General Mathematics Units 1 & 2

Chapter 3 — Financial mathematics

Aboriginal and Torres Strait Islander content has been excluded as accuracy cannot be guaranteed, and it is better to be respectful than cover the entire curriculum inaccurately.

Contents

Theory

Almost every financial decision — pricing an item, taking a discount, paying tax, comparing value for money — rests on confident percentage work. A **percentage** is a fraction with denominator 100, so $15\% = \frac{15}{100} = 0.15$. To find a **percentage of a quantity**, convert the percentage to a decimal and multiply: 15% of \$60 is $0.15 \times 60 = \$9$.

One quantity as a percentage of another. Write the first quantity as a fraction of the second (both in the same units) and multiply by 100%:

$$\text{percentage} = \frac{\text{part}}{\text{whole}} \times 100\%.$$

For example, a shopper who spends \$63 of a \$350 weekly grocery budget on fruit and vegetables has spent $\frac{63}{350} \times 100\% = 18\%$ of it.

Percentage change. When a quantity increases or decreases, the change is always compared with the **original** value, never the new one:

$$\text{percentage change} = \frac{\text{change}}{\text{original value}} \times 100\%.$$

The multiplier method. Increasing a quantity by $r\%$ leaves $100\% + r\%$ of it, so an increase is a single multiplication by $\left(1 + \frac{r}{100}\right)$; a decrease by $r\%$ leaves $100\% - r\%$, a multiplication by $\left(1 - \frac{r}{100}\right)$. An increase of 45% is $\times 1.45$ and a decrease of 20% is $\times 0.80$. Successive changes **multiply**: a 45% increase followed by a 20% decrease is $\times 1.45 \times 0.80 = \times 1.16$, an overall increase of 16% — not $45\% - 20\% = 25\%$, because the decrease acts on the already increased amount.

Mark-ups and discounts. A retailer buys stock at the **cost (wholesale) price** and sells it at a higher **selling price**; the increase is the **mark-up**, usually stated as a percentage of the cost price. A **discount** is a percentage decrease applied to the marked price during a sale. Both are handled with multipliers: a 40% mark-up on \$95 gives $95 \times 1.40 = \$133$, and a 12% discount on \$50 gives $50 \times 0.88 = \$44$.

Working backwards to the original price. If a sale price arose from a 30% discount, then

$$\text{original price} \times 0.70 = \text{sale price}, \text{so original price} = \frac{\text{sale price}}{0.70}.$$

In general, divide the new price by the multiplier that produced it. Never add the percentage back on: increasing the sale price by 30% does **not** undo a 30% discount.

GST. Australia's **Goods and Services Tax (GST)** adds 10% to the pre-tax price of most goods and services. So

$$\text{price including GST} = \text{pre-GST price} \times 1.1.$$

A GST-inclusive price is therefore 110% of the pre-GST price, and the GST is the 10 parts out of those 110 — that is, $\frac{10}{110} = \frac{1}{11}$ of the advertised price:

$$\text{GST included} = \frac{\text{GST-inclusive price}}{11}, \text{pre-GST price} = \frac{\text{GST-inclusive price}}{1.1}.$$

The unitary method. To solve many practical problems, first find the value of **one unit**, then scale up. If 6 kg of potatoes cost \$13.20, then 1 kg costs $13.20 \div 6 = \$2.20$, so 10 kg cost $10 \times 2.20 = \$22$. The same idea works with

1% as the unit: if 8% of a price is \$14, then 1% is $14 \div 8 = \$1.75$, so the full price ($100\%$) is $100 \times 1.75 = \$175$. A unit price (such as cost per litre or per 100 g) is a **rate** — a comparison of two quantities measured in *different* units — and unit prices let you compare pack sizes fairly and identify the **best buy**.

Ratio. A **ratio** compares quantities of the same kind, measured in the same units, and is written $a:b$; the order of the terms matters. Multiplying or dividing every term by the same non-zero number gives an **equivalent ratio**, so dividing both terms of $12:18$ by their highest common factor, 6, puts it in **simplest form** as $2:3$. To **share a quantity in a given ratio**, apply the unitary method to the parts: $a:b$ has $a+b$ parts altogether, so divide the quantity by the number of parts to find **one part**, then multiply. Sharing \$540 in the ratio $2:3$ gives $540 \div 5 = \$108$ for one part, and shares of $2 \times 108 = \$216$ and $3 \times 108 = \$324$; as a check, $216 + 324 = 540$.

Proportion. A **proportion** is a statement that two ratios are equal:

$$\frac{a}{b} = \frac{c}{d}.$$

Two quantities are **in proportion** when their ratio stays constant, so scaling one of them scales the other by the same factor. Find an unknown term by multiplying both sides by a denominator, or equivalently by the unitary method: if 4 kg of mince costs \$27, then the cost of 10 kg, x dollars, satisfies $\frac{x}{10} = \frac{27}{4}$, and since 1 kg costs $27 \div 4 = \$6.75$, $x = 10 \times 6.75 = \$67.50$. A percentage is itself a ratio whose second term is 100, which is why the percentage work above and the unitary method are two views of the same idea.

Using CAS. All of these calculations are ordinary arithmetic, which your CAS performs directly — for example, entering 95×1.40 gives the marked-up price at once. For reverse problems the CAS `solve` command is convenient: solving $1.1x = 93.50$ for x gives the pre-GST price $x = 85$. The by-hand skill that matters is choosing the correct multiplier (and knowing whether to multiply or divide by it); learn the methods below so you can set up the calculation and check that the CAS output is reasonable.

Worked examples

Example 1 — scaffolded

At a Wangaratta secondary college, 26 of the 40 Year 11 students walk or ride to school; last year only 20 of them did. At a nearby college, 96 of the 150 Year 11 students walk or ride. Express each college's count as a percentage, state which college has the greater proportion, and find the percentage increase in the first college's count since last year.

Working	Comment
First college: $\frac{26}{40} \times 100\% = 65\%$.	Part over whole, times 100%.
Second college: $\frac{96}{150} \times 100\% = 64\%$.	Same method with the second college's figures.
$65\% > 64\%$, so the first college has the greater proportion.	Percentages allow a fair comparison even though $96 > 26$.
Change $= 26 - 20 = 6$ students.	The increase since last year.
Percentage increase $= \frac{6}{20} \times 100\% = 30\%$.	Divide by the **original** value, 20 — not by 26.

So 65% of the first college's Year 11 students walk or ride (64% at the second), and the first college's count has risen by 30%.

__Example 2 — scaffolded__

A Torquay surf shop buys wetsuits at a wholesale price of \$240 each and marks them up by $45\,\%$. At the end of summer the shop discounts the marked price by $20\,\%$. Find the selling price, the end-of-summer sale price, and the overall percentage change from the wholesale price.

Working	Comment
Mark-up multiplier $=1+\dfrac{45}{100}=1.45$.	A $45\,\%$ increase leaves $145\,\%$ of the price.
Selling price $=240\times1.45=\$348$.	One multiplication applies the whole mark-up.
Discount multiplier $=1-\dfrac{20}{100}=0.80$.	A $20\,\%$ discount leaves $80\,\%$ of the price.
Sale price $=348\times0.80=\$278.40$.	The discount acts on the marked price, \$348.
Overall change $=278.40-240=\$38.40$.	Compare the sale price with the wholesale price.
Overall percentage change $=\dfrac{38.40}{240}\times100\,\%=16\,\%$.	An increase of $16\,\%$, not $45\,\%-20\,\%=25\,\%$: successive changes multiply $(1.45\times0.80=1.16)$.

So the wetsuits sell for \$348, the sale price is \$278.40, and overall the price is $16\,\%$ above wholesale.

__Example 3 — unscaffolded__

A Ballarat landscaper quotes \$3960, including GST, to pave a courtyard. Find the pre-GST price of the job and the amount of GST included. The pavers themselves cost the landscaper \$850 before GST; find their price once GST is added.

The quoted price is $110\,\%$ of the pre-GST price, so the pre-GST price is $3960\div1.1=\$3600$. The GST included is one-eleventh of the GST-inclusive price: $3960\div11=\$360$. As a check, $10\,\%$ of \$3600 is \$360, and $3600+360=3960$. For the pavers, adding $10\,\%$ GST multiplies the price by 1.1, giving $850\times1.1=\$935$ (the GST added is \$85).

∴ the job is \$3600 before GST, which includes \$360 of GST, and the pavers cost \$935 once GST is added.

__Example 4 — unscaffolded__

In a Geelong appliance store's clearance sale, every refrigerator is discounted by $25\,\%$. One refrigerator has a sale price of \$975. Use the unitary method to find its original price, and hence the amount a buyer saves.

After a $25\,\%$ discount, the sale price represents $100\,\%-25\,\%=75\,\%$ of the original price. So $75\,\%$ of the original price is \$975, which makes $1\,\%$ equal to $975\div75=\$13$, and the full original price ($100\,\%$) equal to $100\times13=\$1300$. Equivalently, dividing by the multiplier gives $975\div0.75=\$1300$. As a check, $1300\times0.75=975$. The saving is the discount: $1300-975=\$325$.

∴ the refrigerator's original price was \$1300, so the buyer saves \$325.

Practice questions

Scaffolded

Q1. A Bendigo secondary school has 1200 students. (a) On one day, $4.5\,\%$ of the students were absent. How many students were absent? (b) Of the 1200 students, 780 travel to school by bus. What percentage travel by bus? (c) The school's Year 11 cohort has 210 students. Express this as a percentage of the whole school. (d) Next year the enrolment is expected to grow by $5\,\%$. Find the expected new enrolment.

Q2. At a Coburg grocer, the price of a carton of eggs rose from \$5.60 to \$6.44. (a) Find the increase in price, in dollars. (b) Find the percentage increase. (c) In the same week, a phone company cut its monthly plan from \$65 to \$52. Find the percentage decrease in the plan's price.

Q3. A Frankston skate shop buys skateboards at a wholesale price of \$85 each and marks them up by 60%. (a) Find the mark-up, in dollars. (b) Find the selling price. (c) The shop buys helmets for \$140 and sells them for \$210. Find the mark-up as a percentage of the wholesale price.

Q4. In its summer sale, a Melbourne department store discounts every marked price by 30%. (a) Find the discount on a coat marked at \$180, and its sale price. (b) Use the multiplier 0.70 to find the sale price of a shirt marked at \$65. (c) A customer pays \$84 for a pair of jeans in the sale. Work backwards to find the marked price before the discount.

Q5. A Shepparton electrician's prices attract GST at 10%. (a) She quotes \$480 before GST to install ceiling fans. Find the GST and the total price the customer pays. (b) She receives a supplier's invoice for \$66, including GST. Find the GST included and the pre-GST amount. (c) A reel of cable costs \$253, including GST. Find the GST included and the pre-GST price. (d) Explain why the GST included in a GST-inclusive price is found by dividing by 11, not by 10.

