost is $90+45\times 6=\$360$. (d) At $h=9$: Company A costs $90+45\times 9=\$495$ and Company B costs $150+35\times 9=\$465$. **Company B** is cheaper, by $495-465=\$30$. (e) The models apply for whole numbers of hours up to a 10-hour day, so the domain is $h\in\{1, 2, 3, \ldots, 10\}$.

Q5. (a) S has 2 rows and 3 columns, so its order is 2×3. (b) SP is defined because the number of columns of S (3) equals the number of rows of P (3). PS is not defined because the number of columns of P (1) does not equal the number of rows of S (2). (c) $SP=\begin{bmatrix}20\times 25+35\times 40+12\times 18\\ 18\times 25+28\times 40+15\times 18\end{bmatrix}=\begin{bmatrix}2116\\ 1840\end{bmatrix}$. The entries give the total weekend revenue at each branch: \$2116 at the first branch and \$1840 at the second. (d) Taking the columns as "from" and the rows as "to", with order Airport then City,

$$T = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}.$$

(e) The next-day numbers are

$$T\begin{bmatrix} 60 \\ 40 \end{bmatrix} = \begin{bmatrix} 0.85 \times 60 + 0.25 \times 40 \\ 0.15 \times 60 + 0.75 \times 40 \end{bmatrix} = \begin{bmatrix} 61 \\ 39 \end{bmatrix},$$

so there are **61 cars at the Airport and 39 at the City** the next morning.