General Mathematics Units 1 & 2

Chapter 4 — Linear functions, graphs and models

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Contents

Theory

Many pairs of quantities are related in the simplest possible way: every time one quantity increases by one unit, the other changes by a fixed amount. A relationship of this kind is described by a **linear function**

$$y = a + b\,x,$$

whose graph is a **straight line**. The two numbers a and b tell us everything about the line. (In other subjects the same rule is often written $y = m\,x + c$ — it is the same line; General Mathematics writes the constant term first.)

The y-intercept. The constant a is the value of y when $x = 0$, because $y = a + b \times 0 = a$. Graphically it is the **y-intercept**, the value at which the line crosses the y-axis — the point $(0, a)$. When x counts time, trips or items, a is the **initial value** of y, the value we start from before anything has happened.

The slope. The coefficient b is the **slope** (also called the gradient) of the line: it is the **constant rate of change** of y. Every one-unit increase in x changes y by exactly b. If b is **positive** the line rises from left to right; if b is **negative** the line falls from left to right. For example, the line $y = 20 + 5\,x$ crosses the y-axis at $(0, 20)$ and rises by 5 for each one-unit step in x.

Finding the slope from two points. If a line passes through the points (x_1, y_1) and (x_2, y_2), its slope is

$$b = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}},$$

the change in y divided by the change in x. Subtract the coordinates in the **same order** in the numerator and the denominator; either order then gives the same answer.

Tables of values. To build a **table of values** from a rule, substitute each x-value in turn. Going the other way, a table with **equally spaced** x-values comes from a linear function exactly when the y-values change by a **constant difference** from column to column. The slope is then $b = \dfrac{\text{change in } y}{\text{change in } x}$ for one step of the table, and a is the y-value at $x = 0$.

The intercepts. The line $y = a + b\,x$ crosses the y-axis where $x = 0$, at $(0, a)$. It crosses the x-axis where $y = 0$; solving $a + b\,x = 0$ gives the **x-intercept** $x = -\dfrac{a}{b}$ (provided $b \neq 0$).

Drawing the graph. Because two points determine a straight line, there are two standard methods.

- **Plot two points.** Choose two convenient x-values reasonably far apart, compute y for each from the rule, plot the two points and rule a straight line through them, extending it across the axes. Plotting a third point is a quick check — if it does not lie on the line, one of the calculations is wrong.
- **Plot the two intercepts.** Find the y-intercept $(0, a)$ and the x-intercept, plot both, and rule the line through them. This method is quickest when both intercepts are easy to find and not too close together.

Horizontal lines. When $b = 0$ the rule reduces to $y = a$: whatever x is, y has the same value a. The graph is a **horizontal line** through $(0, a)$, and its slope is 0 — y does not change at all as x increases.

Linear relations in the form $A\,x + B\,y = C$. A straight line is not always given with y by itself. A **linear relation** is often written instead in the form

$$A\,x + B\,y = C,$$

where A, B and C are numbers — for example $x + 2\,y = 10$. There are two ways to graph a relation written like this.

- **Rearrange it into $y=a+bx$.** Move the x-term to the right-hand side, then divide **every** term by the coefficient of y. For $x+2y=10$: $2y=10-x$, so $y=5-0.5x$ — the same line, now with $a=5$ and $b=-0.5$ on display. Take care when that coefficient is negative, because dividing by it changes the sign of every term.
- **Substitute to find the intercepts.** Putting $x=0$ leaves $By=C$, and putting $y=0$ leaves $Ax=C$. For $x+2y=10$ this gives $(0,5)$ and $(10,0)$ straight away. Two substitutions and a ruled line — usually the quickest method for this form.

Going the other way, $y=a+bx$ can always be rewritten as $Ax+By=C$: multiply through by whatever clears the fractions, then collect the x- and y-terms on the left. So $y=5-0.5x$ becomes $2y=10-x$, that is $x+2y=10$ again.

Equivalent forms. Multiplying every term of a linear relation by the same non-zero number gives an **equivalent form** — a different-looking equation with exactly the same graph. For instance $x+2y=10$, $3x+6y=30$ and $y=5-0.5x$ are three equivalent forms of one line, and each has the intercepts $(0,5)$ and $(10,0)$. Two special cases are worth noting: if $B=0$ the relation becomes $Ax=C$, that is $x=\frac{C}{A}$, a **vertical line** — the one kind of straight line that cannot be written as $y=a+bx$; and if $A=0$ it becomes $y=\frac{C}{B}$, a horizontal line.

Using CAS. General Mathematics is a technology-active, open-book course. Your CAS will graph a linear function as soon as you enter its rule, display a table of values alongside the graph, and locate the axis intercepts exactly. A graph screen normally needs the rule with y by itself, so a relation given as $Ax+By=C$ must be rearranged first — the CAS solve command will do that rearrangement for you. This is a fast check, but you must also be able to plot a line quickly by hand from two computed points, and to read a and b directly from a rule, a graph or a table, so that you can tell at a glance whether the calculator's picture is reasonable.

Worked examples

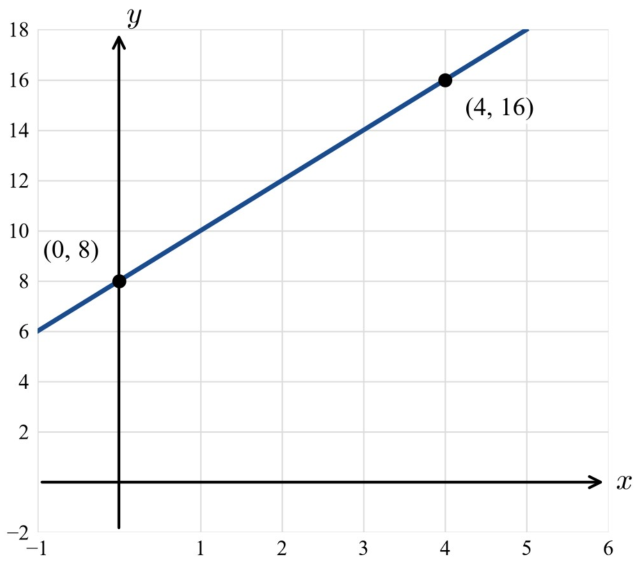
Example 1 — scaffolded

Consider the linear function $y=8+2x$.

(a) State the y-intercept and the slope. (b) Construct a table of values for $x=0,1,2,3,4$. (c) Draw the graph by plotting two points.

Working	Comment
Comparing $y=8+2x$ with $y=a+bx$: $a=8$ and $b=2$.	a is the constant term; b is the coefficient of x.
The y-intercept is 8, at the point $(0,8)$; the slope is 2.	The slope is positive, so the line rises from left to right.
Substitute each x-value: e.g. $x=3$ gives $y=8+2\times3=14$ (see the table below).	Each one-unit step in x increases y by $b=2$ — the constant differences confirm the working.
Plot $(0,8)$ and $(4,16)$ and rule a straight line through them.	Two well-separated points fix the line; the middle entry $(2,12)$ is a quick check that it is correct.

x	0	1	2	3	4
y	8	10	12	14	16



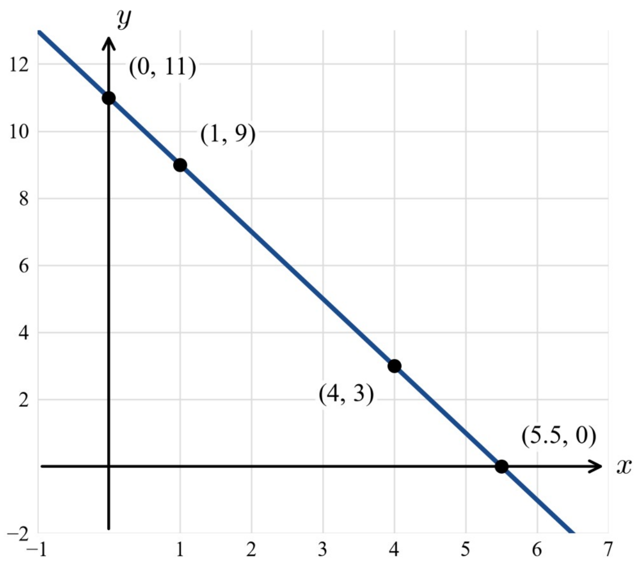
So the line has y-intercept $(0, 8)$, slope 2, and passes through every point in the table.

Example 2 — scaffolded

A straight line passes through the points $(1, 9)$ and $(4, 3)$.

(a) Find the slope. (b) Find the rule of the line. (c) Find the y-intercept and the x-intercept. (d) Sketch the line.

Working	Comment
$b = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-9}{4-1} = \frac{-6}{3} = -2.$	Subtract the coordinates in the same order top and bottom. The slope is negative: the line falls.
Substitute $(1, 9)$ into $y = a - 2x$: $9 = a - 2 \times 1$, so $a = 11$.	Either point can be used to find a; the rule is $y = 11 - 2x$.
Check with $(4, 3)$: $11 - 2 \times 4 = 3$, which is correct.	Always check the rule with the point you did not use.
y-intercept: $a = 11$, the point $(0, 11)$. x-intercept: solve $0 = 11 - 2x$, so $x = 5.5$, the point $(5.5, 0)$.	Set $x = 0$ for the y-intercept and $y = 0$ for the x-intercept.
Plot $(0, 11)$ and $(5.5, 0)$ and rule the line through them.	The two intercepts are convenient points for the sketch.

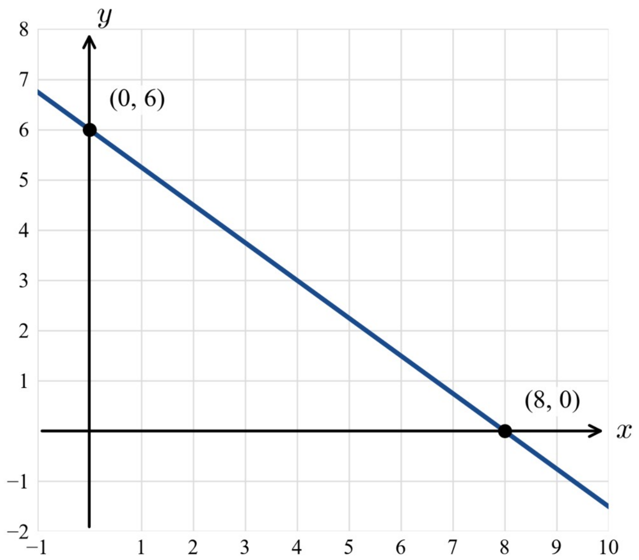


So the line is $y = 11 - 2x$, with slope -2, y-intercept $(0, 11)$ and x-intercept $(5.5, 0)$.

Example 3 — unscaffolded

Consider the linear relation $3x+4y=24$. Find the coordinates of both intercepts, sketch its graph, and write the relation in the form $y=a+bx$.

Substituting $x=0$ leaves $4y=24$, so $y=6$ and the y-intercept is $(0,6)$. Substituting $y=0$ leaves $3x=24$, so $x=8$ and the x-intercept is $(8,0)$. Plotting those two points and ruling a line through them gives the graph below.



To obtain the form $y=a+bx$, make y the subject: $4y=24-3x$, and dividing every term by 4 gives $y=6-0.75x$. The constant term 6 agrees with the y-intercept already found, and the slope $-\frac{3}{4}=-0.75$ is negative, matching the falling line in the sketch. Checking the x-intercept from this form: $0=6-0.75x$ gives $x=8$, as before.

∴ the intercepts are $(0,6)$ and $(8,0)$, and the relation is $y=6-0.75x$.

Example 4 — unscaffolded

A pump is switched on to empty a full rainwater tank. The table shows the amount of water, y litres, in the tank x minutes after the pump starts.

x (minutes)	0	2	4	6
y (litres)	500	440	380	320

Explain why the table represents a linear function, find its rule, interpret a and b, and find how much water is in the tank after 10 minutes.

The x-values are equally spaced (2 minutes apart) and the y-values fall by the constant difference 60 from column to column (500 → 440 → 380 → 320), so y is a linear function of x. The slope is $b=\frac{\text{change in } y}{\text{change in } x}=\frac{-60}{2}=-30$, and a is the y-value at $x=0$, namely 500. The rule is $y=500-30x$. Here $a=500$ is the initial value — the tank holds 500 litres when the pump starts — and $b=-30$ is the constant rate of change — the pump removes 30 litres of water each minute. After 10 minutes, $y=500-30\times 10=200$.

∴ the rule is $y=500-30x$; the tank starts with 500 litres, loses 30 litres per minute, and holds 200 litres after 10 minutes.

Practice questions

Scaffolded

Q1. Consider the linear function $y=3+2x$. (a) State the y-intercept and the slope. (b) Construct a table of values for $x=0,1,2,3,4$. (c) Draw the graph of the function by plotting two points from your table. (d) Use the rule to find y when $x=10$.

Q2. Find the slope of the straight line passing through each pair of points. (a) $(2,5)$ and $(6,13)$ (b) $(0,7)$ and $(5,-3)$ (c) $(-1,4)$ and $(3,4)$. What does this slope tell you about the line?

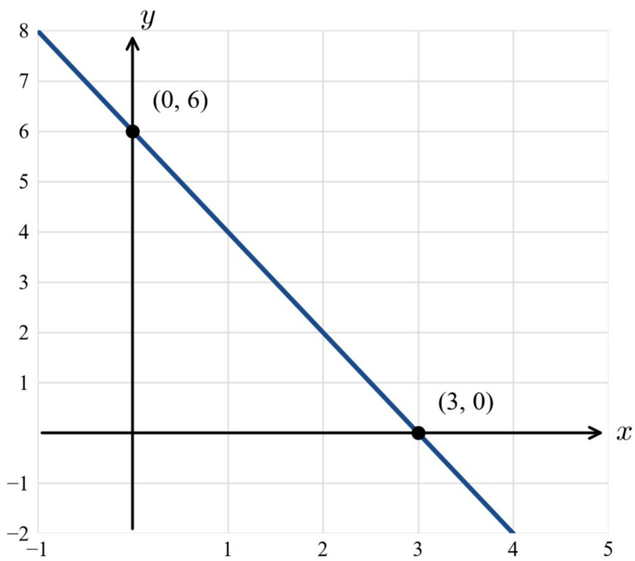
Q3. Consider the linear relation $2x+5y=20$. (a) Find the y-intercept by substituting $x=0$. (b) Find the x-intercept by substituting $y=0$. (c) Draw the graph using the two intercepts. (d) Rearrange the relation into the form $y=a+bx$ and state its slope. Does the line rise or fall from left to right?

Q4. A linear function has the following table of values.

x	0	1	2	3	4
y	5	9	13	17	21

(a) Explain how the table shows that the function is linear.
(b) State the slope b.
(c) Write down the rule.
(d) Use the rule to find y when $x=8$.

Q5. The graph below shows a straight line; the points $(0,6)$ and $(3,0)$ on the line are marked.



(a) Write down the y-intercept.
(b) Use the two marked points to find the slope.
(c) Write down the rule of the line.
(d) Use the rule to find y when $x=2.5$.

Q6. The height, y centimetres, of a burning candle x hours after it is lit is given by the linear function $y=25-4x$. (a) State the initial value a and interpret it in this context. (b) State the slope b and interpret it in this context. (c) Find the height of the candle after 3 hours. (d) Find the x-intercept and interpret it in this context.

Unscaffolded

Q7. For each of the following linear functions, state the slope and the y-intercept, and state whether the line rises from left to right, falls from left to right, or is horizontal. (a) $y=4+5x$ (b) $y=10-2x$ (c) $y=7$ (d) $y=-3+x$

Q8. Find the slope of the line through $(3,11)$ and $(8,31)$, and the slope of the line through $(-2,9)$ and $(4,-9)$.

Q9. Write the linear relation $9x-3y=21$ in the form $y=a+bx$, construct a table of values for $x=0,1,2,3,4$, write down the coordinates of the y-intercept, and find the exact value of the x-intercept.

Q10. Find the coordinates of the x-intercept and the y-intercept of each line. (a) $y=18-3x$ (b) $5x+2y=20$ (c) $10x+4y=40$. What do your answers to (b) and (c) tell you about those two relations?