Q6. Use the unitary method for each part. (a) Five tickets to a cinema cost \$92.50 in total. Find the cost of one ticket, and hence the cost of eight tickets. (b) A supermarket sells muesli in a 500 g box for \$6.40 and an 800 g box for \$9.92. Find the cost per 100 g of each box, and state which box is the better buy. (c) A 30% discount on a food processor saves a shopper \$45. Find the value of 1% of the original price, and hence the original price. (d) Two friends run a weekend market stall and share the day's takings of \$920 in the ratio of the hours each worked, 10 hours to 6 hours. Write that ratio in simplest form, find the value of one part, and hence each friend's share.

Unscaffolded

Q7. A Hobart gym raised its annual membership from \$980 to \$1078. Find the percentage increase.

Q8. Nia, a first-year apprentice in Launceston, saves \$132 of her \$480 weekly pay. Express her savings as a percentage of her pay.

Q9. An Adelaide jeweller buys a watch for \$650 and marks it up by 84%. Find the mark-up in dollars and the selling price.

Q10. A Cairns camping store discounts a \$440 tent by 15%, then takes a further 10% off the reduced price at the end of the season. Find the price after each discount, the total saving, and the overall percentage discount. Explain why the overall discount is not 25%.

Q11. After a 35% mark-up, a Perth kitchenware store sells a coffee machine for \$364.50. Find the wholesale price the store paid.

Q12. A Canberra clothing outlet advertises "28% off everything". Ruby pays \$61.20 for a jacket. Find the jacket's original price.

Q13. A Darwin restaurant's bill for a family dinner comes to \$217.80, including GST. Find the GST included in the bill and the pre-GST amount. The menu lists a seafood platter at \$38.50 (GST inclusive); find the GST included in the platter's price.

Q14. Two builders quote for the same deck on a Sunshine Coast home. Archway Decks quotes \$14800 plus GST; Boardwalk Builders quotes \$16170 including GST. Find the total (GST-inclusive) price of each quote and the GST in each, then state which quote is cheaper for the homeowner and by how much.

Q15. A supermarket sells the same orange juice in three sizes: 1.5 L for \$5.70, 2 L for \$7.40 and 3 L for \$11.40. Find the cost per litre of each size, and state which size is the best buy. Suggest one practical reason a shopper might still choose a different size.

Q16. A travel agent requires a 15% deposit to hold a holiday booking. Zoe's deposit is \$387. Use the unitary method to find the full price of the holiday, and the balance she still owes. Zoe is travelling with a friend, and the two of them split the full price in the ratio $7:5$ because Zoe takes the larger room. Find each person's share.

Q17. A Fitzroy boutique buys dresses for \$160 each, marks them up by 65%, and later discounts the marked price by 25% in a sale. All its prices include GST. Find the marked price, the sale price, the GST included in the sale price, and the overall percentage change from the \$160 cost to the sale price.

Q18. The price of a board game falls from \$80 to \$60. Ari says the price has decreased by 25%; Bree divides the \$20 change by \$60 and claims the decrease is $33\,1/3\%$. Who is correct, and why? Then explain, with a calculation, why increasing the \$60 price by 25% would not return it to \$80.

Answers

Q1. (a) 54 students. (b) 65%. (c) 17.5%. (d) 1260 students. **Q2.** (a) $0.84. (b) 15%. (c) 20%. **Q3.** (a) $51. (b) $136. (c) 50%. **Q4.** (a) Discount $54; sale price $126. (b) $45.50. (c) $120. **Q5.** (a) GST $48; total $528. (b) GST $6; pre-GST $60. (c) GST $23; pre-GST $230. (d) The inclusive price is 110% of the pre-GST price, and GST is 10 parts of those 110, i.e. $\frac{1}{11}$ of the inclusive price. **Q6.** (a) $18.50; $148. (b) 500 g: $1.28 per 100 g; 800 g: $1.24 per 100 g; the 800 g box. (c) $1\% = \$1.50$; original price $150. (d) $10:6 = 5:3$; one part $115; shares $575 and $345. **Q7.** 10%. **Q8.** 27.5%. **Q9.** Mark-up $546; selling price $1196. **Q10.** $374, then $336.60; saving $103.40; overall 23.5% — the second discount acts on the already reduced price, so the discounts do not add. **Q11.** $270. **Q12.** $85. **Q13.** Bill: GST $19.80, pre-GST $198. Platter: GST $3.50. **Q14.** Archway: $16280 incl. (GST $1480); Boardwalk: $16170 incl. (GST $1470). Boardwalk is cheaper by $110. **Q15.** $3.80, $3.70 and $3.80 per litre; the 2 L bottle is the best buy. (Practical reason: e.g. a shopper may not use 2 L before it spoils, or may need more than 2 L.) **Q16.** Full price $2580; balance $2193; one part $215, so Zoe pays $1505 and her friend $1075. **Q17.** Marked price $264; sale price $198; GST $18; overall change $+23.75\%$. **Q18.** Ari is correct ($\frac{20}{80} \times 100\% = 25\%$): percentage change is measured against the original price, $80. Increasing $60 by 25% gives $60 \times 1.25 = \$75 \neq \$80$, because the 25% is now taken of the smaller price.

Fully-worked solutions

Q1. (a) 4.5% of $1200 = 0.045 \times 1200 = 54$ students. (b) $\frac{780}{1200} \times 100\% = 65\%$. (c) $\frac{210}{1200} \times 100\% = 17.5\%$. (d) A 5% increase is a multiplication by 1.05: $1200 \times 1.05 = 1260$ students.

Q2. (a) Increase $= 6.44 - 5.60 = \$0.84$. (b) Percentage increase $= \frac{0.84}{5.60} \times 100\% = 15\%$ (divide by the original price, $5.60). (c) Decrease $= 65 - 52 = \$13$, so the percentage decrease is $\frac{13}{65} \times 100\% = 20\%$.

Q3. (a) Mark-up $= 60\%$ of $85 = 0.60 \times 85 = \$51$. (b) Selling price $= 85 + 51 = \$136$ (or $85 \times 1.60 = 136$). (c) Mark-up $= 210 - 140 = \$70$, so the percentage mark-up is $\frac{70}{140} \times 100\% = 50\%$.

Q4. (a) Discount $= 0.30 \times 180 = \$54$, so the sale price is $180 - 54 = \$126$. (b) Sale price $= 65 \times 0.70 = \$45.50$. (c) The sale price is 70% of the marked price, so marked price $= \frac{84}{0.70} = \$120$. Check: $120 \times 0.70 = 84$.

Q5. (a) GST $= 10\%$ of $480 = \$48$; total $= 480 \times 1.1 = \$528$. (b) GST $= \frac{66}{11} = \$6$; pre-GST amount $= \frac{66}{1.1} = \$60$. Check: $60 + 6 = 66$. (c) GST $= \frac{253}{11} = \$23$; pre-GST price $= \frac{253}{1.1} = \$230$. (d) The GST-inclusive price is 110% of the pre-GST price: 100 parts are the price and 10 parts are GST. The GST is therefore $\frac{10}{110} = \frac{1}{11}$ of the inclusive price. Dividing by 10 would take 10% of the inclusive price, which already contains the GST, and would give an answer that is too large.

Q6. (a) One ticket costs $\frac{92.50}{5} = \$18.50$, so eight cost $8 \times 18.50 = \$148$. (b) 500 g box: $\frac{6.40}{5} = \$1.28$ per 100 g. 800 g box: $\frac{9.92}{8} = \$1.24$ per 100 g. The 800 g box is cheaper per 100 g, so it is the better buy. (c) 30% of the price is $45, so 1% is $\frac{45}{30} = \$1.50$ and the original price (100%) is $100 \times 1.50 = \$150$. (d) The hours are in the ratio $10:6$;

dividing both terms by their highest common factor, 2, gives the simplest form $5:3$. That is $5+3=8$ parts altogether, so one part is $\frac{920}{8}=\$115$. The shares are $5\times115=\$575$ and $3\times115=\$345$ (check: $575+345=920$).

Q7. Increase $=1078-980=\$98$. Percentage increase $=\frac{98}{980}\times100\%=10\%$.

Q8. $\frac{132}{480}\times100\%=27.5\%$ of her pay.

Q9. Mark-up $=0.84\times650=\$546$; selling price $=650+546=\$1196$ (or $650\times1.84=1196$).

Q10. After the first discount the tent costs $440\times0.85=\$374$; after the further 10% it costs $374\times0.90=\$336.60$. The total saving is $440-336.60=\$103.40$, so the overall discount is $\frac{103.40}{440}\times100\%=23.5\%$. It is not 25% because the second discount is 10% of the already reduced price \$374, not of the original \$440 (equivalently, $0.85\times0.90=0.765$, a 23.5% reduction).

Q11. The selling price is 135% of the wholesale price, so wholesale price $=\frac{364.50}{1.35}=\$270$. Check: $270\times1.35=364.50$.

Q12. Ruby paid $100\%-28\%=72\%$ of the original price, so original price $=\frac{61.20}{0.72}=\$85$. Check: $85\times0.72=61.20$.

Q13. GST in the bill $=\frac{217.80}{11}=\$19.80$; pre-GST amount $=\frac{217.80}{1.1}=\$198$. Check: $198\times1.1=217.80$. For the platter, GST $=\frac{38.50}{11}=\$3.50$.

Q14. Archway: total $=14800\times1.1=\$16280$, which includes GST of $0.1\times14800=\$1480$. Boardwalk: the quote already includes GST, so the GST is $\frac{16170}{11}=\$1470$ and the pre-GST amount is $\frac{16170}{1.1}=\$14700$. Comparing the GST-inclusive totals, Boardwalk's \$16170 is cheaper than Archway's \$16280 by $16280-16170=\$110$. The quotes must be compared on the same basis — both including GST. Before tax the difference is $14800-14700=\$100$, and adding 10% GST scales that difference to $100\times1.1=\$110$.

Q15. 1.5 L: $\frac{5.70}{1.5}=\$3.80$ per litre. 2 L: $\frac{7.40}{2}=\$3.70$ per litre. 3 L: $\frac{11.40}{3}=\$3.80$ per litre. The 2 L bottle has the lowest cost per litre, so it is the best buy. A shopper might still prefer another size — for example, a household that cannot finish 2 L before the use-by date wastes money on the larger bottle, while a large household might buy the 3 L for fewer shopping trips.

Q16. 15% of the price is \$387, so 1% is $\frac{387}{15}=\$25.80$ and the full price is $100\times25.80=\$2580$. The balance owing is the remaining 85%: $2580-387=\$2193$ (check: $0.85\times2580=2193$). Splitting the full price in the ratio $7:5$ uses $7+5=12$ parts, so one part is $\frac{2580}{12}=\$215$. Zoe's share is $7\times215=\$1505$ and her friend's is $5\times215=\$1075$ (check: $1505+1075=2580$).

Q17. Marked price $=160\times1.65=\$264$. Sale price $=264\times0.75=\$198$. The sale price includes GST of $\frac{198}{11}=\$18$ (pre-GST \$180). Overall change $=198-160=\$38$, so the percentage change is $\frac{38}{160}\times100\%=23.75\%$ — an overall increase of 23.75% on the cost price.

Q18. Percentage change is always measured against the **original** value. The change is $80 - 60 = \$20$ and the original price is \$80, so the decrease is $\frac{20}{80} \times 100\% = 25\%$ — Ari is correct. Bree divided by the new price, \$60, which gives $\frac{20}{60} \times 100\% = 33\,1/3\%$, the wrong base for a decrease from \$80. For the second part, $60 \times 1.25 = \$75$, not \$80: the 25% increase is taken of the smaller price \$60, so it adds back only \$15 of the \$20 that was removed. To return to \$80 the price would need to rise by $\frac{20}{60} \times 100\% = 33\,1/3\%$.

Theory

Money that is invested earns **interest**, money that is borrowed is charged interest, and equipment that is owned loses value. In this section all of these changes happen by the **same amount every period**, so the values form the sequences studied in Chapter 2: a starting value with a fixed amount added (or subtracted) at each step. We write V_n for the value, in dollars, after n periods, with **starting value** V_0.

Simple interest. The **principal** P is the amount originally invested or borrowed, and the **interest** is the extra money paid for the use of that money — paid *to* you by the institution when you invest with it, and paid *by* you to the lender when you borrow. Under **simple interest** the interest for each period is calculated on the original principal only, so the same amount

$$I = \frac{r}{100} \times P$$

is added every period, where $r\,\%$ is the interest rate per period (usually per annum). The value of the investment therefore satisfies the recurrence relation

$$V_0 = P, V_{n+1} = V_n + I.$$

After n periods, exactly n lots of interest have been added, so the value is given directly by the rule

$$V_n = P + nI = P\left(1 + \frac{rn}{100}\right),$$

and the **total interest** earned so far is $V_n - P = nI$.

Simple interest loans. Money that is *borrowed* under a **flat-rate loan** — also called a **simple interest loan** — is charged interest on the amount borrowed only, so the same $I = \frac{r}{100} \times P$ is added to the debt every period. The amount owed therefore obeys exactly the same recurrence and the same rule,

$$V_0 = P, V_{n+1} = V_n + I, V_n = P + nI,$$

except that V_n is now the total the borrower must repay after n periods and nI is the total interest *charged*. Interest rates on loans are higher than the rates paid on investments, but the arithmetic is identical — the only difference is who pays the interest to whom.

Finding the time or the rate. Because the same interest is added every period, unknown quantities are found by simple division. The number of years needed to earn a required amount of interest is the required interest divided by the interest per year, $n = \frac{\text{total interest}}{I}$. If the rate is unknown, first find the interest added per year, then $r = \frac{I}{P} \times 100$. If the principal is unknown, substitute into $V_n = P\left(1 + \frac{rn}{100}\right)$ and solve for P. For example, \$2000 invested at 3% per annum earns $I = \frac{3}{100} \times 2000 = \$60$ every year, and the value grows \$2000, \$2060, \$2120, …

Flat-rate depreciation. Most assets — vehicles, machines, computers — lose value as they age. This loss of value is called **depreciation**, and the asset's current value is its **book value**. Under **flat-rate depreciation** (also called **straight-line depreciation**) the book value falls by the same amount D every year. The annual depreciation is often quoted as a fixed percentage of the **purchase price** P, in which case $D = \frac{r}{100} \times P$. The book value satisfies

$$V_0 = P, V_{n+1} = V_n - D, \text{so } V_n = P - nD.$$

An asset is usually **written off** — removed from the accounts and sold or scrapped — when its book value falls to its **scrap value** S. The time this takes is the asset's **useful life**: solving $P - nD = S$ gives $n = \frac{P-S}{D}$ years.

Unit-cost depreciation. For some assets the loss in value depends on how much the asset is **used**, not on how old it is: a delivery van wears out by the kilometre, a printing press by the page. Under **unit-cost depreciation** the value falls by a fixed amount for every unit of use, so

$$V = P - (\text{cost per unit}) \times (\text{units used}).$$

Here the natural step is one unit of use (one kilometre, one page) rather than one year, but the calculation is the same: a constant amount is subtracted for each unit. To convert to a value after n years, multiply the cost per unit by the expected number of units used per year.

Linear growth and decay. All three models change by a **constant amount** each step, so each is a first-order linear recurrence $t_{n+1} = a\,t_n + b$ from Chapter 2 with $a = 1$: simple interest has $b = I > 0$ (linear growth) and flat-rate depreciation has $b = -D < 0$ (linear decay). The values $V_0, V_1, V_2, \ldots$ therefore form an **arithmetic sequence**, and a graph of V_n against n is a set of separate points lying on a straight line — rising for an investment, falling for a depreciating asset. (When the value is instead **multiplied** by a constant each period — as in compound interest and reducing-balance depreciation — the growth is geometric; that is the subject of section 3C.)

Using CAS. A CAS calculator generates the values of any of these recurrences with its answer key: type the starting value, press ENTER, then type $\text{Ans} + 140$ (for interest of \$140 per year) or $\text{Ans} - 4750$ (for depreciation of \$4750 per year), and each further press of ENTER gives the next year's value. The rules $V_n = P + nI$ and $V_n = P - nD$ can be tabulated or solved directly in the calculator application for an unknown time, rate or principal. Learn the by-hand methods below so that you understand what the calculator is doing and can check that its output is reasonable.

Worked examples

Example 1 — scaffolded

Priya invests \$4000 in a term deposit paying 3.5% per annum simple interest for 5 years. Find the interest added each year, write a recurrence relation for the value V_n of the deposit after n years, and find the value of the deposit and the total interest earned at the end of the 5 years.

Working	Comment
$I = \frac{r}{100} \times P = \frac{3.5}{100} \times 4000 = \$140.$	Simple interest is calculated on the original principal, so the same \$140 is added every year.
$V_0 = 4000$, $V_{n+1} = V_n + 140.$	The Chapter 2 recurrence $t_{n+1} = t_n + b$ with starting value 4000 and $b = 140$: linear growth.
$V_5 = P + 5I = 4000 + 5 \times 140 = \$4700.$	After 5 years, 5 lots of \$140 have been added; the rule $V_n = P + nI$ gives the value directly.
Total interest $= V_5 - P = 4700 - 4000 = \$700.$	Equivalently, total interest $= nI = 5 \times 140 = 700.$

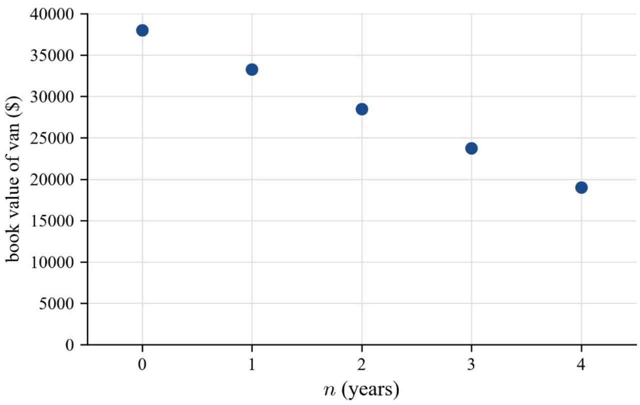
So the deposit earns \$140 interest each year, and after 5 years it is worth \$4700, having earned \$700 in total interest.

Example 2 — scaffolded

A florist buys a delivery van for \$38,000. For tax purposes the van is depreciated by the flat-rate method at 12.5% of its purchase price each year. Find the annual depreciation, write a recurrence relation for the book value V_n after n years, tabulate and plot the values V_0 to V_4, and state the book value of the van after 4 years.

Working	Comment
$D = \frac{12.5}{100} \times 38\,000 = \$4750$.	Flat-rate depreciation subtracts the same amount every year — here a fixed percentage of the purchase price.
$V_0 = 38\,000$, $V_{n+1} = V_n - 4750$.	Linear decay: the Chapter 2 recurrence with $b = -4750$.
$V_1 = 38\,000 - 4750 = 33\,250$; $V_2 = 28\,500$; $V_3 = 23\,750$; $V_4 = 19\,000$.	Iterate — subtract \$4750 at each step (see the table below).
Check with the rule: $V_4 = P - 4D = 38\,000 - 4 \times 4750 = 19\,000$.	The rule $V_n = P - nD$ gives any book value directly.
Plot the points $(0, 38\,000)$ to $(4, 19\,000)$ (below).	Constant change: the separate points lie on a straight line — an arithmetic sequence.

n	0	1	2	3	4
V_n (\$)	$38\,000$	$33\,250$	$28\,500$	$23\,750$	$19\,000$



So the van loses \$4750 of value every year, and its book value after 4 years is \$19,000.

Example 3 — unscaffolded

Marcus borrows \$8400 from a credit union under a flat-rate loan that charges simple interest at a fixed annual rate. After 4 years the total amount he owes has grown to \$10,920. Find the annual interest rate, and determine after how many years the amount owed would reach \$13,440.

Over the 4 years the loan was charged $10\,920 - 8400 = \$2520$ in interest. Simple interest adds the same amount each year, so the interest per year is $I = \frac{2520}{4} = \$630$, and the rate is $r = \frac{I}{P} \times 100 = \frac{630}{8400} \times 100 = 7.5$, that is, $7.5\,\%$ per annum. To reach \$13,440 the loan must be charged $13\,440 - 8400 = \$5040$ in interest in total, which takes $\frac{5040}{630} = 8$ years. (Check with the rule: $V_8 = 8400 + 8 \times 630 = 13\,440$.)

∴ the loan is charged $7.5\,\%$ per annum simple interest, and the amount owed reaches \$13,440 after 8 years.

Example 4 — unscaffolded

A tour operator buys a minibus for \$58,000. Because its loss in value depends on how far it travels rather than on its age, it is depreciated by the unit-cost method at 28 cents per kilometre. Find the book value of the minibus after it has travelled $90\,000$ km, and the total distance it will have travelled when it is written off at its scrap value of \$13,200.

The cost per unit is \$0.28 per kilometre, so after $90\,000$ km the depreciation is $0.28 \times 90\,000 = \$25\,200$, and the book value is $V = P - 25\,200 = 58\,000 - 25\,200 = \$32\,800$. The minibus is written off when its book value reaches \$13,200, by which time the total depreciation is $58\,000 - 13\,200 = \$44\,800$. At 28 cents per kilometre this represents $\frac{44\,800}{0.28} = 160\,000$ kilometres of travel.

∴ the minibus's book value after $90\,000$ km is $32,800, and it is written off after travelling $160\,000$ km in total.

Practice questions

Scaffolded

Q1. Tan invests $3000 in an online savings account paying 4% per annum simple interest. (a) Show that the interest added to the account each year is $120. (b) Write a recurrence relation for the value V_n of the account after n years. (c) Generate V_1, V_2 and V_3. (d) Use the rule $V_n = P + nI$ to find the value of the account after 8 years.

Q2. A café buys an espresso machine for $12,000 and depreciates it by the flat-rate method at 15% of the purchase price each year. (a) Find the annual depreciation D. (b) Write a recurrence relation for the book value V_n after n years. (c) Find the book value of the machine after 3 years. (d) The machine will be written off when its book value falls to $3000. After how many years does this happen?