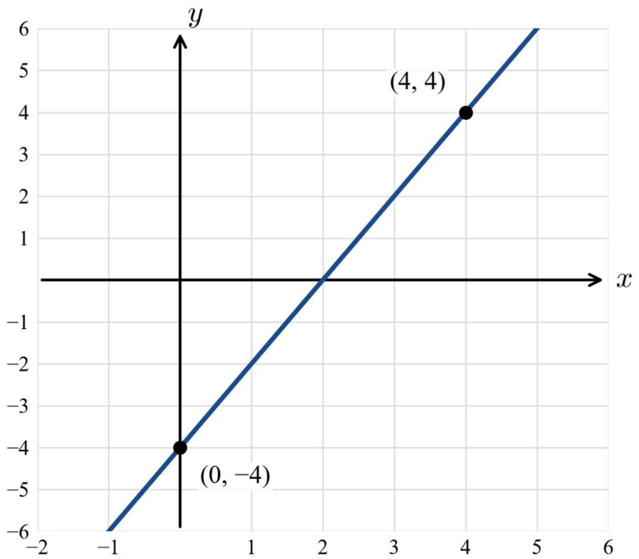
Q11. A linear function has the following table of values.

x	0	3	6	9
y	2	11	20	29

Find the rule of the function, and use it to find y when $x=20$.

Q12. Write down the rule of each of the following straight lines. (a) The line with slope 4 and y-intercept -3. (b) The line with slope $-\frac{1}{2}$ and y-intercept 6. (c) The horizontal line through the point $(0,-5)$.

Q13. The graph below shows a straight line; the points $(0,-4)$ and $(4,4)$ on the line are marked. Find the rule of the line, then find its x-intercept from the rule and check that it agrees with the graph.



Q14. Find the rule of the straight line passing through the points $(2,7)$ and $(5,16)$, and use it to find y when $x=10$.

Q15. Two tables of values are shown below. Decide which table could come from a linear function and explain why the other could not. Then find the rule for the linear one.

Table A:

x	0	1	2	3	4
y	6	10	14	18	22

Table B:

x	0	1	2	3	4
y	3	4	6	9	13

Q16. The amount of data, y gigabytes, left on Priya's monthly mobile plan x days after the month begins is given by $y=60-1.5x$. Interpret the values $a=60$ and $b=-1.5$ in this context, find how much data is left after 20 days, and find after how many days the data runs out.

Q17. A straight line passes through the points $(1,9)$, $(4,9)$ and $(7,9)$. Show that its slope is 0, write down its rule, and state the coordinates of its y-intercept.

Q18. A straight line has slope 2 and passes through the point $(3, 10)$. Find the value of a, write down the rule, and find the x-intercept of the line.

Answers

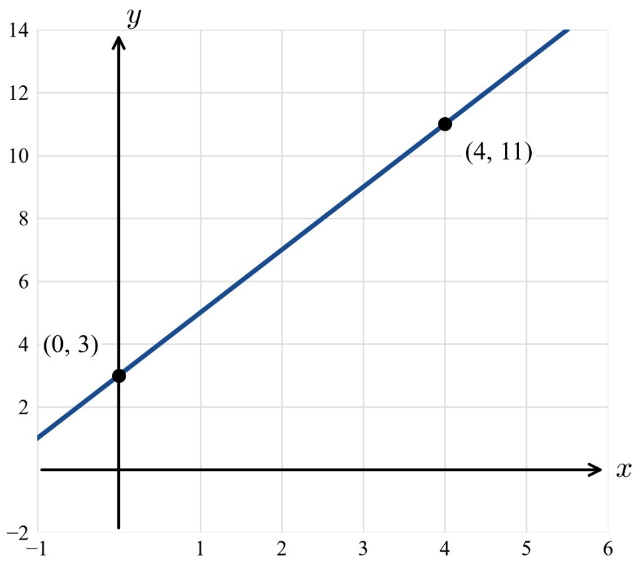
Q1. (a) y-intercept $(0,3)$; slope 2. (b) y-row: 3, 5, 7, 9, 11. (c) Line through $(0,3)$ and $(4,11)$ — see solutions. (d) $y=23$. **Q2.** (a) 2. (b) -2. (c) 0; the line is horizontal. **Q3.** (a) $(0,4)$. (b) $x=10$, i.e. $(10,0)$. (c) Line through $(0,4)$ and $(10,0)$ — see solutions. (d) $y=4-0.4x$; slope -0.4; the line falls. **Q4.** (a) Constant difference of 4 for each unit step in x. (b) $b=4$. (c) $y=5+4x$. (d) $y=37$. **Q5.** (a) 6. (b) -2. (c) $y=6-2x$. (d) $y=1$. **Q6.** (a) $a=25$: the candle is 25 cm tall when lit. (b) $b=-4$: it burns down 4 cm each hour. (c) 13 cm. (d) $x=6.25$: the candle burns away completely after 6.25 hours. **Q7.** (a) Slope 5, y-intercept 4; rises. (b) Slope -2, y-intercept 10; falls. (c) Slope 0, y-intercept 7; horizontal. (d) Slope 1, y-intercept -3; rises. **Q8.** 4 and -3. **Q9.** $y=-7+3x$; y-row: $-7,-4,-1,2,5$; y-intercept $(0,-7)$; x-intercept $x=\frac{7}{3}$. **Q10.** (a) $(6,0)$ and $(0,18)$. (b) $(4,0)$ and $(0,10)$. (c) $(4,0)$ and $(0,10)$ — the same intercepts, so (b) and (c) are equivalent forms of the same line. **Q11.** $y=2+3x$; $y=62$. **Q12.** (a) $y=-3+4x$. (b) $y=6-\frac{1}{2}x$. (c) $y=-5$. **Q13.** $y=-4+2x$; x-intercept $x=2$, matching the graph. **Q14.** $y=1+3x$; $y=31$. **Q15.** Table A is linear (constant difference 4); Table B is not (differences 1, 2, 3, 4). Rule for A: $y=6+4x$. **Q16.** $a=60$: 60 GB at the start of the month; $b=-1.5$: 1.5 GB used per day; 30 GB left after 20 days; the data runs out after 40 days. **Q17.** Slope $=0$; rule $y=9$; y-intercept $(0,9)$. **Q18.** $a=4$; $y=4+2x$; x-intercept $x=-2$.

Fully-worked solutions

Q1. (a) Comparing $y=3+2x$ with $y=a+bx$ gives $a=3$ and $b=2$, so the y-intercept is $(0,3)$ and the slope is 2. (b) Substituting $x=0,1,2,3,4$ gives

x	0	1	2	3	4
y	3	5	7	9	11

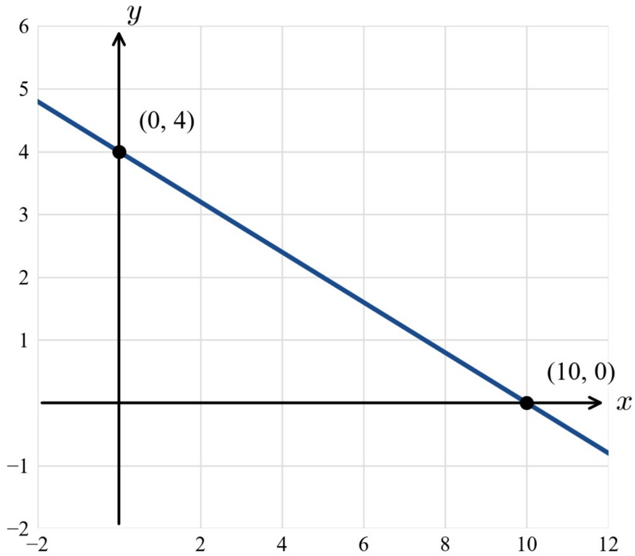
(c) Plot two well-separated points from the table, say $(0,3)$ and $(4,11)$, and rule a straight line through them; the point $(2,7)$ lies on the line, confirming the graph.



(d) $y=3+2\times10=23$.

Q2. (a) $b=\frac{13-5}{6-2}=\frac{8}{4}=2$. (b) $b=\frac{-3-7}{5-0}=\frac{-10}{5}=-2$. (c) $b=\frac{4-4}{3-(-1)}=\frac{0}{4}=0$. A slope of 0 means y does not change as x increases: the line is horizontal.

Q3. (a) Substituting $x=0$ into $2x+5y=20$ leaves $5y=20$, so $y=4$; the y-intercept is $(0,4)$. (b) Substituting $y=0$ leaves $2x=20$, so $x=10$; the x-intercept is $(10,0)$. (c) Plot $(0,4)$ and $(10,0)$ and rule the straight line through them.



(d) Make y the subject: $5y=20-2x$, and dividing every term by 5 gives $y=4-0.4x$. The slope is $-\frac{2}{5}=-0.4$, which is negative, so the line falls from left to right. The constant term 4 agrees with the y-intercept found in (a).

Q4. (a) The x-values increase in equal steps of 1 and the y-values increase by the constant difference 4 each step ($5 \to 9 \to 13 \to 17 \to 21$), which is exactly the pattern produced by a linear function. (b) $b=\frac{\text{change in } y}{\text{change in } x}=\frac{4}{1}=4$. (c) The y-value at $x=0$ is 5, so $a=5$ and the rule is $y=5+4x$. (d) $y=5+4\times 8=37$.

Q5. (a) The line crosses the y-axis at $(0,6)$, so the y-intercept is 6. (b) $b=\frac{0-6}{3-0}=\frac{-6}{3}=-2$. (c) With $a=6$ and $b=-2$, the rule is $y=6-2x$. (d) $y=6-2\times 2.5=6-5=1$.

Q6. (a) $a=25$ is the initial value: when the candle is lit ($x=0$) it is 25 cm tall. (b) $b=-4$ is the constant rate of change: the height decreases by 4 cm for every hour of burning. (c) When $x=3$, $y=25-4\times 3=13$, so the candle is 13 cm tall. (d) Solve $0=25-4x$: $4x=25$, so $x=6.25$. The height reaches zero after 6.25 hours — the candle has burnt away completely.

Q7. Comparing each rule with $y=a+bx$: (a) $y=4+5x$: slope 5, y-intercept 4; the slope is positive, so the line rises. (b) $y=10-2x$: slope -2, y-intercept 10; the slope is negative, so the line falls. (c) $y=7$: slope 0, y-intercept 7; the line is horizontal. (d) $y=-3+x$: slope 1, y-intercept -3; the slope is positive, so the line rises.

Q8. Through $(3,11)$ and $(8,31)$: $b=\frac{31-11}{8-3}=\frac{20}{5}=4$. Through $(-2,9)$ and $(4,-9)$: $b=\frac{-9-9}{4-(-2)}=\frac{-18}{6}=-3$.

Q9. Make y the subject of $9x-3y=21$: $-3y=21-9x$, and dividing every term by -3 — which changes the sign of every term — gives $y=-7+3x$. (Dividing the original relation through by 3 first gives the equivalent form $3x-y=7$, from which $y=-7+3x$ follows just as quickly.) Substituting $x=0,1,2,3,4$ into $y=-7+3x$ gives

x	0	1	2	3	4
y	-7	-4	-1	2	5

The y-intercept is $(0,-7)$. For the x-intercept, solve $0=-7+3x$: $3x=7$, so $x=\frac{7}{3}$ exactly.

Q10. (a) $y = 18 - 3x$: the y-intercept is $(0, 18)$; solving $0 = 18 - 3x$ gives $x = 6$, so the x-intercept is $(6, 0)$. (b) $5x + 2y = 20$: substituting $y = 0$ leaves $5x = 20$, so $x = 4$ and the x-intercept is $(4, 0)$; substituting $x = 0$ leaves $2y = 20$, so $y = 10$ and the y-intercept is $(0, 10)$. (c) $10x + 4y = 40$: substituting $y = 0$ leaves $10x = 40$, so $x = 4$; substituting $x = 0$ leaves $4y = 40$, so $y = 10$. The intercepts are $(4, 0)$ and $(0, 10)$ — exactly those of (b). Every term of $10x + 4y = 40$ is twice the corresponding term of $5x + 2y = 20$, and multiplying a relation through by a non-zero number does not change its graph, so (b) and (c) are equivalent forms of the same line. (Both rearrange to $y = 10 - 2.5x$.)

Q11. The x-values step by 3 and the y-values rise by the constant difference 9 each step, so the function is linear with $b = \frac{\text{change in } y}{\text{change in } x} = \frac{9}{3} = 3$. The y-value at $x = 0$ is 2, so $a = 2$ and the rule is $y = 2 + 3x$. When $x = 20$, $y = 2 + 3 \times 20 = 62$.

Q12. (a) With $b = 4$ and $a = -3$: $y = -3 + 4x$. (b) With $b = -\frac{1}{2}$ and $a = 6$: $y = 6 - \frac{1}{2}x$. (c) A horizontal line has $b = 0$, and passing through $(0, -5)$ means $a = -5$, so the rule is $y = -5$.

Q13. The line crosses the y-axis at $(0, -4)$, so $a = -4$. Using the marked points, $b = \frac{4 - (-4)}{4 - 0} = \frac{8}{4} = 2$, so the rule is $y = -4 + 2x$. Solving $0 = -4 + 2x$ gives $x = 2$, so the x-intercept is $(2, 0)$ — and on the graph the line does indeed cross the x-axis at $x = 2$.

Q14. The slope is $b = \frac{16 - 7}{5 - 2} = \frac{9}{3} = 3$. Substituting $(2, 7)$ into $y = a + 3x$: $7 = a + 3 \times 2$, so $a = 1$ and the rule is $y = 1 + 3x$. Checking with $(5, 16)$: $1 + 3 \times 5 = 16$, which is correct. When $x = 10$, $y = 1 + 3 \times 10 = 31$.

Q15. In both tables the x-values increase in equal steps of 1. In Table A the y-values increase by the constant difference 4 every step (6 → 10 → 14 → 18 → 22), so Table A could come from a linear function. In Table B the differences are $1, 2, 3, 4$ — they grow from step to step — so Table B does not have a constant rate of change and cannot come from a linear function. For Table A, $b = 4$ and the y-value at $x = 0$ is 6, so the rule is $y = 6 + 4x$.

Q16. $a = 60$ is the initial value: Priya has 60 GB of data at the start of the month. $b = -1.5$ is the constant rate of change: she uses 1.5 GB of data each day. After 20 days, $y = 60 - 1.5 \times 20 = 60 - 30 = 30$, so 30 GB remain. The data runs out when $y = 0$: solving $0 = 60 - 1.5x$ gives $1.5x = 60$, so $x = 40$ — the data runs out after 40 days.

Q17. Using $(1, 9)$ and $(4, 9)$: $b = \frac{9 - 9}{4 - 1} = \frac{0}{3} = 0$ (and the pair $(4, 9)$, $(7, 9)$ gives 0 as well). Every point on the line has $y = 9$, so the rule is $y = 9$, a horizontal line, and the y-intercept is $(0, 9)$.

Q18. The rule has the form $y = a + 2x$. Substituting $(3, 10)$: $10 = a + 2 \times 3$, so $a = 4$ and the rule is $y = 4 + 2x$. For the x-intercept, solve $0 = 4 + 2x$: $2x = -4$, so $x = -2$; the x-intercept is $(-2, 0)$.

Theory

Many practical quantities begin at a fixed value and then increase or decrease at a **steady rate**: a tradesperson's bill grows by the same amount for each hour worked, and the water in a draining tank falls by the same number of litres each minute. Whenever a quantity changes at a constant rate, the relationship between the two variables is described by a **linear model**

$$y = a + b\,x\,,$$

where x is the **independent variable** (the input — often, but not always, time) and y is the **dependent variable** (the quantity being modelled). The graph of a linear model is a straight line, which is why the model is called *linear*.

The intercept and the slope in context. The two constants in the rule each have a direct practical meaning, and interpreting them — **with units** — is a key skill.

- a is the **intercept**: the value of y when $x = 0$. In a model it is the **initial value** of the quantity, or the **fixed component** of a cost — the part that is charged regardless of the value of x.
- b is the **slope** (also called the gradient): the **rate of change** of y — the amount by which y changes for each increase of **one unit** in x. If $b > 0$ the quantity increases; if $b < 0$ it decreases. The units of the slope are always "units of y **per** unit of x".

For example, if a gym charges a $60 joining fee plus $18 per week, the total cost of w weeks of membership is $C = 60 + 18\,w$: the intercept 60 is the joining fee in dollars (paid even for $w = 0$ weeks), and the slope 18 means the cost rises by $18 for each extra week.

Formulating a model from a worded description. Read the description and identify two numbers: the **initial (or fixed) value**, which becomes a, and the **rate of change**, which becomes b — taking b negative when the quantity decreases. Then write the rule, defining both variables and their units.

Formulating a model from data. When the data come as a table of values with **equally spaced** x-values, compute the **first differences** — the change in y from each column to the next. If every difference is the same, the data change at a constant rate and a linear model is appropriate. The slope is

$$b = \frac{\text{change in } y}{\text{change in } x}\,,$$

and the intercept a is the value of y when $x = 0$ — read it from the table if $x = 0$ is shown, or step backwards to it. If instead the model must pass through two known data points (x_1, y_1) and (x_2, y_2), first find the slope

$$b = \frac{y_2 - y_1}{x_2 - x_1}\,,$$

then substitute either point into $y = a + b\,x$ to find a.

Predictions, and solving for the independent variable. Once the model is written, substituting a value of x **predicts** the corresponding value of y. Working the other way, if the value of y is known, solve the equation for x:

$$y = a + b\,x \Rightarrow x = \frac{y - a}{b}\,.$$

This answers questions such as "how long until the tank is empty?" or "how many guests can we afford?".

The domain of interpretation. A linear rule such as $y = a + b\,x$ makes sense for every value of x, but the *situation being modelled* usually does not. The **domain of interpretation** is the set of values of the independent variable for which the model makes practical sense. Negative time usually means nothing; a tank cannot hold less than 0 litres or more than its capacity; a stall cannot sell more items than it has made. State the domain as an inequality, for instance $0 \le t \le 40$, and draw the graph of the model as a **line segment** over that domain only.

When is a linear model appropriate? A linear model is appropriate only when the quantity changes at (at least approximately) a **constant rate**. Checking the first differences of a data table is the standard test: if the differences grow or shrink, no single rate of change exists and a linear model should not be used. Even when a linear model fits the data well, be cautious about using it far outside the range of the data that produced it — a plant that grew 10 cm per week this month will not keep that up for two years. Finally, some independent variables (such as the number of items sold) can only take whole-number values; the line is still drawn, but it shows the trend, and answers may need rounding to a whole number to make sense.

Using CAS. General Mathematics is a technology-active, open-book course. Enter the model's rule as a function in your CAS, then use a **table of values** to make several predictions at once, **graph** the rule over its domain of interpretation, and use **numerical solve** to find x when y is known (for example, solving $435 = 75 + 90t$ for t). Always be able to carry out the by-hand steps shown below, so that you can interpret what the CAS returns and check that it is reasonable.

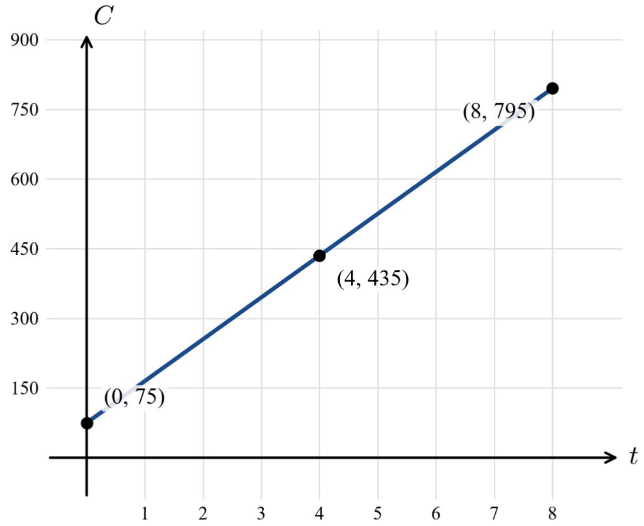
Worked examples

Example 1 — scaffolded

A Ballarat plumber charges a \$75 call-out fee plus \$90 per hour on the job, and never takes a job longer than 8 hours. Write a linear model for the total charge C dollars for a job lasting t hours, interpret the intercept and the slope, find the cost of a 3.5-hour job, find how long a job costing \$435 lasted, and state the domain of interpretation.

Working	Comment
Fixed component: the \$75 call-out fee, so $a = 75$.	The intercept is the amount charged even when $t = 0$.
Rate of change: \$90 per hour, so $b = 90$.	The charge rises by \$90 for each extra hour.
Model: $C = 75 + 90t$.	Intercept–slope form $y = a + bx$.
Interpretation: the intercept 75 is the fixed call-out fee in dollars; the slope 90 means each extra hour adds \$90 to the bill.	Always interpret a and b with units, in context.
When $t = 3.5$: $C = 75 + 90(3.5) = 75 + 315 = 390$.	Substitute to make a prediction: a 3.5-hour job costs \$390.
When $C = 435$: $435 = 75 + 90t$, so $90t = 360$ and $t = 4$.	Solve for the independent variable: the job lasted 4 hours.
Domain of interpretation: $0 \le t \le 8$.	Jobs last between 0 and 8 hours, so the model only applies there.

The graph of the model is a line segment over the domain of interpretation:



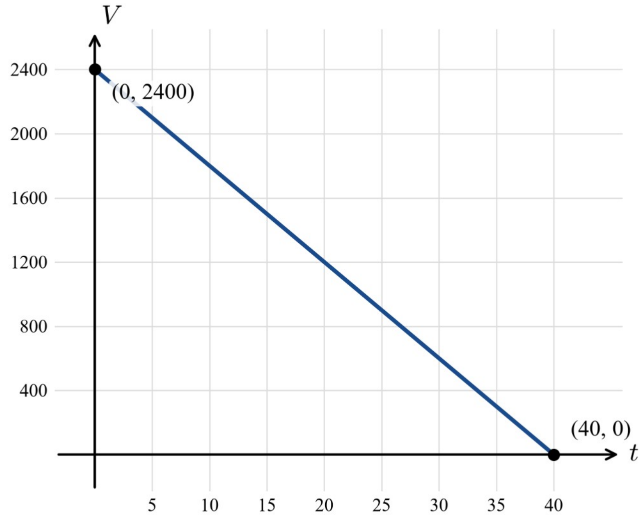
Example 2 — scaffolded

A contractor opens the valve of a full rainwater tank to drain it for cleaning and records the volume of water remaining at five-minute intervals.

t (minutes)	0	5	10	15
V (litres)	2400	2100	1800	1500

Show that a linear model is appropriate, write the model, find how long the tank takes to empty, and state the domain of interpretation.

Working	Comment
First differences: $2100-2400=-300$, $1800-2100=-300$, $1500-1800=-300$.	The t-values are equally spaced, so compare successive V-values.
Every difference is -300 litres per 5 minutes, so the rate of change is constant and a linear model is appropriate.	A constant rate of change is exactly what a linear model requires.
Slope: $b=\dfrac{-300}{5}=-60$.	The tank loses 60 litres each minute.
Intercept: $a=2400$, the volume at $t=0$.	Read directly from the table.
Model: $V=2400-60t$.	Negative slope: the volume is decreasing.
Empty when $V=0$: $0=2400-60t$, so $60t=2400$ and $t=40$.	The tank takes 40 minutes to empty.
Domain of interpretation: $0\le t\le 40$.	Before $t=0$ the valve was closed; after $t=40$ the tank is empty.

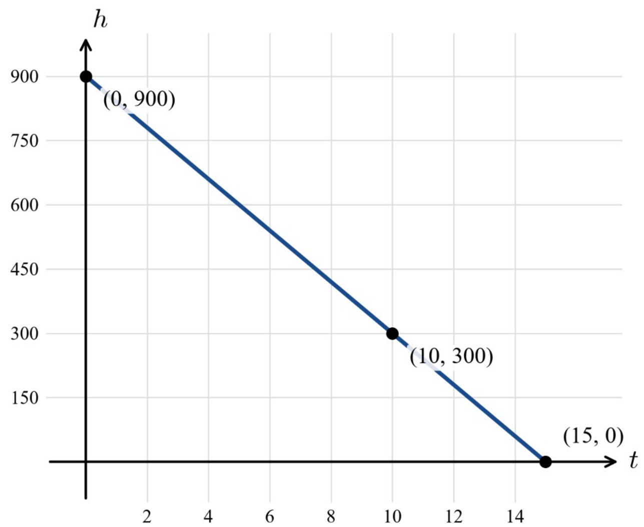


Example 3 — unscaffolded

A hot-air balloon over the Yarra Valley begins its descent at a constant rate. Three minutes after the descent begins its altitude is 720 m, and eight minutes after it begins its altitude is 420 m. Write a linear model for the altitude h metres after t minutes of descent, predict the altitude at $t=10$, find when the balloon lands, and state the domain of interpretation.

The descent is at a constant rate, so a linear model $h=a+bt$ applies, passing through the points $(3,720)$ and $(8,420)$. The slope is $b=\frac{420-720}{8-3}=\frac{-300}{5}=-60$, so the balloon descends 60 metres each minute. Substituting $(3,720)$ into $h=a-60t$ gives $720=a-60(3)=a-180$, so $a=900$: the descent began from an altitude of 900 m. The model is $h=900-60t$. At $t=10$, $h=900-60(10)=300$, an altitude of 300 m. The balloon lands when $h=0$: $0=900-60t$ gives $t=\frac{900}{60}=15$ minutes. The model applies from the start of the descent to the landing.

$\therefore$ $h=900-60t$; the altitude at $t=10$ is 300 m, the balloon lands after 15 minutes, and the domain of interpretation is $0\le t\le 15$.



Example 4 — unscaffolded

A Landcare group runs a sausage sizzle. It spends \$120 on supplies — enough for 400 sausages — and sells sausages for \$2.50 each. Write a linear model for the profit P dollars after selling n sausages, interpret the intercept, find the profit after 30 sales, find how many sausages must be sold to break even, find the greatest possible profit, and comment on the domain of interpretation.

Each sale adds \$2.50, and the \$120 spent on supplies is a fixed cost, so $P=-120+2.50n$. The intercept -120 means that with no sausages sold the group has a loss of \$120 — the money already spent on supplies. After 30 sales, $P=-120+2.50(30)=-120+75=-45$, a loss of \$45: the first sales only recover part of the outlay. The group **breaks even** when $P=0$: solving $0=-120+2.50n$ gives $n=\frac{120}{2.50}=48$, so the 48th sausage sold brings the group level, and every sale after that is pure profit. Only 400 sausages can be sold, so the greatest profit is $P=-120+2.50(400)=-120+1000=880$ dollars. The domain of interpretation is $0\le n\le 400$; moreover n counts sausages, so only whole-number values of n make sense — the line shows the trend, and answers must be whole numbers.

$\therefore$ $P=-120+2.50n$ for whole numbers $0\le n\le 400$; the group breaks even at 48 sausages and the greatest possible profit is \$880.

Practice questions

Scaffolded

Q1. A Melbourne taxi charges a flagfall of \$4.60 plus \$2.20 per kilometre travelled. Let C be the fare in dollars for a trip of d kilometres. (a) State the initial (fixed) value and the rate of change. (b) Write a linear model for the fare. (c) Find the fare for a 15 km trip. (d) A passenger's fare was \$26.60. How far did she travel? (e) Interpret the slope of the model in context.

Q2. A Geelong function centre quotes the following total costs C (in dollars) for a reception with n guests.

n (guests)	20	40	60	80
C (dollars)	3600	5200	6800	8400

(a) Compute the first differences and explain why a linear model is appropriate.
(b) Find the rate of change (the cost per extra guest).
(c) Find the intercept, interpret it, and write the model.
(d) Predict the cost of a reception with 150 guests.

Q3. A backyard pool in Darwin holds 9000 litres when full. It already contains 1500 litres when a hose is turned on, adding water at 25 litres per minute. Let V litres be the volume after t minutes. (a) Write a linear model for V. (b) Find the volume after one hour. (c) Find how long the pool takes to fill, in minutes and in hours. (d) State the domain of interpretation of the model.

Q4. A gardener plants a bamboo screening plant and assumes it grows at a constant rate. Two weeks after planting it is 45 cm tall, and five weeks after planting it is 75 cm tall. Let h cm be the height w weeks after planting. (a) Find the rate of growth in centimetres per week. (b) Find the height of the plant when it was planted. (c) Write the linear model. (d) Predict the height eight weeks after planting. (e) In which week does the model say the plant reaches 125 cm? (f) Explain why the model should not be used to predict the height two years after planting.

Q5. The fuel remaining in a car's tank on a country drive is modelled by $F=48-0.08d$, where F is in litres and d is the distance driven in kilometres. (a) Interpret the intercept 48 in context. (b) Interpret the slope in context, with units, and express it in litres per 100 km. (c) Find the fuel remaining after 250 km. (d) How far can the car drive before the tank is empty, and what is the domain of interpretation?

Q6. Mia and Noah each open a savings account with \$300. The balances at the end of each of the next four weeks are shown.

week w	0	1	2	3	4
Mia (\$)	300	380	460	540	620
Noah (\$)	300	360	450	570	720

(a) Compute the first differences for each account.
(b) Whose balance is suited to a linear model? Explain why, and why the other is not.

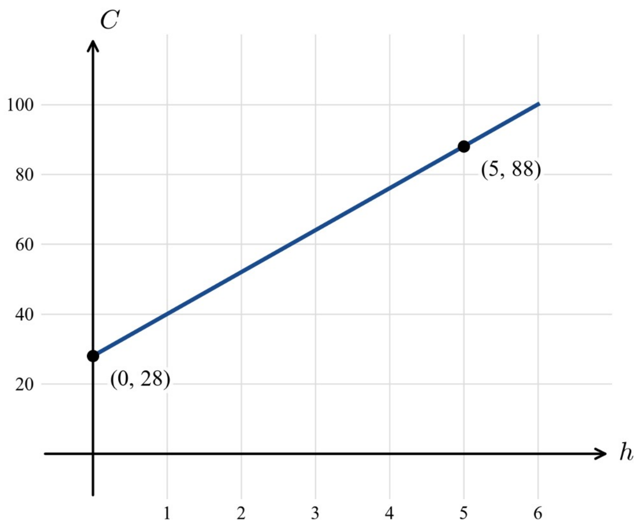
(c) Write a linear model for that person's balance.
(d) Use your model to predict that person's balance at the end of week 10.

Unscaffolded

Q7. A St Kilda shop hires out bicycles for a fixed fee of \$15 plus \$6 per hour. Write a linear model for the cost C dollars of a hire lasting t hours, find the cost of a five-hour hire, and find how many hours a rider can afford with \$51.

Q8. A phone is plugged in to charge and its battery percentage P is recorded every 10 minutes: at $t=0, 10, 20, 30$ minutes the battery shows $22, 34, 46, 58$ per cent. Show that a linear model is appropriate, write the model, and find how long after being plugged in the phone reaches full charge (100%).

Q9. The graph below shows the cost C dollars of hiring a kayak on the Noosa River for h hours. The line passes through $(0, 28)$ and $(5, 88)$.



(a) Find the intercept and the slope of the line, and interpret each in context.
(b) Write the linear model for the cost.
(c) Find the cost of a six-hour hire.
(d) For how many hours can a paddler hire a kayak with \$76?

Q10. At the start of winter a Ballarat household has 2.4 tonnes of firewood and burns 0.15 tonnes each week. Write a linear model for the mass M tonnes of firewood left after w weeks, find the mass left after 8 weeks, find how many weeks the firewood lasts, and state the domain of interpretation.

Q11. A family drives at a steady speed from Melbourne towards their holiday house. One hour into the trip they have 450 km left to travel; three hours in they have 270 km left. Write a linear model for the distance D km remaining after t hours, find the distance remaining after 4 hours, find how long the whole trip takes, state the domain of interpretation, and interpret the slope in context.