Q3. A landscaper buys a ride-on mower for $9750. Because the mower wears out with use rather than age, it is depreciated by the unit-cost method at $1.30 per hour of use. (a) Find the depreciation over the mower's first 800 hours of use. (b) Find the book value of the mower at that point. (c) Find the book value once the mower has been used for a total of 2100 hours. (d) The mower will be written off at its scrap value of $1950. Find the total number of hours of use when this happens.

Q4. Nadia borrows $6400 under a flat-rate loan charging 12% per annum simple interest. (a) Find the interest charged each year. (b) After how many years has the loan been charged $3840 in interest in total? (c) What is the total amount owed at that time?

Q5. A carpenter buys a work ute for $41,500 and depreciates it by the flat-rate method at $4300 per year. (a) Copy and complete the table of book values below.

n (years)	0	1	2	3	4
V_n ($)	$41\,500$	$37\,200$	?	$28\,600$	?

(b) Write a recurrence relation for the book value V_n.

(c) The book values are graphed as separate points. Explain why the points lie on a straight line, and state what kind of sequence the book values form.

(d) The ute's scrap value is $2800. Find its useful life.

Q6. For each situation below, state whether it is modelled by simple interest, flat-rate depreciation or unit-cost depreciation, write a rule for the value, and answer the question. (a) Ava invests $2000 and the account adds $90 interest at the end of each year. What annual interest rate is the account paying? (b) An office photocopier is bought for $5600 and loses $700 of value each year. What is its book value after 6 years? (c) A delivery scooter is bought for $4600 and loses 8 cents of value for every kilometre ridden. What is its book value after it has been ridden $15\,000$ km?

Unscaffolded

Q7. Aisha invests $7500 in government bonds paying 3.2% per annum simple interest. Find the interest she receives each year, the total interest she receives over 6 years, and the value of her investment at the end of the 6 years.

Q8. A fishing charter business buys a boat for $64,000 and depreciates it by the flat-rate method at $5500 per year. Write a recurrence relation for the book value V_n after n years, and find the book value of the boat after 5 years and after 8 years.

Q9. Ben invested $12,000 in a fixed-term deposit paying simple interest. After 4 years the deposit was worth $14,160. Find the total interest earned, the interest earned each year, and the annual interest rate.

Q10. A dental surgery buys a treatment chair for $28,000 and depreciates it by the flat-rate method at 11% of the purchase price per year. The chair will be written off when its book value reaches its scrap value of $3360. Find the annual depreciation and the useful life of the chair.

Q11. A printing business buys a digital press for $150,000 and depreciates it by the unit-cost method at 5 cents per page printed. Find the book value of the press after it has printed $900\,000$ pages, and the number of further pages it can print before its book value falls to $60,000.

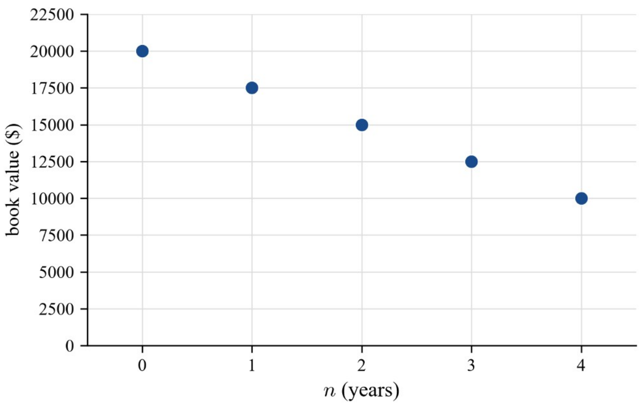
Q12. The recurrence relation $V_0 = 9000$, $V_{n+1} = V_n + 315$ models the value V_n, in dollars, of an investment after n years. (a) Interpret the numbers 9000 and 315 in the context of the model. (b) Find the annual simple interest rate being paid. (c) Find the value of the investment after 10 years.

Q13. The recurrence relation $V_0 = 22\,400$, $V_{n+1} = V_n - 2240$ models the book value V_n, in dollars, of a hatchback n years after it was bought. (a) Interpret the numbers $22\,400$ and 2240 in the context of the model. (b) What percentage of its purchase price does the hatchback lose each year? (c) After how many years is the car's book value exactly half its purchase price?

Q14. A courier business buys a van for $42,000 and must choose between two depreciation methods: flat-rate depreciation of $4800 per year, or unit-cost depreciation at 34 cents per kilometre. The van is expected to travel $18\,000$ km each year. (a) Find the book value of the van after 3 years under each method. (b) Suppose that instead the van travels only $11\,000$ km each year. Which method now gives the lower book value after 3 years? (c) The business argues that the van loses value mainly through the distance it is driven. Which method better reflects the van's value, and why?

Q15. A sum of money is borrowed under a flat-rate loan charging 9% per annum simple interest. At the end of 5 years the total amount owed is $14,500. Use the rule $V_n = P\left(1 + \frac{rn}{100}\right)$ to find the amount that was borrowed.

Q16. The graph below shows the book value V_n, in dollars, of a commercial coffee roaster over its first four years, depreciated by the flat-rate method.



Find the purchase price and the annual depreciation, write a recurrence relation for V_n, find the book value of the roaster after 6 years, and find after how many years the roaster is written off at its scrap value of $2500.

Q17. A courier rides a motorbike bought for $9800, which is depreciated by the unit-cost method at 15 cents per kilometre. The courier rides 8000 km each year. (a) Find the depreciation over one year of riding. (b) Find the book value of the motorbike after 3 years of riding. (c) The motorbike is written off when its book value reaches $800. Find the total distance travelled when this happens, and hence the motorbike's useful life in years.

Q18. The value, in dollars, of an investment paying simple interest is given by the rule $V_n = 6500 + 260n$, where n is the number of years since the money was invested. (a) State the principal and the interest added each year, and find the annual interest rate. (b) Explain why the values $V_0, V_1, V_2, \ldots$ form an arithmetic sequence. (c) Find the first year at the end of which the investment is worth more than $9000.

Answers

Q1. (a) $I=\frac{4}{100}\times 3000=\$120$. (b) $V_0=3000$, $V_{n+1}=V_n+120$. (c) $V_1=3120$, $V_2=3240$, $V_3=3360$. (d) $V_8=\$3960$. **Q2.** (a) $D=\$1800$. (b) $V_0=12\,000$, $V_{n+1}=V_n-1800$. (c) $V_3=\$6600$. (d) After 5 years. **Q3.** (a) \$1040. (b) \$8710. (c) \$7020. (d) 6000 hours. **Q4.** (a) \$768. (b) 5 years. (c) \$10,240 owed. **Q5.** (a) $V_2=32\,900$, $V_4=24\,300$. (b) $V_0=41\,500$, $V_{n+1}=V_n-4300$. (c) The value falls by the same \$4300 each year, so the points fall in a straight line; the book values form an arithmetic sequence. (d) 9 years. **Q6.** (a) Simple interest; $V_n=2000+90n$; rate $=4.5\%$ per annum. (b) Flat-rate depreciation; $V_n=5600-700n$; $V_6=\$1400$. (c) Unit-cost depreciation; value $=4600-0.08\times(\text{km ridden})$; \$3400. **Q7.** \$240 each year; \$1440 over 6 years; value \$8940. **Q8.** $V_0=64\,000$, $V_{n+1}=V_n-5500$; $V_5=\$36\,500$; $V_8=\$20\,000$. **Q9.** Total interest \$2160; \$540 each year; rate $=4.5\%$ per annum. **Q10.** $D=\$3080$; useful life 8 years. **Q11.** \$105,000; a further $900\,000$ pages. **Q12.** (a) \$9000 is the principal invested; \$315 is the interest added each year. (b) 3.5% per annum. (c) $V_{10}=\$12\,150$. **Q13.** (a) \$22,400 is the purchase price; \$2240 is the annual (flat-rate) depreciation. (b) 10% of the purchase price. (c) After 5 years ($V_5=11\,200$). **Q14.** (a) Flat-rate \$27,600; unit-cost \$23,640. (b) Flat-rate (unit-cost now gives \$30,780). (c) Unit-cost, because it links the loss of value to the distance driven, the main cause of wear. **Q15.** $P=\$10\,000$. **Q16.** Purchase price \$20,000; $D=\$2500$ per year; $V_0=20\,000$, $V_{n+1}=V_n-2500$; $V_6=\$5000$; written off after 7 years. **Q17.** (a) \$1200. (b) \$6200. (c) $60\,000$ km; useful life 7.5 years. **Q18.** (a) Principal \$6500; interest \$260 each year; rate $=4\%$ per annum. (b) Each value is 260 more than the one before — a constant difference. (c) At the end of the 10th year ($V_{10}=9100$).

Fully-worked solutions

Q1. (a) $I=\frac{r}{100}\times P=\frac{4}{100}\times 3000=\$120$. (b) The account starts at \$3000 and grows by \$120 each year: $V_0=3000$, $V_{n+1}=V_n+120$. (c) $V_1=3000+120=3120$; $V_2=3120+120=3240$; $V_3=3240+120=3360$. (d) $V_8=P+8I=3000+8\times 120=3000+960=\$3960$.

Q2. (a) $D=\frac{15}{100}\times 12\,000=\$1800$. (b) $V_0=12\,000$, $V_{n+1}=V_n-1800$. (c) $V_3=P-3D=12\,000-3\times 1800=12\,000-5400=\$6600$. (d) The total depreciation allowed is $12\,000-3000=\$9000$, so the machine is written off after $n=\frac{9000}{1800}=5$ years. (Check: $V_5=12\,000-5\times 1800=3000$.)

Q3. (a) Depreciation $=1.30\times 800=\$1040$. (b) Book value $=9750-1040=\$8710$. (c) After 2100 hours the depreciation is $1.30\times 2100=\$2730$, so the book value is $9750-2730=\$7020$. (d) The total depreciation allowed is $9750-1950=\$7800$, which at \$1.30 per hour represents $\frac{7800}{1.30}=6000$ hours of use.

Q4. (a) $I=\frac{12}{100}\times 6400=\$768$ is charged each year. (b) The same \$768 is charged every year, so \$3840 of interest takes $\frac{3840}{768}=5$ years. (c) The amount owed is the principal plus the interest charged: $V_5=6400+3840=\$10\,240$.

Q5. (a) The value falls by \$4300 each year, so $V_2=37\,200-4300=32\,900$ and $V_4=28\,600-4300=24\,300$. (b) $V_0=41\,500$, $V_{n+1}=V_n-4300$. (c) The book value changes by the same amount, $-\$4300$, every year, so equal steps across (one year) always produce equal steps down (\$4300) — the points line up on a straight line. A sequence with a constant difference between consecutive terms is an arithmetic sequence. (d) The total depreciation allowed is $41\,500-2800=\$38\,700$, so the useful life is $n=\frac{P-S}{D}=\frac{38\,700}{4300}=9$ years.