Q12. A catering company prices a buffet using the model $C=350+28n$, where C is the total cost in dollars for a function with n guests. (a) Interpret the intercept and the slope of the model in context, with units. (b) Find the cost of catering for 80 guests. (c) What does the model give when $n=0$, and what does this represent?

Q13. A new bakery's social-media account had 120 followers when it opened, and $240, 480, 960, 1920$ followers at the ends of months $1, 2, 3, 4$. By computing first differences, explain why a linear model is **not** appropriate for these data.

Q14. During a drought, a reservoir holding 850 gigalitres loses 12 gigalitres each week. Write a linear model for the volume W gigalitres after t weeks, find the volume after 20 weeks, and find after how many weeks the volume falls to 250 gigalitres, the level at which water restrictions begin.

Q15. A courier charges a flat fee of \$9.50 plus \$3.20 per kilogram to deliver a parcel. Write a linear model for the cost C dollars of delivering a parcel of mass m kilograms, find the cost for a 2.5 kg parcel, and find the mass of a parcel that costs \$25.50 to deliver.

Q16. A student spends \$160 on materials and a stall fee to sell handmade soap at a weekend market, makes 60 bars, and sells them for \$8 each. Let P dollars be her profit after selling n bars. (a) Write a linear model for P. (b) Find P when $n = 10$, and interpret the answer. (c) How many bars must she sell to break even? (d) Find her greatest possible profit, and state the domain of interpretation of the model.

Q17. On a Snowy Mountains hike, the temperature at an altitude of 500 m is 21°C and at 1500 m it is 15°C. Assume temperature falls at a constant rate with altitude. (a) Write a linear model for the temperature T (in °C) at altitude h metres. (b) Predict the temperature at 2200 m, near the summit of Mt Kosciuszko. (c) At what altitude does the model predict a temperature of 0°C? (d) The data came from altitudes between 500 m and 1500 m. Why should predictions far outside this range be treated with caution?

Q18. Aisha's mobile plan gives her 50 GB of data for a 31-day billing month, and she uses 1.6 GB every day. (a) Write a linear model for the data R GB remaining after d days. (b) How much data remains after 20 days? (c) After how many days does the model say her data runs out? Will her allowance last the month? (d) State the domain of interpretation of the model.

Answers

Q1. (a) Initial value \$4.60 (flagfall); rate \$2.20 per km. (b) $C=4.60+2.20d$. (c) \$37.60. (d) 10 km. (e) Each extra kilometre adds \$2.20 to the fare. **Q2.** (a) Differences all $+1600$ per 20 guests — constant rate, so linear. (b) \$80 per guest. (c) $a=2000$ (fixed venue-hire cost); $C=2000+80n$. (d) \$14 000. **Q3.** (a) $V=1500+25t$. (b) 3000 L. (c) 300 minutes $=5$ hours. (d) $0\le t\le 300$. **Q4.** (a) 10 cm per week. (b) 25 cm. (c) $h=25+10w$. (d) 105 cm. (e) Week 10. (f) A plant will not keep growing at a constant 10 cm/week for two years, so the constant-rate assumption fails far outside the data. **Q5.** (a) The tank holds 48 L when the drive starts. (b) The car uses 0.08 L of fuel per km, i.e. 8 L per 100 km. (c) 28 L. (d) 600 km; $0\le d\le 600$. **Q6.** (a) Mia: $80, 80, 80, 80$; Noah: $60, 90, 120, 150$. (b) Mia's — her balance grows at a constant \$80/week; Noah's differences grow, so no single rate exists. (c) $M=300+80w$. (d) \$1100. **Q7.** $C=15+6t$; \$45; 6 hours. **Q8.** Differences all $+12$ per 10 min, so linear; $P=22+1.2t$; full charge at $t=65$ minutes. **Q9.** (a) Intercept 28 — the fixed hire fee of \$28; slope 12 — each extra hour costs \$12. (b) $C=28+12h$. (c) \$100. (d) 4 hours. **Q10.** $M=2.4-0.15w$; 1.2 tonnes; 16 weeks; $0\le w\le 16$. **Q11.** $D=540-90t$; 180 km; 6 hours; $0\le t\le 6$; the slope -90 means the remaining distance falls 90 km each hour, i.e. the family travels at 90 km/h. **Q12.** (a) Intercept 350: a fixed cost of \$350 charged for any function; slope 28: each guest adds \$28. (b) \$2590. (c) $C=350$; the fixed cost of putting on a function even with no guests. **Q13.** Differences $120, 240, 480, 960$ — the monthly increase itself grows each month, so there is no constant rate of change and a linear model is not appropriate. **Q14.** $W=850-12t$; 610 GL; 50 weeks. **Q15.** $C=9.50+3.20m$; \$17.50; 5 kg. **Q16.** (a) $P=-160+8n$. (b) $P=-80$: after 10 sales she is still \$80 short of covering her costs. (c) 20 bars. (d) \$320; $0\le n\le 60$ (whole numbers). **Q17.** (a) $T=24-0.006h$. (b) 10.8°C. (c) 4000 m. (d) Far outside 500–1500 m the constant-rate assumption may fail (extrapolation), so predictions there are unreliable. **Q18.** (a) $R=50-1.6d$. (b) 18 GB. (c) $d=31.25$ days — just after the 31-day month ends, so the allowance lasts the month. (d) $0\le d\le 31$ (the 31-day billing month).

Fully-worked solutions

Q1. (a) The flagfall \$4.60 is charged regardless of distance, so it is the initial (fixed) value; the rate of change is \$2.20 per kilometre. (b) With $a=4.60$ and $b=2.20$, the model is $C=4.60+2.20d$. (c) When $d=15$: $C=4.60+2.20(15)=4.60+33.00=37.60$, so the fare is \$37.60. (d) When $C=26.60$: $26.60=4.60+2.20d$, so $2.20d=22.00$ and $d=\frac{22.00}{2.20}=10$. She travelled 10 km. (e) The slope 2.20 is the rate of change of the fare: each extra kilometre travelled adds \$2.20 to the fare.

Q2. (a) The n-values step evenly by 20 guests. The first differences are $5200-3600=1600$, $6800-5200=1600$ and $8400-6800=1600$. Every difference is the same, so the cost rises at a constant rate and a linear model is appropriate. (b) The cost rises \$1600 for every 20 extra guests, so the rate is $b=\frac{1600}{20}=80$ dollars per guest. (c) Stepping back 20 guests from $n=20$: $a=3600-1600=2000$. (Equivalently $a=3600-80\times 20=2000$.) The intercept \$2000 is the fixed cost — venue hire and staffing charged even before any guests are catered for. The model is $C=2000+80n$. (d) When $n=150$: $C=2000+80(150)=2000+12\,000=14\,000$, so a 150-guest reception costs \$14 000.

Q3. (a) The initial volume is 1500 L and water is added at 25 L per minute, so $V=1500+25t$. (b) One hour is $t=60$: $V=1500+25(60)=1500+1500=3000$, so the pool holds 3000 L. (c) The pool is full when $V=9000$: $9000=1500+25t$, so $25t=7500$ and $t=\frac{7500}{25}=300$ minutes, which is $300\div 60=5$ hours. (d) The model applies from the moment the hose is turned on until the pool is full: $0\le t\le 300$.

Q4. (a) Between week 2 and week 5 the plant grew $75-45=30$ cm in $5-2=3$ weeks, so the rate is $b=\frac{30}{3}=10$ cm per week. (b) Working back from week 2: $a=45-10\times 2=25$. The plant was 25 cm tall when planted. (c) $h=25+10w$. (d) When $w=8$: $h=25+10(8)=105$, so the model predicts 105 cm. (e) When $h=125$: $125=25+10w$, so $10w=100$ and $w=10$. The model says the plant reaches 125 cm in week 10. (f) Two years is about 104 weeks, far beyond the few weeks of observations. Plants slow down and stop growing as they mature, so the constant-rate assumption behind the linear model will fail; the model would predict $25+10\times104=1065$ cm — over ten metres — which is not credible for a screening plant.

Q5. (a) The intercept 48 is the value of F when $d=0$: the tank holds 48 litres of fuel at the start of the drive. (b) The slope -0.08 means the fuel remaining falls by 0.08 litres for each kilometre driven — the car consumes 0.08 L/km, which is $0.08\times100=8$ litres per 100 km. (c) When $d=250$: $F=48-0.08(250)=48-20=28$, so 28 litres remain. (d) The tank is empty when $F=0$: $0=48-0.08d$, so $0.08d=48$ and $d=\frac{48}{0.08}=600$. The car can drive 600 km, and the model makes sense for $0\le d\le600$.

Q6. (a) Mia: $380-300=80$, $460-380=80$, $540-460=80$, $620-540=80$ — every difference is \$80. Noah: $360-300=60$, $450-360=90$, $570-450=120$, $720-570=150$ — the differences grow each week. (b) Mia's balance is suited to a linear model because it rises by the same amount (\$80) every week — a constant rate of change. Noah's deposits grow every week, so his balance has no single rate of change and a linear model is not appropriate for it. (c) Mia starts with $a=300$ and adds $b=80$ per week: $M=300+80w$. (d) When $w=10$: $M=300+80(10)=300+800=1100$, so the model predicts \$1100.

Q7. The fixed fee is \$15 and each hour adds \$6, so $C=15+6t$. A five-hour hire costs $C=15+6(5)=15+30=45$ dollars. With \$51: $51=15+6t$, so $6t=36$ and $t=6$ — the rider can afford six hours.

Q8. The times step evenly by 10 minutes and the first differences are $34-22=12$, $46-34=12$ and $58-46=12$ — all equal, so the battery charges at a constant rate and a linear model is appropriate. The rate is $b=\frac{12}{10}=1.2$ percentage points per minute, and the intercept is the starting charge $a=22$, so $P=22+1.2t$. Full charge is $P=100$: $100=22+1.2t$, so $1.2t=78$ and $t=\frac{78}{1.2}=65$. The phone is fully charged 65 minutes after being plugged in.

Q9. (a) The line meets the vertical axis at $(0,28)$, so the intercept is 28: hiring a kayak costs a fixed \$28 before any time is added. From $(0,28)$ to $(5,88)$ the cost rises $88-28=60$ dollars over 5 hours, so the slope is $\frac{60}{5}=12$: each hour of hire costs \$12. (b) $C=28+12h$. (c) When $h=6$: $C=28+12(6)=28+72=100$, so a six-hour hire costs \$100. (d) When $C=76$: $76=28+12h$, so $12h=48$ and $h=4$. The paddler can hire the kayak for 4 hours.

Q10. The initial mass is 2.4 tonnes and 0.15 tonnes is burnt each week, so $M=2.4-0.15w$. After 8 weeks, $M=2.4-0.15(8)=2.4-1.2=1.2$ tonnes. The firewood runs out when $M=0$: $0=2.4-0.15w$, so $0.15w=2.4$ and $w=\frac{2.4}{0.15}=16$ — it lasts 16 weeks. The model applies while there is firewood: $0\le w\le16$.

Q11. The model passes through $(1,450)$ and $(3,270)$. The slope is $b=\frac{270-450}{3-1}=\frac{-180}{2}=-90$, and substituting $(1,450)$ into $D=a-90t$ gives $450=a-90$, so $a=540$: the holiday house is 540 km from Melbourne. The model is $D=540-90t$. After 4 hours, $D=540-90(4)=540-360=180$ km remain. The trip ends when $D=0$: $0=540-90t$ gives $t=\frac{540}{90}=6$ hours. The domain of interpretation is $0\le t\le6$. The slope -90 means the distance remaining falls by 90 km every hour — the family is travelling at 90 km/h.

Q12. (a) The intercept 350 is the cost when $n=0$: a fixed charge of \$350 (kitchen, staff and equipment) that applies to any function regardless of size. The slope 28 is the rate of change: each additional guest adds \$28 to the total cost. (b) When $n=80$: $C=350+28(80)=350+2240=2590$, so catering for 80 guests costs \$2590. (c) When $n=0$ the model gives $C=350$. This represents the fixed cost of putting on a function even if no guests attend — the venue and staff must still be paid for.

Q13. The months step evenly by 1. The first differences are $240-120=120$, $480-240=240$, $960-480=480$ and $1920-960=960$. The differences are not constant — in fact the increase doubles every month — so the follower count does not change at a constant rate, and a linear model $y=a+bx$ (which requires a single, fixed rate of change) is not appropriate for these data.

Q14. The initial volume is 850 GL and 12 GL is lost each week, so $W = 850 - 12t$. After 20 weeks, $W = 850 - 12(20) = 850 - 240 = 610$ GL. Restrictions begin when $W = 250$: $250 = 850 - 12t$, so $12t = 600$ and $t = \frac{600}{12} = 50$. Restrictions begin after 50 weeks.

Q15. The flat fee is $9.50 and each kilogram adds $3.20, so $C = 9.50 + 3.20m$. For a 2.5 kg parcel, $C = 9.50 + 3.20(2.5) = 9.50 + 8.00 = 17.50$, so the cost is $17.50. When $C = 25.50$: $25.50 = 9.50 + 3.20m$, so $3.20m = 16.00$ and $m = \frac{16.00}{3.20} = 5$. The parcel has a mass of 5 kg.

Q16. (a) The $160 spent on materials and the stall fee is a fixed cost and each bar sold brings in $8, so $P = -160 + 8n$. (b) When $n = 10$: $P = -160 + 8(10) = -160 + 80 = -80$. After selling 10 bars she has made a loss of $80 — her takings so far cover only half of her costs. (c) Break-even is $P = 0$: $0 = -160 + 8n$ gives $8n = 160$, so $n = 20$. She must sell 20 bars to break even. (d) She made 60 bars, so at most $n = 60$ can be sold: $P = -160 + 8(60) = -160 + 480 = 320$, a greatest possible profit of $320. The domain of interpretation is $0 \le n \le 60$, with n a whole number of bars.

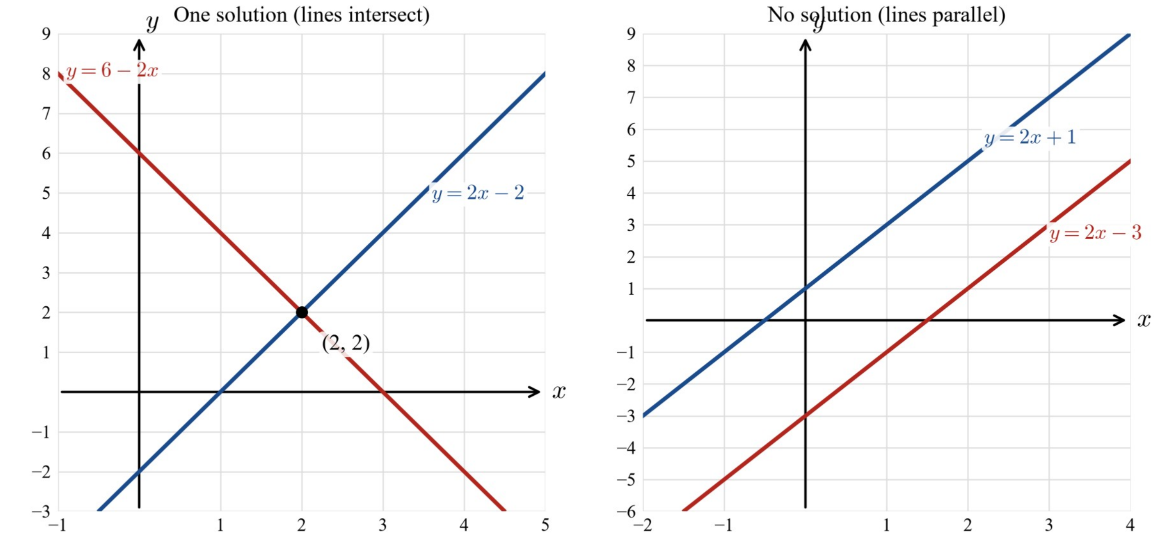
Q17. (a) The model passes through $(500, 21)$ and $(1500, 15)$. The slope is $b = \frac{15 - 21}{1500 - 500} = \frac{-6}{1000} = -0.006$, so the temperature falls 0.006°C per metre of altitude (6°C per 1000 m). Substituting $(500, 21)$ into $T = a - 0.006h$ gives $21 = a - 0.006(500) = a - 3$, so $a = 24$, and the model is $T = 24 - 0.006h$. (b) When $h = 2200$: $T = 24 - 0.006(2200) = 24 - 13.2 = 10.8$, so the model predicts 10.8°C. (c) When $T = 0$: $0 = 24 - 0.006h$, so $0.006h = 24$ and $h = \frac{24}{0.006} = 4000$. The model predicts 0°C at an altitude of 4000 m. (d) The model was built from data between 500 m and 1500 m. Using it far outside that range is **extrapolation**: there is no evidence the temperature keeps falling at the same constant rate well beyond the data (weather conditions and other factors change with altitude), so predictions such as the answer to part (c) — an altitude higher than any Australian mountain — should be treated with caution.