Q6. (a) A fixed amount is added each year: simple interest, with $V_n = 2000 + 90n$. The rate is $r = \frac{I}{P} \times 100 = \frac{90}{2000} \times 100 = 4.5$, that is, 4.5% per annum. (b) A fixed amount is subtracted each year: flat-rate depreciation, with $V_n = 5600 - 700n$. After 6 years, $V_6 = 5600 - 6 \times 700 = 5600 - 4200 = \$1400$. (c) The value falls per kilometre of use: unit-cost depreciation, with value $= 4600 - 0.08 \times (\text{km ridden})$. After $15\,000$ km the depreciation is $0.08 \times 15\,000 = \$1200$, so the book value is $4600 - 1200 = \$3400$.

Q7. $I = \frac{3.2}{100} \times 7500 = \$240$ each year. Over 6 years the total interest is $6 \times 240 = \$1440$, so the investment is worth $V_6 = 7500 + 1440 = \$8940$.

Q8. The boat starts at $64,000 and loses $5500 each year: $V_0 = 64\,000$, $V_{n+1} = V_n - 5500$. Then $V_5 = 64\,000 - 5 \times 5500 = 64\,000 - 27\,500 = \$36\,500$ and $V_8 = 64\,000 - 8 \times 5500 = 64\,000 - 44\,000 = \$20\,000$.

Q9. Total interest $= 14\,160 - 12\,000 = \$2160$. Simple interest adds the same amount each year, so the interest per year is $I = \frac{2160}{4} = \$540$, and the rate is $r = \frac{I}{P} \times 100 = \frac{540}{12\,000} \times 100 = 4.5$, that is, 4.5% per annum.

Q10. $D = \frac{11}{100} \times 28\,000 = \$3080$ per year. The total depreciation allowed is $28\,000 - 3360 = \$24\,640$, so the useful life is $n = \frac{24\,640}{3080} = 8$ years. (Check: $V_8 = 28\,000 - 8 \times 3080 = 3360$.)

Q11. After $900\,000$ pages the depreciation is $0.05 \times 900\,000 = \$45\,000$, so the book value is $150\,000 - 45\,000 = \$105\,000$. To fall from $105,000 to $60,000 the press must depreciate a further $105\,000 - 60\,000 = \$45\,000$, which at 5 cents per page represents $\frac{45\,000}{0.05} = 900\,000$ further pages.

Q12. (a) The 9000 is the starting value V_0 — the principal of $9000 invested — and the 315 is the fixed amount added each year: the investment earns $315 of simple interest per year. (b) $r = \frac{I}{P} \times 100 = \frac{315}{9000} \times 100 = 3.5$, that is, 3.5% per annum. (c) $V_{10} = 9000 + 10 \times 315 = 9000 + 3150 = \$12\,150$.

Q13. (a) The $22\,400$ is the starting value V_0 — the car's purchase price of $22,400 — and the 2240 is the fixed amount subtracted each year: the car depreciates by $2240 per year (flat-rate depreciation). (b) $\frac{2240}{22\,400} \times 100 = 10$, so the car loses 10% of its purchase price each year. (c) Half the purchase price is $\frac{22\,400}{2} = \$11\,200$, so the depreciation required is $22\,400 - 11\,200 = \$11\,200$, which takes $\frac{11\,200}{2240} = 5$ years. (Check: $V_5 = 22\,400 - 5 \times 2240 = 11\,200$.)

Q14. (a) Flat-rate: $V_3 = 42\,000 - 3 \times 4800 = 42\,000 - 14\,400 = \$27\,600$. Unit-cost: at $18\,000$ km per year the depreciation is $0.34 \times 18\,000 = \$6120$ per year, so after 3 years (that is, $54\,000$ km) the book value is $42\,000 - 3 \times 6120 = \$23\,640$. (b) At $11\,000$ km per year the unit-cost depreciation is $0.34 \times 11\,000 = \$3740$ per year, giving a book value after 3 years of $42\,000 - 3 \times 3740 = \$30\,780$. The flat-rate value of $27,600 is now the lower one. (c) The unit-cost method better reflects the van's value, because it ties the loss of value directly to the distance driven — the main cause of the van's wear. Under flat-rate depreciation the book value falls by $4800 in a year even if the van barely leaves the depot, whereas the unit-cost value falls quickly in a busy year and slowly in a quiet one.

Q15. Substituting into $V_n = P\left(1+\frac{rn}{100}\right)$ with $r=9$ and $n=5$: $14\,500 = P\left(1+\frac{9\times 5}{100}\right) = 1.45\,P$, so $P = \frac{14\,500}{1.45} = \$\,10\,000$ was borrowed.

Q16. From the graph, the value at $n=0$ is $V_0 = \$\,20\,000$ (the purchase price), and the points fall by the same amount each year: $20\,000 - 17\,500 = \$\,2500$, so $D = \$\,2500$ per year. The recurrence relation is $V_0 = 20\,000$, $V_{n+1} = V_n - 2500$. After 6 years, $V_6 = 20\,000 - 6\times 2500 = 20\,000 - 15\,000 = \$\,5000$. The total depreciation allowed before write-off is $20\,000 - 2500 = \$\,17\,500$, so the useful life is $\frac{17\,500}{2500} = 7$ years.

Q17. (a) One year of riding is 8000 km, so the depreciation is $0.15 \times 8000 = \$\,1200$. (b) After 3 years the motorbike has travelled $24\,000$ km, so the depreciation is $3 \times 1200 = \$\,3600$ and the book value is $9800 - 3600 = \$\,6200$. (c) The total depreciation allowed is $9800 - 800 = \$\,9000$, which at 15 cents per kilometre represents $\frac{9000}{0.15} = 60\,000$ km. At 8000 km per year this takes $\frac{60\,000}{8000} = 7.5$ years.

Q18. (a) The rule $V_n = 6500 + 260\,n$ has starting value $V_0 = 6500$, the principal, and adds $260 each year, the annual interest. The rate is $r = \frac{I}{P} \times 100 = \frac{260}{6500} \times 100 = 4$, that is, $4\,\%$ per annum. (b) Each value is obtained from the one before by adding the same amount, 260 (the recurrence is $V_{n+1} = V_n + 260$), so consecutive values have a constant difference — the defining property of an arithmetic sequence. (c) We need $6500 + 260\,n > 9000$, that is, $260\,n > 2500$, so $n > 9.6\ldots$ At the end of the 9th year $V_9 = 6500 + 2340 = 8840$, which is not more than $9000; at the end of the 10th year $V_{10} = 6500 + 2600 = \$\,9100$, which is. So the investment first exceeds $9000 at the end of the 10th year.

Theory

Money that is invested earns **interest**, and money that is borrowed is charged interest; either way, the way the interest is calculated matters. With **simple interest** the interest for every period is calculated on the original amount invested or borrowed — the **principal** — so the same dollar amount is added each period and the value grows in equal steps. With **compound interest** the interest earned in each period is added to the balance, and the next period's interest is calculated on this new, larger balance. Interest is earned on interest, so the balance grows by more each period. Interest and depreciation rates are quoted **per annum** (p.a. — per year) unless stated otherwise.

The compound interest recurrence. Let V_n be the value of an investment after n compounding periods, starting from the principal $V_0 = P$. If interest is paid at the rate $r\,\%$ per compounding period, then each period the balance keeps its full $100\,\%$ and gains a further $r\,\%$, so it is multiplied by the **growth multiplier** $R = 1 + \frac{r}{100}$:

$$V_0 = P\,,\, V_{n+1} = R\,V_n\,,\, R = 1 + \frac{r}{100}\,.$$

At $4\,\%$ p.a. compounding annually, for example, $R = 1.04$. This is exactly the multiplying recurrence $t_{n+1} = R\,t_n$ met in Chapter 2: the balances form a **geometric sequence** with common ratio R, and compound interest is geometric growth.

The value after n periods. Each step multiplies the balance by R, so after n steps the principal has been multiplied by R^n:

$$V_n = P\,R^n = P\left(1 + \frac{r}{100}\right)^n.$$

This rule jumps straight to any term without iterating step by step. The total **interest earned** over the n periods is $I = V_n - P$. Give money answers correct to the nearest cent, and round only at the end of a calculation, never part-way through.

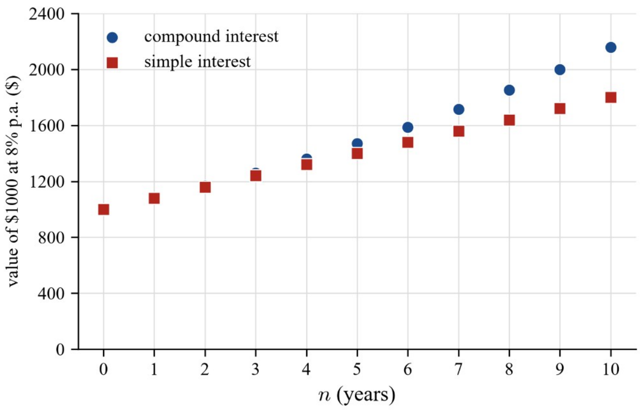
Compound interest on a loan or debt. Compound interest works the same way when money is owed. If the interest charged on a debt is added to the amount owed each period and no repayments are made, then the amount owed follows the same recurrence $V_{n+1} = R\,V_n$ and the same rule $V_n = P\,R^n$, with $R = 1 + \frac{r}{100}$ and P the amount **borrowed**. Each period's interest is charged on an amount that already includes the earlier interest, so an unpaid debt grows by more each period, just as an investment does — which is why a debt left to run can end up far larger than the amount borrowed. The total **interest charged** is again $I = V_n - P$. Every debt in this subtopic runs without repayments, so its balance simply compounds.

Compounding periods other than a year. Interest may be compounded yearly, quarterly or monthly. When the compounding period is not a year, convert the quoted annual rate to the rate **per compounding period**, and count n in periods, not years:

$$\text{rate per period} = \frac{\text{annual rate}}{\text{number of periods per year}}.$$

For example, $6\,\%$ p.a. compounding monthly is $6 \div 12 = 0.5\,\%$ per month, so $R = 1.005$, and two years is $n = 24$ monthly periods. At the same annual rate, more frequent compounding earns slightly more interest, because interest starts earning interest sooner.

Simple versus compound interest. At the same rate, a simple-interest investment grows by the same dollar amount every period, so its graph is a set of points lying on a straight line — linear growth. A compound-interest investment grows by an increasing dollar amount every period, so its points curve upward and pull further ahead each year — geometric growth. For example, \$1000 invested at 8% p.a. grows to $1000+10\times80=\$1800$ after 10 years with simple interest, but to $1000(1.08)^{10}\approx\$2158.92$ with annual compounding, as the graph below shows.



Reducing-balance depreciation. Many assets — vehicles, machinery, computers — lose value as they age; this loss of value is called **depreciation**. In **reducing-balance depreciation** the value falls by a fixed percentage $r\%$ of its *current* value each period, so each period the asset keeps $(100-r)\%$ of its value:

$$V_0=P, V_{n+1}=RV_n, R=1-\frac{r}{100}, V_n=PR^n,$$

where P is the purchase price. Because R is less than 1, the values form a *decreasing* geometric sequence — geometric decay. The largest dollar fall happens in the first period, and each later fall is smaller, because $r\%$ is taken of an ever-smaller value. The value never quite reaches zero; in practice an owner retires or sells the asset when its value first falls below some planned amount, such as its **scrap value** — the amount the asset is expected to be worth when it is disposed of.