Q18. (a) The allowance starts at 50 GB and falls by 1.6 GB each day, so $R = 50 - 1.6d$. (b) When $d = 20$: $R = 50 - 1.6(20) = 50 - 32 = 18$, so 18 GB remain. (c) The data run out when $R = 0$: $0 = 50 - 1.6d$, so $1.6d = 50$ and $d = \frac{50}{1.6} = 31.25$. The model says the data run out 31.25 days into the billing month — but the month is only 31 days long, so the allowance (just) lasts the whole month. (d) The model makes practical sense over the billing month only. Since the month is 31 days long and the data never run out within it (part (c)), the domain of interpretation is $0 \le d \le 31$; beyond day 31 the allowance resets, so the rule no longer applies.

Theory

A linear equation in two variables, such as $y = 2x - 2$, has infinitely many solutions — every point on its straight-line graph. Many practical questions, however, give **two** pieces of information about the same two quantities, and each piece of information becomes a linear equation. A pair of linear equations that must hold at the same time is called a pair of **simultaneous linear equations**, and to **solve** the pair means to find the values of the two variables that satisfy **both** equations at once.

The geometric meaning. Each equation is the rule of a straight line, so a pair of values that satisfies both equations is a point that lies on both lines — the **point of intersection**. Two lines with **different slopes** must cross exactly once, so the pair of equations has exactly **one solution**. For example, the lines $y = 2x - 2$ and $y = 6 - 2x$ intersect at $(2, 2)$, and this point satisfies both rules: $2 \times 2 - 2 = 2$ and $6 - 2 \times 2 = 2$. If the two lines have the **same slope** but different y-intercepts they are **parallel**: they never meet, and the pair of equations has **no solution**. The lines $y = 2x + 1$ and $y = 2x - 3$, for instance, both have slope 2, so no pair of values can satisfy both equations.



Solving by graphing. Draw both lines accurately on the same axes (plot two or three points of each line, or use its intercepts), read off the coordinates of the point of intersection, and **check** the reading by substituting it into both equations. Graphing gives a picture of the whole situation, but it is only as accurate as the drawing — it works best when the solution is a pair of small whole numbers that sit on the grid.

Solving by substitution. Substitution suits pairs in which one equation already has y (or x) as its subject, such as $y = 2x - 1$. Substitute the expression for that variable into the other equation, giving a single linear equation in one unknown; solve it; then substitute back to find the other variable. Finish by checking the pair in both original equations.

Solving by elimination. Elimination suits pairs written with both variables on the same side, such as $2x + 3y = 23$ and $5x - 2y = 10$. Add or subtract the equations so that one variable cancels out: if the chosen variable has the **same coefficient** in both equations, **subtract** one equation from the other; if its coefficients are **equal in size but opposite in sign**, **add** the equations. If neither happens as the equations stand, first multiply one or both equations by suitable numbers to make matching coefficients. Solve the one-variable equation that results, substitute back to find the other variable, and check the pair in both original equations.

Linear models and break-even analysis. To solve a worded problem, define the two variables (with their units), translate each piece of given information into a linear equation, solve the pair by the most convenient method, and **interpret the solution in context**. A common business application compares the **cost** of producing n items, $C = \text{fixed costs} + \text{cost per item} \times n$, with the **revenue** earned by selling them, $R = \text{selling price} \times n$. The **break-even point** is where $R = C$: selling fewer items than this leaves a **loss**, selling more than this earns a **profit**, and the profit at any level of sales is $R - C$.

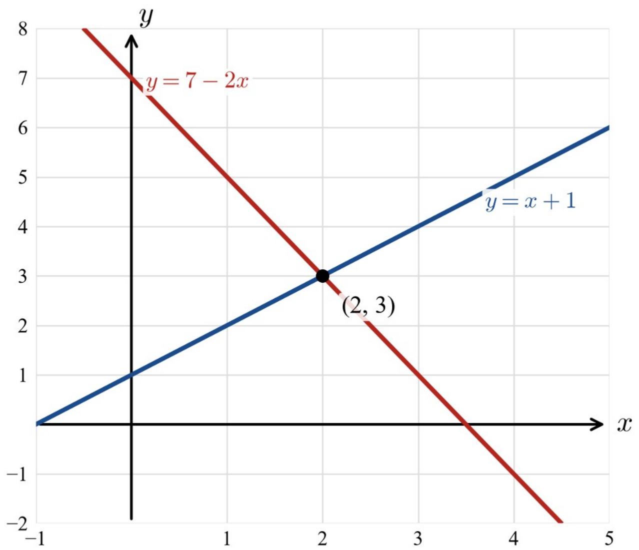
Using CAS. General Mathematics is a technology-active, open-book course, and a CAS calculator solves a pair of simultaneous linear equations directly: use the simultaneous-equation template (or a solve command entered with the two equations and the two variables), or graph both lines and use the intersection tool. The by-hand methods of this section show what the calculator is doing, and let you check that its answer — and your entry of the equations — is correct.

Worked examples

Example 1 — scaffolded

Solve the simultaneous equations $y=x+1$ and $y=7-2x$ by drawing the graphs of both equations and finding their point of intersection.

Working	Comment
For $y=x+1$: when $x=0$, $y=1$; when $x=1$, $y=2$; when $x=3$, $y=4$.	Two points fix a straight line; the third point is a check. Plot $(0,1)$, $(1,2)$ and $(3,4)$ and rule the line.
For $y=7-2x$: when $x=0$, $y=7$; when $x=1$, $y=5$; when $x=3$, $y=1$.	Plot $(0,7)$, $(1,5)$ and $(3,1)$ and rule the second line.
The lines cross at the single point $(2,3)$ — see the graph.	The point of intersection lies on both lines, so it satisfies both equations.
Check in $y=x+1$: $2+1=3$. Check in $y=7-2x$: $7-2\times2=3$.	Always check a graphical reading by substituting into **both** equations.



So the solution is $x=2$, $y=3$: the point $(2,3)$ is the only point that lies on both lines.

Example 2 — scaffolded

Solve the simultaneous equations $y=2x-1$ and $3x+2y=19$ by substitution.

Working	Comment
Substitute $y=2x-1$ into the second equation: $3x+2(2x-1)=19$.	The first equation has y as its subject, so replace y in the other equation by $2x-1$.
Expand and simplify: $3x+4x-2=19$, so $7x-2=19$.	The equation now has only the one unknown x.
Solve: $7x=21$, so $x=3$.	Add 2 to both sides, then divide both sides by 7.
Substitute back: $y=2\times3-1=5$.	Use the subject equation to find y.
Check in $3x+2y=19$: $3\times3+2\times5=9+10=19$.	The pair satisfies both equations.

So the solution is $x=3$, $y=5$.

Example 3 — unscaffolded

Solve the simultaneous equations $2x+3y=23$ and $5x-2y=10$ by elimination.

Adding or subtracting the equations as they stand does not remove either variable, so first make the coefficients of y equal in size and opposite in sign. Multiplying the first equation by 2 gives $4x+6y=46$, and multiplying the second equation by 3 gives $15x-6y=30$. The y terms are now $+6y$ and $-6y$, so adding the two new equations eliminates y:

$$4x+6y+15x-6y=46+30\text{, so }19x=76\text{ and }x=4\text{.}$$

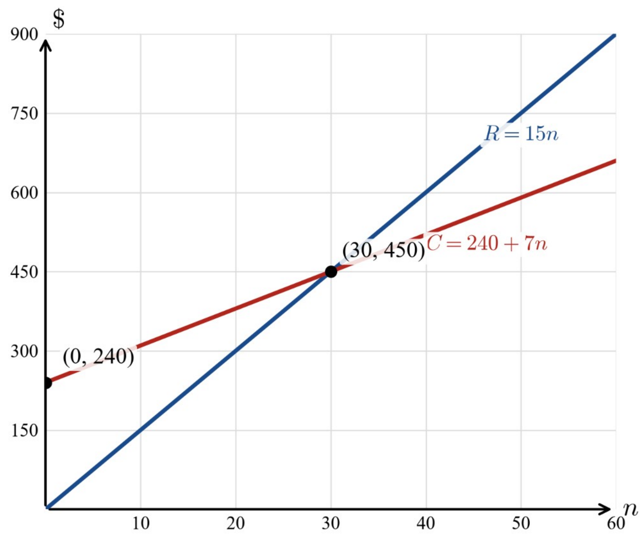
Substituting $x=4$ into $2x+3y=23$ gives $8+3y=23$, so $3y=15$ and $y=5$. Checking the pair in the other original equation: $5\times4-2\times5=20-10=10$, as required.

∴ the solution is $x=4$, $y=5$.

Example 4 — unscaffolded

Mia sells hand-poured soy candles at a monthly makers' market. Her fixed costs are \$240 per month (stall fee and equipment hire), the materials for each candle cost \$7, and she sells each candle for \$15. Let n be the number of candles she makes and sells in a month. Find equations for her monthly cost and revenue, find the break-even point, and find her profit if she sells 45 candles.

The monthly cost is the fixed \$240 plus \$7 for each candle, $C=240+7n$, and the revenue is \$15 for each candle sold, $R=15n$ (both in dollars). At the break-even point $R=C$, so $15n=240+7n$, giving $8n=240$ and $n=30$; there both sides equal $15\times30=\$450$. Each candle sold contributes $15-7=\$8$ towards the fixed costs, and $240\div8=30$ confirms the break-even number. On the graph the cost line starts at $\$240$ and rises \$7 per candle, the revenue line starts at the origin and rises \$15 per candle, and the lines cross at $(30,450)$: for fewer than 30 candles the cost line is above the revenue line (a loss), and beyond 30 candles the revenue line is on top (a profit). If Mia sells 45 candles, $R=15\times45=675$ and $C=240+7\times45=555$, so her profit is $R-C=675-555=\$120$.

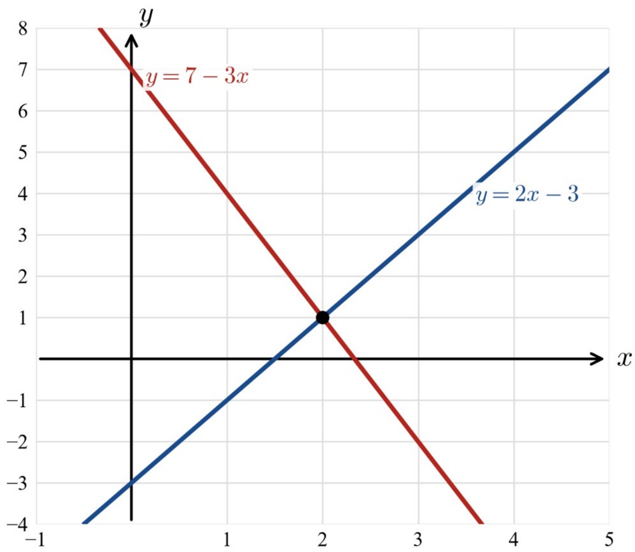


∴ with $C=240+7n$ and $R=15n$, Mia breaks even at 30 candles (\$450 of sales), and selling 45 candles earns a profit of \$120.

Practice questions

Scaffolded

Q1. The simultaneous equations $y=2x-3$ and $y=7-3x$ are to be solved graphically. The graphs of both equations are drawn below.



(a) For the line $y=2x-3$, find y when $x=0$, when $x=1$ and when $x=3$.
(b) For the line $y=7-3x$, find y when $x=0$, when $x=1$ and when $x=3$.
(c) Use the graph to write down the coordinates of the point of intersection of the two lines.
(d) Check your reading by substituting it into both equations, and hence state the solution of the simultaneous equations.

Q2. Consider the simultaneous equations $y=x+4$ and $3x+y=16$. (a) Substitute $y=x+4$ into $3x+y=16$ to obtain an equation in x only. (b) Solve this equation for x. (c) Find the value of y. (d) Check that your pair of values satisfies both equations.

Q3. Consider the simultaneous equations $2x+y=11$ and $2x-y=5$. (a) Add the two equations. Which variable is eliminated? (b) Solve the resulting equation for x. (c) Substitute your value of x into $2x+y=11$ to find y. (d) Check the solution in the equation $2x-y=5$.

Q4. Consider the simultaneous equations $2x+3y=16$ and $x+2y=9$. (a) Multiply the second equation by 2 and write down the new equation. (b) Subtract the first equation from your answer to part (a) to eliminate x, and hence find y. (c) Find the value of x. (d) Check the solution in the equation $2x+3y=16$.

Q5. At a school production, Raf buys two adult tickets and three child tickets for $47, while Elena buys one adult ticket and two child tickets for $26. (a) Using a for the price of an adult ticket and c for the price of a child ticket, both in dollars, write down two simultaneous equations. (b) Solve the equations. (c) State the price of each type of ticket. (d) Find the total cost for a family that needs three adult tickets and four child tickets.

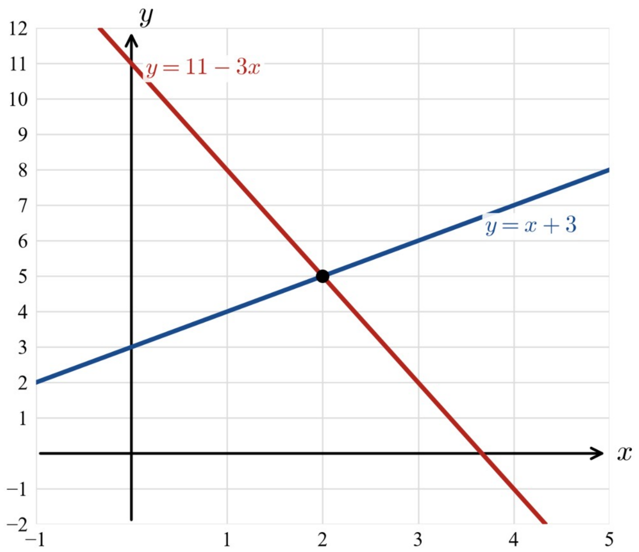
Q6. Priya sells home-baked brownies at a weekend market. The stall costs $90 to hire for the day, the ingredients for each brownie cost $1.50, and she sells each brownie for $4.00. Let n be the number of brownies she makes and sells. (a) Write an equation for her total cost C (in dollars) for the day. (b) Write an equation for her revenue R (in dollars). (c) Solve $R=C$ to find the break-even number of brownies. (d) Interpret the break-even point in the context of Priya's stall. (e) Find her profit or loss for the day if she sells 50 brownies.

Unscaffolded

Q7. Solve the simultaneous equations $y=3x-2$ and $x+y=14$ by substitution, and check your solution in both equations.

Q8. Solve the simultaneous equations $5x+2y=31$ and $3x-2y=9$ by elimination.

Q9. The graphs of $y=x+3$ and $y=11-3x$ are drawn below. Use the graph to solve the simultaneous equations, and check the solution by substituting it into both equations.



Q10. Solve the simultaneous equations $3x + 4y = 25$ and $2x + 3y = 18$ by elimination. (You will need to multiply both equations before adding or subtracting.)

Q11. At a bakery, three meat pies and two sausage rolls cost $27.60, while two meat pies and five sausage rolls cost $33.80. Find the price of a meat pie and the price of a sausage roll, and hence the cost of one meat pie and one sausage roll together.