Using CAS. Compound values come from the power key: enter $P\times R^n$ directly — for example 6250×1.04^3 — rather than multiplying period by period. To watch a balance grow or an asset's value fall, use the answer key: type the starting value, press ENTER, then type $\text{Ans}\times R$ and press ENTER repeatedly, counting presses to keep track of n; the sequence application of a CAS will tabulate or plot the same recurrence. Learn the by-hand methods below so that you understand what the calculator is doing and can check that its output is reasonable.

Worked examples

Example 1 — scaffolded

Mia invests \$6250 in a term deposit with a regional credit union paying 4% p.a. interest, compounding annually. Write a recurrence relation for the value V_n of the deposit after n years, find the value at the end of each of the first three years, and find the total interest earned over the three years.

Working	Comment
$R=1+\frac{4}{100}=1.04$.	Each year the balance keeps 100% and gains 4%, so it is multiplied by the growth multiplier R.
$V_0=6250$, $V_{n+1}=1.04V_n$.	The starting value is the principal; the rule multiplies the current balance by 1.04.
$V_1=1.04\times6250=6500$.	End of year 1: \$250 interest has been added.
$V_2=1.04\times6500=6760$ and $V_3=1.04\times6760=7030.40$.	Iterate. The second year's interest (\$260) exceeds the first year's, because it is earned on \$6500, not \$6250.
Check: $V_3=6250(1.04)^3=6250\times1.124864=7030.40$.	The rule $V_n=PR^n$ gives the same value directly.

Working	Comment
Interest $= 7030.40 - 6250 = 780.40$.	Total interest = final value − principal.

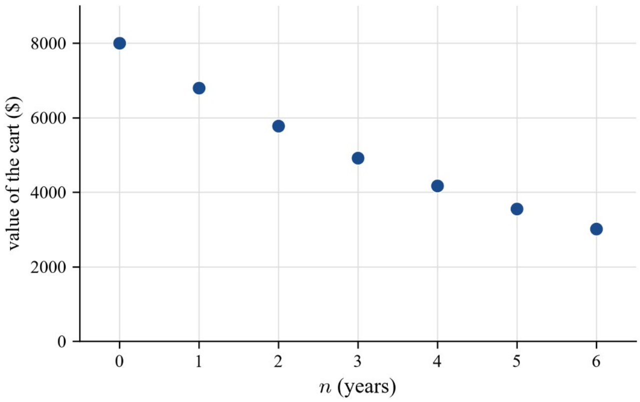
So the deposit is worth \$7030.40 after three years, and the total interest earned is \$780.40.

Example 2 — scaffolded

A food-van operator buys a mobile coffee cart for \$8000. The cart depreciates at 15% p.a. on a reducing-balance basis. Write a recurrence relation for the value V_n of the cart after n years, find its value at the end of each of the first three years, and find the total depreciation over the three years.

Working	Comment
$R = 1 - \frac{15}{100} = 0.85$.	The cart loses 15% of its current value each year, so it keeps 85%: multiply by 0.85.
$V_0 = 8000$, $V_{n+1} = 0.85\,V_n$.	The starting value is the purchase price; the rule multiplies the current value by 0.85.
$V_1 = 0.85 \times 8000 = 6800$.	End of year 1: the value has fallen by \$1200.
$V_2 = 0.85 \times 6800 = 5780$ and $V_3 = 0.85 \times 5780 = 4913$.	The falls are \$1020 then \$867 — each smaller than the last, being 15% of a smaller value.
Check: $V_3 = 8000\left(0.85\right)^3 = 8000 \times 0.614125 = 4913$.	The rule $V_n = P\,R^n$ gives the same value directly.
Depreciation $= 8000 - 4913 = 3087$.	Total depreciation = purchase price − current value.

The graph below shows the value over the first six years: a decreasing geometric sequence, falling by less each year.



So the cart is worth \$4913 at the end of the third year, and the total depreciation over the three years is \$3087.

Example 3 — unscaffolded

Jack borrows \$1600 from a finance company at 18% p.a. interest, compounding monthly. He makes no repayments, and the loan and all of the interest are cleared in a single payment 6 months later. Find the amount owing after 6 months and the interest charged, correct to the nearest cent.

A debt on which no repayments are made grows by the same rule as an investment: each month's interest is added to the amount owed, and the next month's interest is charged on that larger amount. The rate is quoted per annum but interest is compounded monthly, so the rate per compounding period is $18 \div 12 = 1.5\%$ per month, giving $R = 1.015$, and 6 months is $n = 6$ periods. By the rule $V_n = P\,R^n$, $V_6 = 1600\left(1.015\right)^6 = 1600 \times 1.093443\ldots = 1749.509\ldots$, which is \$1749.51 to the nearest cent. The interest charged is $1749.51 - 1600 = \$\,149.51$.

∴ after 6 months Jack owes \$1749.51, of which \$149.51 is interest.

Example 4 — unscaffolded

Priya inherits \$6000 and can invest it for four years at 5% p.a. **simple** interest, or at 5% p.a. **compound** interest compounding annually. Compare the two options and find how much more the better option returns.

Simple interest pays 5% of the principal, $0.05 \times 6000 = \$300$, every year — the same amount each year, since it is always calculated on the original \$6000. Over four years the interest is $4 \times 300 = \$1200$, so the investment grows to $6000 + 1200 = \$7200$. Compound interest instead multiplies the balance by $R = 1.05$ each year, giving balances $6300, 6615, 6945.75$ and finally $V_4 = 6000(1.05)^4 = 6000 \times 1.21550625 = 7293.0375$, which is \$7293.04 to the nearest cent. The yearly additions under compounding are $300, 315, 330.75$ and 347.29 — growing every year, because each year's interest is earned on a larger balance — whereas simple interest adds a constant \$300. The compound option finishes $7293.04 - 7200 = \$93.04$ ahead.

∴ the compound option is better: it returns \$7293.04 against \$7200, or \$93.04 more, because the interest itself earns interest.

Practice questions

Scaffolded

Q1. Ari deposits \$4000 with a community bank paying 5% p.a. interest, compounding annually. (a) Write down the growth multiplier R. (b) Write a recurrence relation for the balance V_n after n years. (c) Find the balance at the end of each of the first three years. (d) Find the total interest earned over the three years.

Q2. A courier buys a food-delivery scooter for \$5200. It depreciates at 10% p.a. on a reducing-balance basis. (a) Write down the multiplier R. (b) Write a recurrence relation for the value V_n of the scooter after n years. (c) Find the value at the end of each of the first three years. (d) Find the total depreciation over the three years.

Q3. Leila borrows \$7000 to set up a market stall. The loan is charged 9% p.a. interest, compounding annually, and she makes no repayments — the whole debt is cleared in a single payment at the end of the term. (a) Write down the multiplier R and a rule for the amount owing V_n after n years. (b) Use the rule to find the amount owing after 4 years, correct to the nearest cent. (c) Find the interest charged over the 4 years.

Q4. An online saver account pays 8% p.a. interest, compounding quarterly. Tom deposits \$4500 for one year. (a) Find the interest rate per quarter. (b) State the number of compounding periods in one year. (c) Find the balance at the end of the year, correct to the nearest cent. (d) A rival account pays 8% p.a. **simple** interest. Which account pays more over one year, and by how much?

Q5. Two apprentices each invest \$2000 for three years. Anh's account pays 10% p.a. **simple** interest; Bree's account pays 10% p.a. **compound** interest, compounding annually. The table shows the value of each investment at the end of each year, with two entries missing.

End of year	1	2	3
Anh (simple)	2200	?	2600
Bree (compound)	2200	2420	?

(a) Copy and complete the table.
(b) Find the total interest each apprentice earns over the three years.
(c) Explain why the two accounts pay the same interest in the first year, but Bree's account pays more in every later year.

Q6. A printing press is bought for \$9500 and depreciates at 20% p.a. on a reducing-balance basis. (a) Write a recurrence relation for the value V_n of the press after n years. (b) Find the value of the press at the end of each of the first four years. (c) Divide each year's value by the previous year's value to show that the values form a geometric sequence, and state the common ratio. (d) The owner will scrap the press when its value first falls below \$5000. At the end of which year does this happen?

Unscaffolded

Q7. Noah borrows \$2500 for a set of tools at 12% p.a. interest, compounding annually, and clears the debt in a single payment two years later. Find the amount owing after two years and the interest charged.

Q8. A landscaping business buys a ride-on mower for \$4800. It depreciates at 25% p.a. on a reducing-balance basis. Find the value of the mower at the end of the third year, and the total depreciation over the three years.

Q9. Sana invests \$3500 at 6% p.a. interest, compounding annually. Write a recurrence relation for the balance V_n after n years, find the balance at the end of each of the first three years (correct to the nearest cent where necessary), and find the total interest earned over the three years.

Q10. Han deposits \$3000 into an account paying 12% p.a. interest, compounding monthly. Find the interest rate per month, the balance after 3 months (correct to the nearest cent), and the interest earned in those 3 months.

Q11. Petra has \$6500 to invest for one year. Account X pays 8% p.a. interest, compounding annually; Account Y pays 8% p.a. interest, compounding quarterly. Find the value of each account at the end of the year, and explain why the account with more frequent compounding finishes ahead even though the annual rates are equal.

Q12. Dev wants to deposit \$5500. Bank A offers 6.5% p.a. **simple** interest; Bank B offers 6% p.a. **compound** interest, compounding annually. Determine which bank leaves Dev better off after 3 years, and show that the other bank is better after 5 years. Explain why the comparison changes.

Q13. The value V_n of a laptop, in dollars, n years after purchase is modelled by

$$V_0 = 3200, V_{n+1} = 0.88\,V_n.$$

State the annual depreciation rate as a percentage, explain what the number 0.88 represents in this context, and find the laptop's value after 3 years, correct to the nearest cent.

Q14. A credit union account pays 9% p.a. interest, compounding monthly. Ruby deposits \$2400. Find the interest rate per month, and the balance and the interest earned after 4 months, correct to the nearest cent.

Q15. A box trailer is bought for \$7500 and depreciates at 20% p.a. on a reducing-balance basis. The owner will replace it when its value first falls below half the purchase price. Find the trailer's value at the end of each year until it is replaced, and state after how many years the replacement happens.

Q16. One year after opening a savings account with a single deposit of \$1500, Zoe's balance is \$1590. Interest compounds annually and she made no other transactions. Find the annual interest rate, write a recurrence relation for the balance V_n after n years, and find the balance after 3 years, correct to the nearest cent.

Q17. Three recurrence relations model the value V_n, in dollars, of three different \$3800 assets after n years: A: $V_0 = 3800$, $V_{n+1} = V_n + 228$; B: $V_0 = 3800$, $V_{n+1} = 1.06\,V_n$; C: $V_0 = 3800$, $V_{n+1} = 0.94\,V_n$. Identify which recurrence describes simple interest, which describes compound interest and which describes reducing-balance depreciation, stating the annual rate in each case. Find each value after 2 years, and explain why B is worth more than A after two years even though both involve 6% p.a.