Q12. A sports club runs a car-wash fundraiser. Hiring the venue and equipment costs $360, the supplies for each car washed cost $2.50, and the club charges $10 per wash. Write equations for the club's total cost and revenue for n washed cars, find the break-even number of cars, explain what this number means for the club, and find the profit if 70 cars are washed.

Q13. Nadia is comparing two pilates studios. Studio A charges a $45 joining fee plus $9 per class; Studio B charges a $120 joining fee plus $6 per class. For what number of classes is the total cost the same at both studios, and what is that cost? Which studio is cheaper for Nadia if she plans to attend 30 classes, and by how much? Who should choose Studio A?

Q14. For each pair of simultaneous equations, state which method — graphing, substitution or elimination — is the most efficient, and give a brief reason. Then solve pairs (a) and (b). (a) $y = 4x - 7$ and $2x + y = 17$. (b) $3x + 2y = 23$ and $3x - 2y = 7$. (c) A pair of equations whose graphs are already drawn accurately on the same set of axes.

Q15. Explain why the simultaneous equations $y = 3x + 2$ and $6x - 2y = 5$ have no solution, and describe what a graph of the two equations would show.

Q16. The pair of values $x = 2,\ y = 5$ is the solution of the simultaneous equations $ax + y = 13$ and $x + by = 17$. Find the values of a and b.

Q17. A landscaper buys four bags of mulch and three bags of gravel for $101 on Monday, and six bags of mulch and two bags of gravel for $109 on Tuesday, at the same prices both days. Find the price of a bag of mulch and the price of a bag of gravel.

Q18. A small business prints custom tote bags. Fixed costs are $1240 per month, each bag costs $3.80 to produce, and each bag sells for $11.80. Write equations for the monthly cost and revenue for n bags, find the break-even number of bags, and find how many bags must be sold in a month to make a profit of $600. Interpret your answers for the owner.

Answers

Q1. (a) $-3,\ -1,\ 3$. (b) $7,\ 4,\ -2$. (c) $(2,1)$. (d) $x=2,\ y=1$. **Q2.** (a) $4x+4=16$. (b) $x=3$. (c) $y=7$. (d) $3(3)+7=16$ and $7=3+4$. **Q3.** (a) $4x=16$; the variable y is eliminated. (b) $x=4$. (c) $y=3$. (d) $2(4)-3=5$. **Q4.** (a) $2x+4y=18$. (b) $y=2$. (c) $x=5$. (d) $2(5)+3(2)=16$. **Q5.** (a) $2a+3c=47$ and $a+2c=26$. (b) $a=16,\ c=5$. (c) Adult \$16, child \$5. (d) \$68. **Q6.** (a) $C=90+1.5n$. (b) $R=4n$. (c) $n=36$. (d) Selling 36 brownies makes revenue exactly cover the day's costs (\$144 each). (e) Profit of \$35. **Q7.** $x=4,\ y=10$. **Q8.** $x=5,\ y=3$. **Q9.** The lines cross at $(2,5)$, so $x=2,\ y=5$. **Q10.** $x=3,\ y=4$. **Q11.** Meat pie \$6.40, sausage roll \$4.20; together \$10.60. **Q12.** $C=360+2.5n$, $R=10n$; break-even at $n=48$ cars (\$480 each side); profit for 70 cars $=\$165$. **Q13.** Equal at 25 classes, costing \$270 at either studio; for 30 classes Studio B is cheaper (\$300 against \$315, a saving of \$15); anyone planning fewer than 25 classes should choose Studio A. **Q14.** (a) Substitution (y is already the subject); $x=4,\ y=9$. (b) Elimination (the y coefficients are equal and opposite); $x=5,\ y=4$. (c) Graphing — read off the point of intersection, then check by substitution. **Q15.** Both lines have slope 3 but different y-intercepts (2 and -2.5), so they are parallel and never meet: no solution. **Q16.** $a=4,\ b=3$. **Q17.** Mulch \$12.50 per bag, gravel \$17.00 per bag. **Q18.** $C=1240+3.8n$, $R=11.8n$; break-even at $n=155$ bags; a \$600 profit needs $n=230$ bags.

Fully-worked solutions

Q1. (a) Substituting $x=0$, 1 and 3 into $y=2x-3$ gives $y=-3$, $y=-1$ and $y=3$. (b) Substituting the same values into $y=7-3x$ gives $y=7$, $y=4$ and $y=-2$. (c) The graphs cross at the single point $(2,1)$. (d) Check in $y=2x-3$: $2\times2-3=1$. Check in $y=7-3x$: $7-3\times2=1$. Both equations are satisfied, so the solution is $x=2,\ y=1$.

Q2. (a) Replacing y by $x+4$ gives $3x+(x+4)=16$, that is $4x+4=16$. (b) $4x=12$, so $x=3$. (c) $y=x+4=3+4=7$. (d) In $3x+y=16$: $3\times3+7=9+7=16$; and in $y=x+4$: $7=3+4$. Both hold, so $x=3,\ y=7$.

Q3. (a) Adding gives $(2x+y)+(2x-y)=11+5$, that is $4x=16$; the $+y$ and $-y$ terms cancel, so y is eliminated. (b) $x=4$. (c) $2\times4+y=11$ gives $8+y=11$, so $y=3$. (d) $2\times4-3=8-3=5$, as required. The solution is $x=4,\ y=3$.

Q4. (a) Multiplying $x+2y=9$ by 2 gives $2x+4y=18$. (b) Subtracting $2x+3y=16$ from $2x+4y=18$ gives $(2x+4y)-(2x+3y)=18-16$, that is $y=2$. (c) Substituting into $x+2y=9$ gives $x+4=9$, so $x=5$. (d) $2\times5+3\times2=10+6=16$, as required. The solution is $x=5,\ y=2$.

Q5. (a) Raf's purchase gives $2a+3c=47$ and Elena's gives $a+2c=26$. (b) From the second equation $a=26-2c$. Substituting into the first: $2(26-2c)+3c=47$, so $52-4c+3c=47$, giving $52-c=47$ and $c=5$. Then $a=26-2\times5=16$. Check: $2\times16+3\times5=32+15=47$, as required. (c) An adult ticket costs \$16 and a child ticket costs \$5. (d) $3\times16+4\times5=48+20=68$, so the family pays \$68.

Q6. (a) The day costs the \$90 hire plus \$1.50 per brownie: $C=90+1.5n$. (b) Each brownie sells for \$4.00, so $R=4n$. (c) $4n=90+1.5n$ gives $2.5n=90$, so $n=36$. (d) If Priya sells exactly 36 brownies, her revenue ($4\times36=\$144$) exactly covers the cost of the day ($90+1.5\times36=\$144$): she neither makes nor loses money. Selling fewer than 36 leaves a loss; selling more earns a profit. (e) For $n=50$: $R=4\times50=200$ and $C=90+1.5\times50=165$, so she makes a profit of $200-165=\$35$.

Q7. Substituting $y=3x-2$ into $x+y=14$ gives $x+(3x-2)=14$, so $4x-2=14$, $4x=16$ and $x=4$. Then $y=3\times4-2=10$. Check: $4+10=14$ and $10=3\times4-2$. The solution is $x=4,\ y=10$.

Q8. The y coefficients are $+2$ and -2, so adding the equations eliminates y: $8x=40$, giving $x=5$. Substituting into $5x+2y=31$ gives $25+2y=31$, so $2y=6$ and $y=3$. Check: $3\times5-2\times3=15-6=9$, as required. The solution is $x=5,\ y=3$.

Q9. The two lines cross at the point $(2,5)$. Check in $y=x+3$: $2+3=5$. Check in $y=11-3x$: $11-3\times2=5$. Both equations are satisfied, so the solution is $x=2,\ y=5$.

Q10. To eliminate y, multiply $3x+4y=25$ by 3 to get $9x+12y=75$, and multiply $2x+3y=18$ by 4 to get $8x+12y=72$. Subtracting the second new equation from the first gives $x=3$. Substituting into $3x+4y=25$: $9+4y=25$, so $4y=16$ and $y=4$. Check: $2\times3+3\times4=6+12=18$, as required. The solution is $x=3,\ y=4$.

Q11. Let p be the price of a meat pie and s the price of a sausage roll, in dollars. Then $3p+2s=27.60$ and $2p+5s=33.80$. Multiplying the first equation by 5 gives $15p+10s=138.00$, and multiplying the second by 2 gives $4p+10s=67.60$. Subtracting, $11p=70.40$, so $p=6.40$. Then $3\times6.40+2s=27.60$ gives $2s=27.60-19.20=8.40$, so $s=4.20$. Check: $2\times6.40+5\times4.20=12.80+21.00=33.80$, as required. A meat pie costs \$6.40, a sausage roll costs \$4.20, and together they cost $6.40+4.20=\$10.60$.

Q12. The cost of washing n cars is $C=360+2.5n$ and the revenue is $R=10n$ (both in dollars). At break-even $10n=360+2.5n$, so $7.5n=360$ and $n=48$; both sides there equal $10\times48=\$480$. The club must wash 48 cars just to cover its costs — washing fewer than 48 loses money, and every car beyond the 48th adds to the profit. For 70 cars, $R=10\times70=700$ and $C=360+2.5\times70=535$, so the profit is $700-535=\$165$.

Q13. For n classes, Studio A costs $A=45+9n$ and Studio B costs $B=120+6n$ (both in dollars). Setting $A=B$: $45+9n=120+6n$, so $3n=75$ and $n=25$. The common cost is $45+9\times25=\$270$ (check: $120+6\times25=270$). For 30 classes, $A=45+270=\$315$ and $B=120+180=\$300$, so Studio B is cheaper by $315-300=\$15$. Studio A's smaller joining fee makes it the cheaper choice for anyone planning **fewer than 25** classes; beyond 25 classes Studio B's lower per-class price wins.

Q14. (a) **Substitution** is most efficient, because the first equation already has y as its subject. Substituting: $2x+(4x-7)=17$, so $6x-7=17$, $6x=24$ and $x=4$; then $y=4\times4-7=9$. Check: $2\times4+9=17$. The solution is $x=4,\ y=9$. (b) **Elimination** is most efficient, because the y coefficients $+2$ and -2 are already equal and opposite. Adding: $6x=30$, so $x=5$; then $3\times5+2y=23$ gives $2y=8$ and $y=4$. Check: $3\times5-2\times4=15-8=7$, as required. The solution is $x=5,\ y=4$. (c) **Graphing** — the lines are already drawn, so simply read off the point of intersection and check it by substituting into both equations.

Q15. Rearranging the second equation: $6x-2y=5$ gives $2y=6x-5$, so $y=3x-2.5$. Both lines therefore have slope 3, but their y-intercepts, 2 and -2.5, are different: the lines are parallel, so they never meet and no pair of values satisfies both equations. Algebraically, substituting $y=3x+2$ into $6x-2y=5$ gives $6x-6x-4=5$, that is $-4=5$, which is false — so there is no solution. A graph would show two parallel lines with no point of intersection.

Q16. Since $x=2,\ y=5$ satisfies $ax+y=13$, we have $2a+5=13$, so $2a=8$ and $a=4$. Since the pair also satisfies $x+by=17$, we have $2+5b=17$, so $5b=15$ and $b=3$. Check: $4\times2+5=13$ and $2+3\times5=17$, as required.

Q17. Let m be the price of a bag of mulch and g the price of a bag of gravel, in dollars. Monday gives $4m+3g=101$ and Tuesday gives $6m+2g=109$. Multiplying the first equation by 2 gives $8m+6g=202$, and multiplying the second by 3 gives $18m+6g=327$. Subtracting, $10m=125$, so $m=12.50$. Then $4\times12.50+3g=101$ gives $3g=101-50=51$, so $g=17.00$. Check: $6\times12.50+2\times17=75+34=109$, as required. Mulch costs \$12.50 per bag and gravel costs \$17.00 per bag.

Q18. The monthly cost of producing n bags is $C=1240+3.8n$ and the revenue is $R=11.8n$ (both in dollars). At break-even $11.8n=1240+3.8n$, so $8n=1240$ and $n=155$: each bag contributes $11.80-3.80=\$8$ towards the fixed costs, and the business must sell 155 bags a month before it starts making money. For a profit of \$600: $R-C=11.8n-(1240+3.8n)=8n-1240=600$, so $8n=1840$ and $n=230$. The owner needs sales of 155 bags a month to cover costs, and 230 bags a month to bank a \$600 profit.

Theory

Many everyday charges cannot be modelled by a single straight line. A pay office uses one hourly rate for ordinary hours and a higher rate for overtime; a water corporation charges more per kilolitre once use passes a threshold; a car park charges one flat fee for each band of time. This section develops the two kinds of graph used to model such situations: **piecewise linear graphs** and **step graphs**.

Piecewise linear graphs. A **piecewise linear graph** (or line-segment graph) is made up of two or more straight-line pieces, each with its **own rule** and each valid on its **own part of the domain**. The pieces meet at **break points**, and in practical models the pieces usually join, so the graph can be drawn without lifting the pen. For example, an e-bike hire scheme charges $40 per hour for the first 2 hours and $25 per hour after that, so the total cost C dollars of a hire lasting t hours is

$$C=\begin{cases} 40t, & 0\le t\le 2 \\ 80+25(t-2), & t>2. \end{cases}$$

The second rule starts from $C=40\times 2=80$, the cost at the break point, and adds $25 for each hour beyond 2; a 3-hour hire costs $80+25\times 1=\$105$.

The slope of each piece is a rate. In a model like the one above, the slope of each piece tells you how fast the output is changing on that part of the domain — dollars per hour, dollars per kilolitre, and so on. A **break in slope** therefore means the **rate changes** at that input: beyond 2 hours, each extra hour of e-bike hire costs $25 instead of $40. The steeper the piece, the faster the output grows. Interpreting a break in slope in context is a key skill in this section.

Evaluating, and finding an input from an output. To evaluate a piecewise model, first decide **which piece** the input belongs to, then substitute into that piece's rule. To find the input that produces a given output, work in the opposite order: calculate the output at each break point, use these to decide which piece the target output lies on, then solve that piece's linear equation.

Step graphs. A **step graph** is constant on each interval of its domain: the output holds one value for a while, then **jumps** to a new value. Step graphs suit charges set by **bracket** — parking fees by band of time, parcel postage by band of weight, tiered entry prices. At the end of each step the endpoint convention matters: a **filled dot** shows the endpoint **is included** in that step, and an **open circle** shows it **is not**. So if the $11 postage step has a filled dot at 1 kg and the $14 step has an open circle there, a parcel of exactly 1 kg costs $11, not $14. Unlike a piecewise linear graph, a step graph has jumps, and within a step the output does not change at all.

Progressive tax scales. Australian income tax is calculated from a **progressive tax scale**: income is divided into **brackets**, and each bracket has its own rate, quoted as so many "cents for each dollar over" the bracket's threshold. The tax payable on a taxable income inside a bracket is

$$\text{tax}=\text{base amount}+\text{rate}\times(\text{income}-\text{threshold}),$$

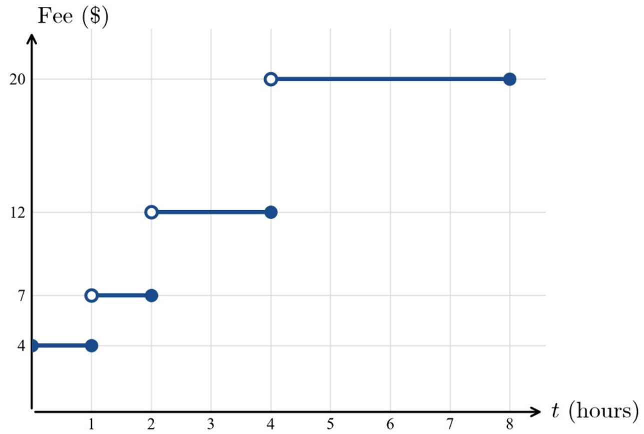
where the base amount is the tax payable at the bottom of the bracket. The graph of tax against income is piecewise linear: the pieces join (earning one more dollar never adds more than a dollar of tax), and the slope of each piece is that bracket's **marginal tax rate**, which increases from bracket to bracket.