Q18. Marco has \$6400 to invest for two years and is offered three term deposits. Option 1: 7% p.a. simple interest. Option 2: 6% p.a. compound interest, compounding annually. Option 3: 6% p.a. compound interest, compounding quarterly. Find the value of each option at the end of the two years (correct to the nearest cent where necessary), rank the options, and recommend one to Marco with a reason. Explain why Option 3 beats Option 2.

Answers

Q1. (a) $R = 1.05$. (b) $V_0 = 4000$, $V_{n+1} = 1.05 V_n$. (c) $V_1 = \$4200$, $V_2 = \$4410$, $V_3 = \$4630.50$. (d) \$630.50. **Q2.** (a) $R = 0.9$. (b) $V_0 = 5200$, $V_{n+1} = 0.9 V_n$. (c) $V_1 = \$4680$, $V_2 = \$4212$, $V_3 = \$3790.80$. (d) \$1409.20. **Q3.** (a) $R = 1.09$, $V_n = 7000(1.09)^n$. (b) $V_4 = \$9881.07$. (c) \$2881.07. **Q4.** (a) 2% per quarter. (b) 4. (c) \$4870.94. (d) The compound (quarterly) account, by \$10.94 (\$4870.94 against \$4860). **Q5.** (a) Anh, end of year 2: \$2400; Bree, end of year 3: \$2662. (b) Anh \$600, Bree \$662. (c) In year 1 both earn 10% of \$2000; after that Bree also earns interest on her interest, while Anh's interest is always 10% of the original \$2000. **Q6.** (a) $V_0 = 9500$, $V_{n+1} = 0.8 V_n$. (b) \$7600, \$6080, \$4864, \$3891.20. (c) Every ratio equals 0.8, the common ratio. (d) End of year 3. **Q7.** \$3136 owing; interest \$636. **Q8.** \$2025; depreciation \$2775. **Q9.** $V_0 = 3500$, $V_{n+1} = 1.06 V_n$; \$3710, \$3932.60, \$4168.56; interest \$668.56. **Q10.** 1% per month; \$3090.90; interest \$90.90. **Q11.** X: \$7020; Y: \$7035.81. Y finishes \$15.81 ahead: interest credited each quarter itself earns interest in the remaining quarters. **Q12.** After 3 years: A \$6572.50, B \$6550.59 — Bank A by \$21.91. After 5 years: A \$7287.50, B \$7360.24 — Bank B by \$72.74. Simple interest adds a constant \$357.50 each year, while compound interest adds a growing amount, so B eventually overtakes. **Q13.** 12% p.a.; 0.88 is the fraction (88%) of its value the laptop keeps each year; \$2180.71. **Q14.** 0.75% per month; balance \$2472.81; interest \$72.81. **Q15.** \$6000, \$4800, \$3840, \$3072; replaced after 4 years (first value below \$3750). **Q16.** 6% p.a.; $V_0 = 1500$, $V_{n+1} = 1.06 V_n$; \$1786.52. **Q17.** A: simple interest at 6% p.a., \$4256. B: compound interest at 6% p.a., \$4269.68. C: reducing-balance depreciation at 6% p.a., \$3357.68. B beats A by \$13.68, because B's second-year interest is 6% of \$4028, not of \$3800. **Q18.** Option 1 \$7296; Option 2 \$7191.04; Option 3 \$7209.55. Ranking: Option 1, then Option 3, then Option 2 — recommend Option 1 over two years. Option 3 beats Option 2 because quarterly compounding credits interest sooner, and that interest then earns interest.

Fully-worked solutions

Q1. (a) Each year the balance grows by 5%, so $R = 1 + \frac{5}{100} = 1.05$. (b) $V_0 = 4000$, $V_{n+1} = 1.05 V_n$. (c) $V_1 = 1.05 \times 4000 = 4200$; $V_2 = 1.05 \times 4200 = 4410$; $V_3 = 1.05 \times 4410 = 4630.50$. The balances are \$4200, \$4410 and \$4630.50. (d) Interest $= 4630.50 - 4000 = \$630.50$.

Q2. (a) The scooter keeps 90% of its value each year, so $R = 1 - \frac{10}{100} = 0.9$. (b) $V_0 = 5200$, $V_{n+1} = 0.9 V_n$. (c) $V_1 = 0.9 \times 5200 = 4680$; $V_2 = 0.9 \times 4680 = 4212$; $V_3 = 0.9 \times 4212 = 3790.80$. The values are \$4680, \$4212 and \$3790.80. (d) Total depreciation $= 5200 - 3790.80 = \$1409.20$.

Q3. (a) The debt grows by 9% each year, so $R = 1 + \frac{9}{100} = 1.09$ and $V_n = 7000(1.09)^n$. (b) $(1.09)^4 = 1.41158161$, so $V_4 = 7000 \times 1.41158161 = 9881.071\ldots$, which is \$9881.07 to the nearest cent. (c) Interest $= 9881.07 - 7000 = \$2881.07$.

Q4. (a) Rate per quarter $= 8 \div 4 = 2\%$. (b) Interest is credited every quarter, so there are 4 compounding periods in one year. (c) $R = 1.02$ and $n = 4$: $V_4 = 4500(1.02)^4 = 4500 \times 1.08243216 = 4870.944\ldots$, which is \$4870.94 to the nearest cent. (d) Simple interest pays 8% of \$4500, i.e. \$360, giving \$4860. The compound account pays $4870.94 - 4860 = \$10.94$ more, because the interest credited at the end of each quarter itself earns interest in the remaining quarters.

Q5. (a) Anh earns simple interest of 10% of \$2000, i.e. \$200 each year, so her balances are $2200, 2400, 2600$: the missing value is \$2400. Bree's balance is multiplied by 1.1 each year: $2200, 2420$, then $2420 \times 1.1 = 2662$: the missing value is \$2662. (b) Anh: $2600 - 2000 = \$600$. Bree: $2662 - 2000 = \$662$. (c) In the first year both accounts pay 10% of the \$2000 principal, i.e. \$200. From the second year on, Bree's 10% is calculated on a balance that already includes her earlier interest (for example 10% of \$2200, i.e. \$220, in year 2), while Anh's interest is always 10% of the original \$2000. Interest on interest makes Bree's account grow faster.

Q6. (a) $R=1-\frac{20}{100}=0.8$, so $V_0=9500$, $V_{n+1}=0.8V_n$. (b) $V_1=0.8\times9500=7600$; $V_2=0.8\times7600=6080$; $V_3=0.8\times6080=4864$; $V_4=0.8\times4864=3891.20$. The values are $7600, $6080, $4864 and $3891.20. (c) $\frac{7600}{9500}=0.8$, $\frac{6080}{7600}=0.8$, $\frac{4864}{6080}=0.8$ and $\frac{3891.20}{4864}=0.8$. Consecutive values always have the same ratio, so the values form a geometric sequence with common ratio 0.8. (d) $V_2=6080$ is above $5000 but $V_3=4864$ is below it, so the value first falls below $5000 at the end of year 3.

Q7. $R=1+\frac{12}{100}=1.12$, so $V_2=2500(1.12)^2=2500\times1.2544=\$3136$. The interest charged is $3136-2500=\$636$.

Q8. $R=1-\frac{25}{100}=0.75$. The values are $V_1=0.75\times4800=3600$, $V_2=0.75\times3600=2700$ and $V_3=0.75\times2700=2025$; equivalently $V_3=4800(0.75)^3=4800\times0.421875=2025$. The mower is worth $2025, and the total depreciation is $4800-2025=\$2775$.

Q9. $R=1.06$, so $V_0=3500$, $V_{n+1}=1.06V_n$. Iterating: $V_1=1.06\times3500=3710$; $V_2=1.06\times3710=3932.60$; $V_3=1.06\times3932.60=4168.556$, which is $4168.56 to the nearest cent. The interest earned is $4168.56-3500=\$668.56$.

Q10. Rate per month $=12\div12=1\%$, so $R=1.01$. Then $V_1=1.01\times3000=3030$, $V_2=1.01\times3030=3060.30$ and $V_3=1.01\times3060.30=3090.903$, which is $3090.90 to the nearest cent. The interest is $3090.90-3000=\$90.90$.

Q11. Account X compounds once: $V_1=6500\times1.08=\$7020$. Account Y compounds quarterly at $8\div4=2\%$ per quarter for $n=4$ quarters: $V_4=6500(1.02)^4=6500\times1.08243216=7035.809\ldots$, i.e. $7035.81. Account Y finishes $7035.81-7020=\$15.81$ ahead. The annual rates are equal, but Y credits interest at the end of every quarter, and each credited amount then earns interest for the remaining quarters of the year; X's interest is credited only once, at the end of the year, and earns nothing extra.

Q12. Bank A pays simple interest of 6.5% of $5500, i.e. $357.50, each year. After 3 years: $5500+3\times357.50=\$6572.50$. Bank B compounds: $V_3=5500(1.06)^3=5500\times1.191016=6550.588$, i.e. $6550.59. So after 3 years Bank A is ahead by $6572.50-6550.59=\$21.91$. After 5 years, Bank A gives $5500+5\times357.50=\$7287.50$, while Bank B gives $V_5=5500(1.06)^5=5500\times1.3382255776=7360.240\ldots$, i.e. $7360.24 — now Bank B is ahead by $7360.24-7287.50=\$72.74$. The comparison changes because Bank A adds the same $357.50 every year (linear growth) while Bank B's yearly interest keeps growing (geometric growth), so the compound account eventually overtakes despite its lower rate.

Q13. The multiplier is $0.88=1-\frac{12}{100}$, so the laptop depreciates at 12% p.a. on a reducing-balance basis. The number 0.88 is the fraction of its value the laptop keeps each year: every year the laptop is worth 88% of what it was worth a year earlier. After 3 years, $V_3=3200(0.88)^3=3200\times0.681472=2180.7104$, which is $2180.71 to the nearest cent.

Q14. Rate per month $=9\div12=0.75\%$, so $R=1.0075$ and 4 months is $n=4$ periods. $V_4=2400(1.0075)^4=2400\times1.030339\ldots=2472.814\ldots$, which is $2472.81 to the nearest cent. The interest earned is $2472.81-2400=\$72.81$.

Q15. $R=1-\frac{20}{100}=0.8$, and half the purchase price is $7500\div2=\$3750$. Iterating: $V_1=0.8\times7500=6000$; $V_2=0.8\times6000=4800$; $V_3=0.8\times4800=3840$; $V_4=0.8\times3840=3072$. The values $6000, $4800 and $3840 are all above $3750, but $V_4=\$3072$ is below it, so the trailer is replaced after 4 years.

Q16. In one year the balance is multiplied once by R, so $R = \frac{1590}{1500} = 1.06 = 1 + \frac{6}{100}$: the rate is 6% p.a. The recurrence is $V_0 = 1500$, $V_{n+1} = 1.06\,V_n$. After 3 years, $V_3 = 1500(1.06)^3 = 1500 \times 1.191016 = 1786.524$, which is $1786.52 to the nearest cent.