Using CAS. General Mathematics is a technology-active, open-book course. Your CAS has a piecewise template that lets you define a function such as the e-bike cost above in one entry; you can then graph it, evaluate it at any input, and solve for the input that gives a required output. To sketch by hand, calculate and plot the break points, join them with line segments of the correct slope, and (for step graphs) mark each endpoint with a filled dot or open circle. Always check a CAS answer against the by-hand method: substitute the answer back into the correct piece.

Worked examples

Example 1 — scaffolded

The step graph below shows the fee charged at a car park near Geelong's waterfront for stays of up to 8 hours. A filled dot shows that the endpoint is included in that step; an open circle shows that it is not.



Use the graph to find (a) the fee for a 45-minute stay, (b) the fee for a stay of exactly 2 hours, (c) the fee for a 2.5-hour stay, and (d) the longest time a driver can stay for a fee of $12.

Working	Comment
(a) 45 minutes is $t = 0.75$ hours; the step above $t = 0.75$ is at $4.	Locate the time on the horizontal axis and read up to the step.
(b) At $t = 2$ the filled dot lies on the $7 step, so the fee is $7.	Exactly 2 hours is included in the $7 step; the open circle at $(2, 12)$ shows that $12 applies only beyond 2 hours.
(c) At $t = 2.5$ the step is at $12.	2.5 hours lies in the interval from 2 to 4 hours (2 excluded, 4 included).
(d) The $12 step ends with a filled dot at $t = 4$, so the longest stay for $12 is 4 hours.	The filled dot means a 4-hour stay is still charged $12; any longer and the fee jumps to $20.

So the fees are $4, $7 and $12, and a driver can stay up to 4 hours for $12.

Example 2 — scaffolded

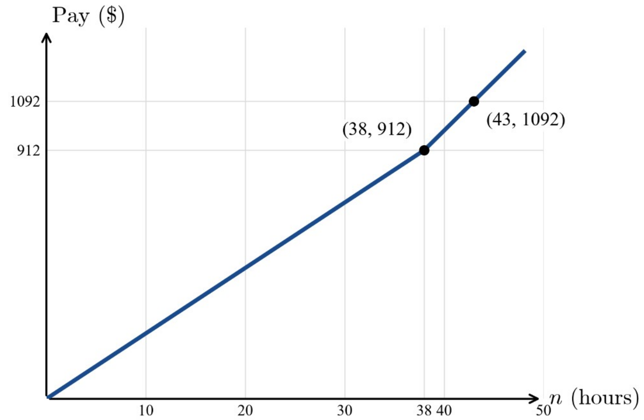
Ari works at a Bendigo café. He is paid $24 per hour for the first 38 hours each week, and time-and-a-half — that is, $36 per hour — for every hour beyond 38. Let P be Ari's weekly pay, in dollars, for n hours of work. Construct a piecewise linear rule for P, and use it to find (a) his pay for a 35-hour week, (b) his pay for a 43-hour week, and (c) the hours he worked in a week he was paid $1020.

Working	Comment
For $0 \le n \le 38$: $P = 24n$.	Ordinary hours: pay grows at $24 per hour, starting from 0.
At the break point $n = 38$: $P = 24 \times 38 = 912$.	Working exactly 38 hours pays $912 — the second piece must start from this value.
For $n > 38$: $P = 912 + 36(n - 38)$.	Overtime: pay grows from $912 at $1.5 \times 24 = \$36$ per extra hour.
(a) $n = 35 \le 38$, so $P = 24 \times 35 = 840$.	Choose the piece containing the input: 35 is in the first piece.
(b) $n = 43 > 38$, so $P = 912 + 36 \times 5 = 912 + 180 = 1092$.	43 hours is 5 hours of overtime beyond the 38.
(c) $1020 > 912$, so $912 + 36(n - 38) = 1020$, giving	$1020 exceeds the $912 earned at the break point, so it lies on the overtime piece; solve that piece's equation.

Working	Comment

$n - 38 = \frac{108}{36} = 3$, so $n = 41$.

$$P = \begin{cases} 24n, & 0 \le n \le 38 \\ 912 + 36(n - 38), & n > 38 \end{cases}$$



So Ari earns \$840 for 35 hours and \$1092 for 43 hours, and a \$1020 week means he worked 41 hours. The break in slope at $n = 38$ shows where the higher overtime rate takes over.

Example 3 — unscaffolded

A regional water corporation charges households \$2.50 per kilolitre (kL) for the first 120 kL used in a year, and \$4.00 per kL for every kilolitre after that. Write a piecewise linear rule for the annual usage charge C dollars on x kL, find the charge for 90 kL and for 150 kL, find the usage that produces a charge of \$500, and interpret the break in slope.

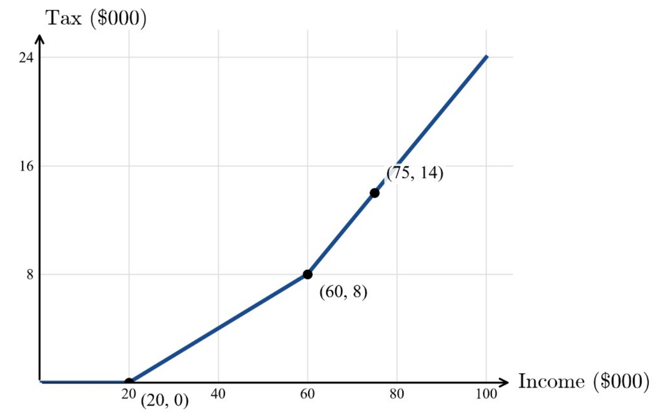
For $0 \le x \le 120$ the charge is $C = 2.5x$. At the break point, $C = 2.5 \times 120 = 300$, so for $x > 120$ the charge is $C = 300 + 4(x - 120)$. For 90 kL, the first rule applies: $C = 2.5 \times 90 = 225$, a charge of \$225. For 150 kL, the second rule applies: $C = 300 + 4 \times 30 = 420$, a charge of \$420. A charge of \$500 exceeds the \$300 charged at the break point, so it lies on the second piece: $300 + 4(x - 120) = 500$ gives $x - 120 = \frac{200}{4} = 50$, so $x = 170$ kL. The break in slope at 120 kL means the rate changes there: each kilolitre beyond 120 costs \$4.00 instead of \$2.50, so the graph becomes steeper — a pricing structure that discourages high water use.

$\therefore$ $C = 2.5x$ for $0 \le x \le 120$ and $C = 300 + 4(x - 120)$ for $x > 120$; the charges are \$225 and \$420, a \$500 charge corresponds to 170 kL, and the steeper piece past 120 kL shows the higher rate for high use.

Example 4 — unscaffolded

The simplified annual income-tax scale of a certain country is shown in the table, and its graph (in thousands of dollars) below.

Taxable income	Tax payable
\$0 – \$20,000	nil
\$20,001 – \$60,000	20 cents for each dollar over \$20,000
over \$60,000	\$8000 plus 40 cents for each dollar over \$60,000



Find the tax payable on a taxable income of $75,000, the taxable income of a resident who paid $5000 in tax, and this country's marginal tax rate for incomes over $60,000. For the $75,000 income, find also the overall percentage of income paid in tax, correct to one decimal place.

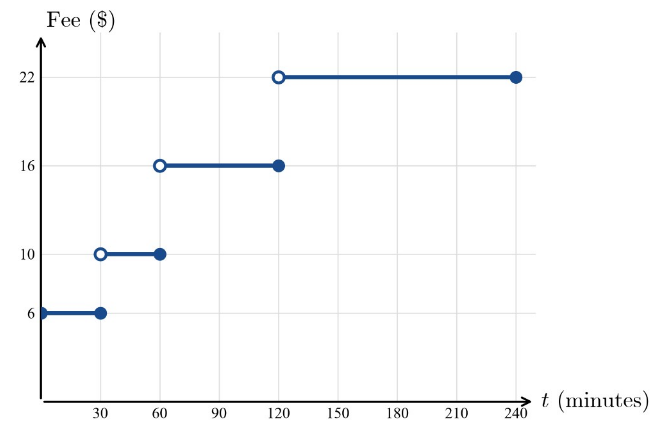
An income of $75,000 is in the top bracket, so the tax is $8000+0.40\times(75\,000-60\,000)=8000+6000=\$14\,000$. For the second question, work out the tax at the bracket boundaries: $0 at $20,000 and $0.20\times 40\,000=\$8000$ at $60,000. Since $0<5000<8000$, a tax bill of $5000 lies in the middle bracket: $0.20(x-20\,000)=5000$ gives $x-20\,000=25\,000$, so $x=\$45\,000$. The marginal rate over $60,000 is the slope of the top piece: 40 cents in the dollar, or 40%. The overall rate on $75,000 is $\dfrac{14\,000}{75\,000}\times 100\approx 18.7\%$ — much less than the marginal 40%, because the first $20,000 is tax-free and the next $40,000 is taxed at only 20%.

∴ the tax on $75,000 is $14,000 (an overall rate of about 18.7%), a $5000 tax bill corresponds to a taxable income of $45,000, and the marginal rate above $60,000 is 40%.

Practice questions

Scaffolded

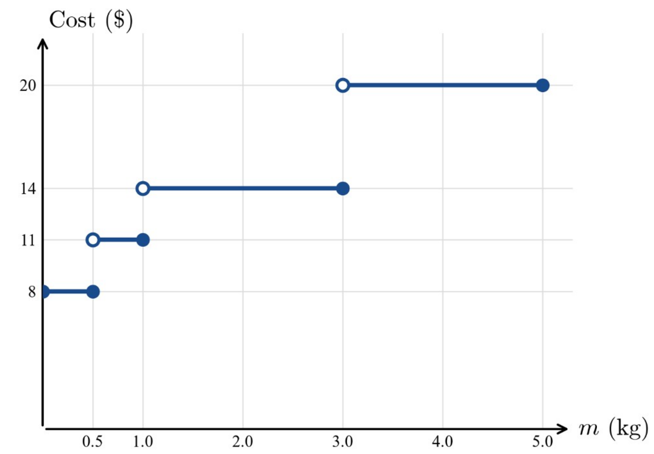
Q1. The step graph shows the fee at an airport pick-up car park for stays of up to 240 minutes.



(a) Find the fee for a 20-minute stay.
(b) Find the fee for a stay of exactly 60 minutes.
(c) Find the fee for a 90-minute stay.
(d) Sana paid $16. What is the longest time she could have stayed?

Q2. Lena works at a plant nursery. She is paid $22 per hour for the first 8 hours in a day and time-and-a-half ($33 per hour) for every hour after that. Let P be her pay, in dollars, for n hours in a day. (a) Write a rule for P for $0\le n\le 8$ and a rule for $n>8$. (b) Find her pay for a 6-hour day. (c) Find her pay for an 11-hour day. (d) On a stocktake day Lena was paid $242. How many hours did she work?

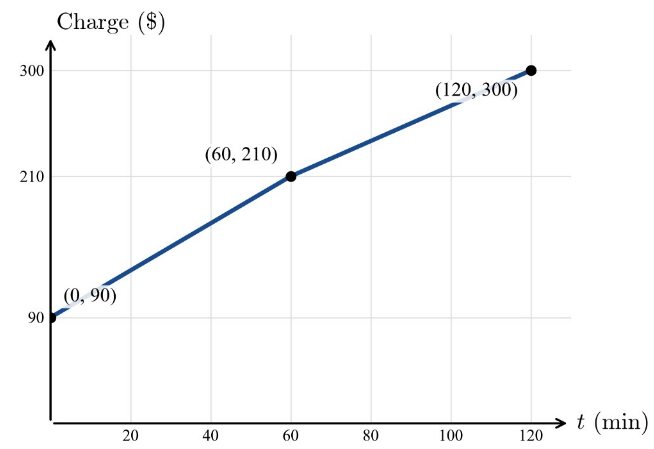
Q3. The step graph shows the cost of sending a parcel with a courier, according to its mass in kilograms.



(a) Find the cost of sending a 250-gram parcel.
(b) Find the cost of sending a parcel weighing exactly 1 kg.
(c) Find the cost of sending a 1.2-kg parcel.
(d) Priya can spend at most \$14. What is the heaviest parcel she can send?
(e) Explain the meaning of the open circle at $(1, 14)$.

Q4. A gas retailer charges 4 cents per megajoule (MJ) for the first 5000 MJ used in a quarter, and 6 cents per MJ after that, so the usage charge in dollars is $C = 0.04x$ for $0 \le x \le 5000$ and $C = 200 + 0.06(x - 5000)$ for $x > 5000$, where x is the number of megajoules used. (a) Find the charge for a quarter in which 3000 MJ is used. (b) Find the charge for a quarter in which 7000 MJ is used. (c) In one quarter a household's usage charge was \$290. How many megajoules did it use? (d) Interpret the break in slope at $x = 5000$ in context.

Q5. The graph shows the total charge C dollars of a plumber's visit lasting t minutes: one line segment from $(0, 90)$ to $(60, 210)$, then a second from $(60, 210)$ to $(120, 300)$.



(a) Write down the call-out fee — the charge at $t = 0$.
(b) Find the slope of the first piece and interpret it in context.
(c) Write a rule for C for $0 \le t \le 60$ and a rule for $60 < t \le 120$.
(d) Find the charge for a visit lasting 100 minutes.

Q6. The annual income-tax scale of a small island nation is shown.

Taxable income	Tax payable
\$0 – \$15,000	nil
\$15,001 – \$50,000	15 cents for each dollar over \$15,000
over \$50,000	\$5250 plus 35 cents for each dollar over \$50,000

(a) Find the tax payable on a taxable income of \$12,000.

(b) Find the tax payable on a taxable income of $40,000.
(c) Find the tax payable on a taxable income of $80,000.
(d) A resident paid $12,250 in tax. Find their taxable income.

Unscaffolded

Q7. A kayak-hire kiosk on the Murray River charges $15 for hires of up to 1 hour, $22 for hires of over 1 and up to 2 hours, $30 for over 2 and up to 4 hours, and $45 for over 4 and up to 8 hours. Find the cost of a 45-minute hire, a hire of exactly 2 hours, and a 2.5-hour hire, and find the longest hire possible for $30.

Q8. An apprentice electrician is paid $25 per hour for the first 38 hours each week and time-and-a-half ($37.50 per hour) beyond that. Find her pay for a 38-hour week and for a 42-hour week, and find the number of hours she worked in a week she was paid $1250.

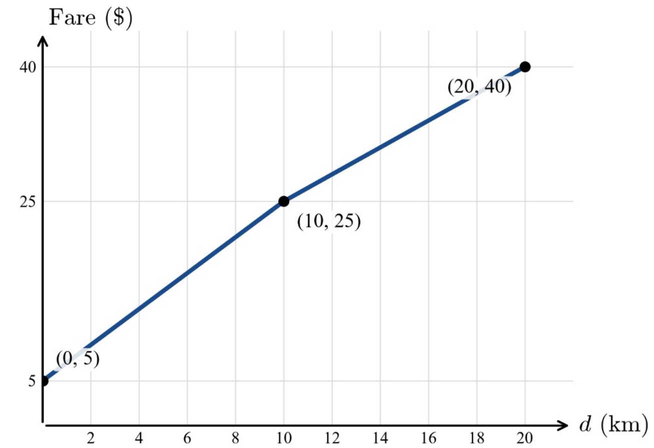
Q9. A rural water authority charges $3 per kilolitre for the first 80 kL used in a year and $4.50 per kL thereafter. Write a piecewise linear rule for the annual charge C dollars on x kL, find the charge on 60 kL and on 100 kL, and find the usage that gives a charge of $420.

Q10. A country's income-tax scale is: nil on taxable income up to $18,000; 25 cents for each dollar over $18,000 up to $55,000; and $9250 plus 45 cents for each dollar over $55,000. Find the tax payable on taxable incomes of $40,000 and $80,000, find the taxable income of a person whose tax bill is $13,750, and explain what the break in slope at $55,000 means for a taxpayer.

Q11. For each situation, state whether a step graph or a piecewise linear graph is the better model, and give a brief reason. (a) The cost of posting a parcel, where the courier charges by weight bracket. (b) A worker's weekly pay, where hours beyond 38 are paid at a higher hourly rate. (c) The fee at a car park that charges by band of time. (d) A household's annual water bill, charged per kilolitre with a higher rate beyond a threshold.

Q12. A washing-machine technician charges a $60 call-out fee that covers the first 30 minutes on site, plus $2 per minute after that. Write a piecewise rule for the charge C dollars for t minutes on site, describe the shape of its graph, find the charge for a 45-minute visit, and find how long the technician stayed if the charge was $150.

Q13. The graph shows a taxi fare F dollars against distance travelled d km: one segment from $(0,5)$ to $(10,25)$ and a second from $(10,25)$ to $(20,40)$.



(a) Find the rule for each piece of the graph.
(b) Find the fare for a 16-km trip.
(c) A passenger paid $37. How far did they travel?
(d) Interpret the break in slope at $d = 10$ in context.

Q14. A trailer-hire yard charges $20 for hires of up to 2 hours, $30 for over 2 and up to 4 hours, $45 for over 4 and up to 8 hours, and $60 for over 8 and up to 24 hours. Describe how to draw the step graph of the charge, including where the dots are open or filled. Then find the cost of a 4-hour hire and of a 5-hour hire, and state the possible hire times of a customer who paid $45.

Q15. Under a shearing-shed award, a rouseabout is paid \$26 per hour for the first 8 hours in a day, \$39 per hour for the next 4 hours, and \$52 per hour beyond 12 hours. Find the pay for an 8-hour day, a 12-hour day and a 14-hour day, and find the length of a day for which the pay was \$429.

Q16. A country's income-tax scale is: nil up to \$20,000; 25 cents for each dollar over \$20,000 up to \$70,000; and \$12,500 plus 40 cents for each dollar over \$70,000. (a) Find the tax payable on a taxable income of \$96,000. (b) For this income, find the overall percentage of income paid in tax (one decimal place), state the marginal tax rate, and explain why the two differ. (c) Find the taxable income of a person who paid \$20,500 in tax.

Q17. An electricity retailer's quarterly bill is a \$100 supply charge plus 25 cents per kilowatt-hour (kWh) for the first 800 kWh, and 35 cents per kWh after that: $B = 100 + 0.25x$ for $0 \le x \le 800$ and $B = 300 + 0.35(x - 800)$ for $x > 800$. Find the bill for 600 kWh and for 1000 kWh, find the usage in a quarter with a bill of \$440, and interpret the vertical intercept of the graph of B in context.

Q18. Nadia is choosing between two mobile-data plans. Plan A costs \$30 per month, which includes 20 GB of data, plus \$5 per extra gigabyte (charged pro rata). Plan B costs a flat \$45 per month for up to 40 GB. Nadia never uses more than 40 GB. (a) Find the cost of each plan in a month in which she uses 30 GB. (b) Find the monthly usage for which the two plans cost the same. (c) Hence advise Nadia which plan to choose, depending on her usage.

Answers

Q1. (a) $6. (b) $10. (c) $16. (d) 120 minutes (2 hours). **Q2.** (a) $P=22n$ for $0\le n\le 8$; $P=176+33(n-8)$ for $n>8$. (b) $132. (c) $275. (d) 10 hours. **Q3.** (a) $8. (b) $11. (c) $14. (d) 3 kg. (e) A parcel of exactly 1 kg is not on the $14 step — it costs $11; $14 applies only to parcels over 1 kg. **Q4.** (a) $120. (b) $320. (c) 6500 MJ. (d) Beyond 5000 MJ each extra megajoule costs 6 cents instead of 4 cents, so the graph becomes steeper. **Q5.** (a) $90. (b) Slope $=2$: the charge rises $2 per minute for the first hour. (c) $C=90+2t$ for $0\le t\le 60$; $C=210+1.5(t-60)$ for $60<t\le 120$. (d) $270. **Q6.** (a) Nil. (b) $3750. (c) $15,750. (d) $70,000. **Q7.** $15; $22; $30; longest hire for $30 is 4 hours. **Q8.** $950; $1100; 46 hours. **Q9.** $C=3x$ for $0\le x\le 80$; $C=240+4.5(x-80)$ for $x>80$; $180; $330; 120 kL. **Q10.** $5500; $20,500; $65,000; beyond $55,000 each extra dollar earned is taxed at 45 cents instead of 25 cents (the marginal rate rises). **Q11.** (a) Step graph — the cost is constant within each weight bracket and jumps at the bracket ends. (b) Piecewise linear — pay grows continuously with hours, with a change of slope at 38. (c) Step graph — one flat fee per band of time. (d) Piecewise linear — the bill grows continuously with each kilolitre, with a steeper slope past the threshold. **Q12.** $C=60$ for $0\le t\le 30$; $C=60+2(t-30)$ for $t>30$; the graph is flat at 60 until $t=30$, then a line of slope 2; $90; 75 minutes. **Q13.** (a) $F=5+2d$ for $0\le d\le 10$; $F=25+1.5(d-10)$ for $10<d\le 20$. (b) $34. (c) 18 km. (d) Beyond 10 km each extra kilometre costs $1.50 instead of $2, so the fare grows more slowly. **Q14.** Four horizontal steps at 20, 30, 45 and 60, each with an open circle at its left end and a filled dot at its right end (at $t=2,4,8,24$); $30; $45; over 4 and up to 8 hours ($4<t\le 8$). **Q15.** $208; $364; $468; 13.25 hours (13 h 15 min). **Q16.** (a) $22,900. (b) Overall rate $\approx 23.9\%$; marginal rate 40%; they differ because only income above $70,000 is taxed at 40 cents — the first $20,000 is tax-free and the next $50,000 is taxed at 25 cents. (c) $90,000. **Q17.** $250; $370; 1200 kWh; the vertical intercept 100 is the $100 supply charge, payable even if no electricity is used. **Q18.** (a) Plan A $80; Plan B $45. (b) 23 GB. (c) Plan A is cheaper if she uses less than 23 GB per month; Plan B is cheaper for more than 23 GB (up to 40 GB); at exactly 23 GB they cost the same.

Fully-worked solutions

Q1. (a) At $t=20$ the step is at $6 (the first step covers stays of up to 30 minutes). (b) At $t=60$ the filled dot lies on the $10 step, so a stay of exactly 60 minutes costs $10; the open circle at $(60,16)$ shows $16 applies only beyond 60 minutes. (c) 90 minutes lies between 60 and 120 minutes, so the fee is $16. (d) The $16 step ends with a filled dot at $t=120$, so the longest stay for $16 is 120 minutes — 2 hours.

Q2. (a) For $0\le n\le 8$, $P=22n$. At the break point, $P=22\times 8=176$, so for $n>8$, $P=176+33(n-8)$. (b) $n=6\le 8$: $P=22\times 6=132$, so $132. (c) $n=11>8$: $P=176+33\times 3=176+99=275$, so $275. (d) $242>176$, so the overtime rule applies: $176+33(n-8)=242$ gives $n-8=\frac{66}{33}=2$, so $n=10$ hours.

Q3. (a) 250 g $=0.25$ kg lies on the first step, so the cost is $8. (b) The filled dot at $(1,11)$ shows a 1-kg parcel is included in the $11 step: $11. (c) 1.2 kg lies between 1 and 3 kg, so the cost is $14. (d) The $14 step ends with a filled dot at 3 kg, so the heaviest parcel for $14 is 3 kg. (e) The open circle at $(1,14)$ means the mass 1 kg is **excluded** from the $14 step: a parcel of exactly 1 kg costs $11, and the $14 charge applies only to parcels over 1 kg.

Q4. (a) $x=3000\le 5000$: $C=0.04\times 3000=120$, a charge of $120. (b) $x=7000>5000$: $C=200+0.06\times 2000=200+120=320$, a charge of $320. (c) At the break point the charge is $0.04\times 5000=\$200$; since $290>200$ the second rule applies: $200+0.06(x-5000)=290$ gives $x-5000=\frac{90}{0.06}=1500$, so $x=6500$ MJ. (d) The break in slope at 5000 MJ means the rate changes there: each megajoule beyond 5000 costs 6 cents rather than 4 cents, so the charge graph becomes steeper for heavy gas use.

Q5. (a) The graph starts at $(0,90)$, so the call-out fee is $90. (b) Slope of the first piece $=\frac{210-90}{60-0}=\frac{120}{60}=2$: the charge increases by $2 for every minute on site during the first hour. (c) First piece: intercept 90, slope 2, so $C=90+2t$ for $0\le t\le 60$. Second piece: slope $\frac{300-210}{120-60}=\frac{90}{60}=1.5$, starting from $(60,210)$, so $C=210+1.5(t-60)$ for $60<t\le 120$. (d) $t=100>60$: $C=210+1.5\times 40=210+60=270$, a charge of $270.

Q6. (a) \$12,000 is below \$15,000, so no tax is payable. (b) $0.15\times(40\,000-15\,000)=0.15\times 25\,000=3750$, so \$3750. (c) $5250+0.35\times(80\,000-50\,000)=5250+10\,500=15\,750$, so \$15,750. (d) The tax at the top of the middle bracket is $0.15\times 35\,000=\$5250$; since $12\,250>5250$, the income is in the top bracket: $5250+0.35(x-50\,000)=12\,250$ gives $x-50\,000=\dfrac{7000}{0.35}=20\,000$, so $x=\$70\,000$.

Q7. A 45-minute hire is on the first step: \$15. A hire of exactly 2 hours is included in the second step (over 1, up to 2 hours): \$22. A 2.5-hour hire is on the third step: \$30. The \$30 step includes its right endpoint, so the longest hire for \$30 is 4 hours.

Q8. For 38 hours: $25\times 38=950$, so \$950. For 42 hours: $950+37.50\times 4=950+150=1100$, so \$1100. A pay of \$1250 exceeds \$950, so $950+37.5(n-38)=1250$ gives $n-38=\dfrac{300}{37.5}=8$, so she worked $n=46$ hours.

Q9. For $0\le x\le 80$, $C=3x$. At the break point $C=3\times 80=240$, so for $x>80$, $C=240+4.5(x-80)$. For 60 kL: $C=3\times 60=180$, a charge of \$180. For 100 kL: $C=240+4.5\times 20=240+90=330$, a charge of \$330. A charge of \$420 exceeds \$240, so $240+4.5(x-80)=420$ gives $x-80=\dfrac{180}{4.5}=40$, so $x=120$ kL.

Q10. Tax on \$40,000: $0.25\times(40\,000-18\,000)=0.25\times 22\,000=\$5500$. Tax on \$80,000: $9250+0.45\times(80\,000-55\,000)=9250+11\,250=\$20\,500$. The tax at \$55,000 is $0.25\times 37\,000=\$9250$; since $13\,750>9250$ the income is in the top bracket: $9250+0.45(x-55\,000)=13\,750$ gives $x-55\,000=\dfrac{4500}{0.45}=10\,000$, so the taxable income is \$65,000. The break in slope at \$55,000 is a change in the marginal rate: each dollar earned beyond \$55,000 attracts 45 cents of tax rather than 25 cents, so the tax graph is steeper there.

Q11. (a) **Step graph**: within each weight bracket the cost is constant, and it jumps at the bracket boundaries — the cost does not change gradually with weight. (b) **Piecewise linear graph**: pay increases continuously with every hour worked; at 38 hours the graph does not jump, but its slope increases to the overtime rate. (c) **Step graph**: one flat fee applies to every stay within a band of time, with jumps between bands. (d) **Piecewise linear graph**: the bill grows with every kilolitre used, at one rate up to the threshold and a steeper rate beyond it, with no jump at the threshold.

Q12. For $0\le t\le 30$ the charge is the flat call-out fee: $C=60$. For $t>30$: $C=60+2(t-30)$. The graph is a horizontal segment at height 60 from $t=0$ to $t=30$, then a line segment of slope 2 — a break in slope from rate \$0 to \$2 per minute, with no jump. For 45 minutes: $C=60+2\times 15=90$, so \$90. For a charge of \$150 (which exceeds 60, so the second rule applies): $60+2(t-30)=150$ gives $t-30=\dfrac{90}{2}=45$, so $t=75$ minutes.

Q13. (a) First piece: intercept 5 (the flagfall) and slope $\dfrac{25-5}{10-0}=2$, so $F=5+2d$ for $0\le d\le 10$. Second piece: slope $\dfrac{40-25}{20-10}=1.5$ from the point $(10,25)$, so $F=25+1.5(d-10)$ for $10<d\le 20$. (b) $d=16>10$: $F=25+1.5\times 6=25+9=34$, a fare of \$34. (c) $37>25$, so $25+1.5(d-10)=37$ gives $d-10=\dfrac{12}{1.5}=8$, so $d=18$ km. (d) The break in slope at 10 km means the per-kilometre rate falls from \$2 to \$1.50 once a trip passes 10 km, so the fare rises more slowly on long trips.

Q14. The graph has four horizontal steps: height 20 for $0<t\le 2$, height 30 for $2<t\le 4$, height 45 for $4<t\le 8$ and height 60 for $8<t\le 24$. Each step has an open circle at its left end and a filled dot at its right end, because the charge for each band includes the band's upper time limit. A 4-hour hire lies at the filled dot of the \$30 step, so it costs \$30. A 5-hour hire is on the \$45 step: \$45. A customer who paid \$45 hired the trailer for more than 4 and up to 8 hours: $4<t\le 8$.

Q15. For 8 hours: $26 \times 8 = 208$, so \$208. For 12 hours: $208 + 39 \times 4 = 208 + 156 = 364$, so \$364. For 14 hours: $364 + 52 \times 2 = 364 + 104 = 468$, so \$468. A pay of \$429 lies between \$364 and the pay for longer days, so the third rule applies: $364 + 52(n - 12) = 429$ gives $n - 12 = \frac{65}{52} = 1.25$, so $n = 13.25$ hours — a 13-hour 15-minute day.

Q16. (a) \$96,000 is in the top bracket: $12\,500 + 0.40 \times (96\,000 - 70\,000) = 12\,500 + 10\,400 = \$22\,900$. (b) Overall rate $= \frac{22\,900}{96\,000} \times 100 \approx 23.9\,\%$. The marginal rate is $40\,\%$ — the slope of the top piece. They differ because the 40-cent rate applies only to the income **above** \$70,000: the first \$20,000 is tax-free and the next \$50,000 is taxed at 25 cents, which pulls the overall rate well below the marginal rate. (c) The tax at \$70,000 is $0.25 \times 50\,000 = \$12\,500$; since $20\,500 > 12\,500$ the income is in the top bracket: $12\,500 + 0.40(x - 70\,000) = 20\,500$ gives $x - 70\,000 = \frac{8000}{0.40} = 20\,000$, so the taxable income is \$90,000.

Q17. For 600 kWh: $B = 100 + 0.25 \times 600 = 100 + 150 = 250$, a bill of \$250. For 1000 kWh: $B = 300 + 0.35 \times 200 = 300 + 70 = 370$, a bill of \$370. The bill at the break point is $100 + 0.25 \times 800 = \$300$; since $440 > 300$ the second rule applies: $300 + 0.35(x - 800) = 440$ gives $x - 800 = \frac{140}{0.35} = 400$, so $x = 1200$ kWh. The vertical intercept is $B = 100$ at $x = 0$: even a household that uses no electricity at all pays the \$100 supply charge for the quarter.

Q18. (a) On Plan A, 30 GB is 10 GB over the included 20 GB: $30 + 5 \times 10 = \$80$. Plan B is a flat \$45. (b) For usage x GB with $20 < x \leq 40$, Plan A costs $30 + 5(x - 20)$. Setting this equal to 45: $5(x - 20) = 15$, so $x - 20 = 3$ and $x = 23$ GB. (For $x \leq 20$ Plan A costs \$30, which is never equal to \$45.) (c) For usage under 23 GB, Plan A is cheaper (at 20 GB or less it costs only \$30); at exactly 23 GB the plans both cost \$45; for usage between 23 GB and 40 GB, Plan B's flat \$45 is cheaper. So Nadia should choose Plan A if she typically uses less than 23 GB a month, and Plan B if she uses more.