Q17. Recurrence A adds the constant amount $228 each year, and 228 is 6% of 3800, so A describes **simple interest** at 6% p.a. — linear growth. Recurrence B multiplies by $1.06 = 1 + \frac{6}{100}$, so B describes **compound interest** at 6% p.a. — geometric growth. Recurrence C multiplies by $0.94 = 1 - \frac{6}{100}$, so C describes **reducing-balance depreciation** at 6% p.a. — geometric decay. After 2 years: A: $3800 + 2 \times 228 = \$4256$; B: $3800(1.06)^2 = 3800 \times 1.1236 = \$4269.68$; C: $3800(0.94)^2 = 3800 \times 0.8836 = \$3357.68$. B beats A by $4269.68 - 4256 = \$13.68$: in the second year B earns 6% of $4028 (the balance including the first year's $228 interest), i.e. $241.68, whereas A earns the same $228 as before — interest on interest.

Q18. Option 1: simple interest of 7% of $6400, i.e. $448 per year, so after two years the value is $6400 + 2 \times 448 = \$7296$. Option 2: $R = 1.06$, $n = 2$: $V_2 = 6400(1.06)^2 = 6400 \times 1.1236 = \$7191.04$. Option 3: rate per quarter $= 6 \div 4 = 1.5\%$, so $R = 1.015$ and two years is $n = 8$ quarters: $V_8 = 6400(1.015)^8 = 6400 \times 1.126492\ldots = 7209.552\ldots$, i.e. $7209.55. The ranking is Option 1 ($7296), then Option 3 ($7209.55), then Option 2 ($7191.04), so over two years Marco should choose **Option 1**: with only two years of growth, the higher simple rate beats both compound options. Option 3 beats Option 2 because, at the same 6% p.a., quarterly compounding credits interest at the end of every quarter and that interest then earns interest, whereas Option 2 credits interest only once each year.

Theory

Prices in Australia tend to rise over time: a trolley of groceries, a tank of petrol or a movie ticket costs more this year than it did a few years ago. This general rise in prices is called **inflation**. The Australian Bureau of Statistics tracks the prices of a fixed basket of everyday goods and services in the **Consumer Price Index (CPI)**, and the **inflation rate** is the percentage by which prices rise in a year. An inflation rate of 3% per annum means that, on average, something that costs $100 now will cost about $103 in a year's time.

Inflation is compound growth. If prices rise by the same percentage each year, each year's increase is calculated on the already-increased price — exactly the compound growth studied in 3C. With a starting price P and an annual inflation rate i written as a decimal $\left(i=\frac{r}{100}\text{ for a rate of } r\,\%\right)$, the expected price after n years is

$$V_n = P\left(1+i\right)^n.$$

At 3% inflation a price is multiplied by 1.03 every year, so over five years it is multiplied by $1.03^5 \approx 1.159$ — a rise of about 15.9%, more than $5\times 3=15\%$, because the increases compound.

Inflation eats spending power. Inflation also works in reverse on money that is not spent. A $50 note hidden in a drawer is still a $50 note years later, but the goods it can buy have become dearer, so the note buys **less** — its **spending power** has shrunk. After n years of inflation at rate i, a fixed amount M buys only what

$$\frac{M}{\left(1+i\right)^n}$$

would buy today. Equivalently, an item priced at M today will cost $M\left(1+i\right)^n$ then. Both calculations use the same factor $\left(1+i\right)^n$: multiply to push a price into the future, divide to pull a future amount back into today's buying terms. This is why keeping savings as cash under the bed loses value, and why fixed payments (a pension or allowance that never changes) buy less each year.

Paying for a purchase: the options. The advertised price of an item is rarely the whole story — the **true cost** depends on how you pay. The everyday options are:

Option	How you pay	Possible extra costs
Cash or debit card	your own money, all at once	none — some sellers even give a discount for paying up front
Credit card	the bank pays the seller now; you repay the bank later	interest (often around 20% p.a.) if the balance is not cleared within the interest-free period; possibly an annual fee
Personal loan	borrow the price from a lender; repay it in equal regular repayments over a fixed term	interest (built into the repayments) plus establishment or service fees
Buy now pay later (BNPL)	a small number of equal instalments, usually fortnightly, the first at purchase	no interest, but late fees for missed instalments and, on some schemes, account-keeping fees

A debit card spends money already in your bank account, so it behaves like cash. Credit-card interest is calculated daily in practice; in this course we model the interest on a card balance held for part of a year with the **simple interest** rule from 3B, $I=P\times\frac{r}{100}\times t$, with t in years.

Comparing options: compute the total cost. To compare purchase options fairly, compute for each one

$$\text{total cost} = \text{price} + \text{interest} + \text{fees} - \text{any discount},$$

then compare the totals: the cheapest option is the one with the smallest total cost, and a good comparison also states **by how much** it is cheaper. Paying over time is a trade-off — a loan or instalment plan spreads the payments so a large purchase fits a weekly budget, but the convenience is bought with interest and fees, so it almost always costs more in total than paying cash.

Using CAS. Your CAS calculator's finance solver (or a recurrence such as $V_{n+1} = 1.03\,V_n$) will project inflated prices, and its arithmetic handles the interest and fee totals directly. Use it to check your work, but set out the by-hand calculation as in the examples below — writing the factor $(1+i)^n$ and the total-cost sum makes it clear what is being compared, and lets you spot an unreasonable answer at a glance.

Worked examples

Example 1 — scaffolded

A supermarket loaf of bread costs \$4.20 today. Suppose inflation runs at 3% per annum. Find the expected price of the loaf after one year, after two years and after five years, each to the nearest cent.

Working	Comment
Here $P = 4.20$ and $i = 0.03$, so each year the price is multiplied by $1 + i = 1.03$.	Inflation at 3% p.a. is compound growth with multiplier 1.03.
After 1 year: $V_1 = 4.20 \times 1.03 = 4.326$, so the price is \$4.33.	One year is one multiplication by 1.03; round to the nearest cent at the end.
After 2 years: $V_2 = 4.20 \times 1.03^2 = 4.20 \times 1.0609 = 4.45578$, so the price is \$4.46.	Work from the exact values, not the rounded \$4.33, and round only the final answer.
After 5 years: $V_5 = 4.20 \times 1.03^5 = 4.20 \times 1.15927\ldots = 4.86895\ldots$, so the price is \$4.87.	$1.03^5 \approx 1.159$: over five years prices rise by about 15.9%, not 15%, because the increases compound.

Year n	0	1	2	5
Expected price	\$4.20	\$4.33	\$4.46	\$4.87

So the loaf is expected to cost \$4.33 in one year, \$4.46 in two years and \$4.87 in five years.

Example 2 — scaffolded

Mia is given a \$100 note for her birthday and leaves it in a drawer for three years. Over that time inflation averages 4% per annum. (a) What will an item that costs \$100 today cost in three years' time? (b) What is the spending power of Mia's note after the three years, in today's terms? (c) How much spending power has the note lost?

Working	Comment
$(1+i)^n = 1.04^3 = 1.124864$.	Three years of 4% inflation; the same factor is used in every part.
(a) $100 \times 1.124864 = 112.4864$, so the item will cost \$112.49.	Multiply by the factor to push a price into the future.
(b) $100 \div 1.124864 = 88.899\ldots$, so the note buys what \$88.90 buys today.	Divide by the factor to pull a future amount back into today's buying terms.
(c) $100 - 88.90 = 11.10$, a loss of \$11.10 of spending power.	The note still says \$100, but it buys about \$11 less than it did.

So prices rise to \$112.49, the note's spending power falls to \$88.90, and Mia has effectively lost \$11.10 by leaving the money idle.

Example 3 — unscaffolded

A pair of basketball shoes is priced at \$320. Jack pays using a buy-now-pay-later scheme: four equal fortnightly instalments, the first at purchase, with no interest. Each instalment that is paid late attracts a \$10 late fee. Jack pays two of his instalments late. Find his total cost, compare it with paying cash, and express the extra amount as a percentage of the price.

Each instalment is $320 \div 4 = \$80$. Had Jack paid every instalment on time his total would have been $4 \times 80 = \$320$ — exactly the cash price, which is the appeal of BNPL. The two late instalments cost $2 \times 10 = \$20$ in fees, so his total cost is $320 + 20 = \$340$, which is \$20 more than paying cash. As a percentage of the price this is $\frac{20}{320} \times 100 = 6.25\%$.

∴ Jack pays \$340 in total — \$20 (or 6.25% of the price) more than the cash price, all of it avoidable late fees.

Example 4 — unscaffolded

Chen is buying a \$2400 laptop and has three options. **Cash:** the store gives a 5% discount for payment in full. **Credit card:** his card charges simple interest at 20% per annum; he would make no repayments for 9 months, then pay the balance and the interest in full. **Personal loan:** 12 monthly repayments of \$212, plus a \$50 establishment fee. Find the total cost of each option, and state which is cheapest and by how much.

Cash: the discount is 5% of \$2400, so Chen pays $2400 \times 0.95 = \$2280$. Credit card: the interest is $I = 2400 \times \frac{20}{100} \times \frac{9}{12} = \$360$, so the total cost is $2400 + 360 = \$2760$. Personal loan: the repayments total $12 \times 212 = \$2544$, and adding the fee gives $2544 + 50 = \$2594$.

Option	Total cost
Cash (5% discount)	\$2280
Credit card	\$2760
Personal loan	\$2594

Paying cash is cheapest: it beats the personal loan by $2594 - 2280 = \$314$ and the credit card by $2760 - 2280 = \$480$. The loan and the card buy time to pay, but that time costs hundreds of dollars.

∴ cash is the cheapest option at \$2280 — \$314 less than the personal loan and \$480 less than the credit card.

Practice questions

Scaffolded

Q1. A cinema ticket costs \$16.00 today. Assume inflation runs at 4% per annum. (a) Find the expected price of the ticket after one year. (b) Find the expected price after two years, to the nearest cent. (c) Use $V_n = P(1+i)^n$ to find the expected price after six years, to the nearest cent. (d) How much more than today's price is the six-year price?

Q2. Sofia's nonna gives her \$500 in cash, and Sofia leaves it in a money box for four years. Inflation averages 3% per annum over that time. (a) What will an item that costs \$500 today cost in four years' time, to the nearest cent? (b) Find the spending power of Sofia's \$500 after the four years, in today's terms, to the nearest cent. (c) How much spending power has the money lost?

Q3. Liam buys an \$840 phone on his credit card, which charges simple interest at 19% per annum. He makes no repayments for 8 months, then pays off the balance and the interest in full. (a) Find the interest charged, using

$I = P \times \frac{r}{100} \times t$ with t in years. (b) Find the total cost of the phone. (c) How much more did Liam pay than if he had paid cash?

Q4. Ava buys a $360 dress using a buy-now-pay-later scheme: four equal fortnightly instalments, the first at purchase, with no interest. A late instalment attracts a $12 fee. (a) Find the size of each instalment. (b) If Ava pays every instalment on time, what is her total cost? How does it compare with paying cash? (c) In fact Ava pays two instalments late. Find her total cost. (d) Express the extra amount she pays as a percentage of the price, to two decimal places.

Q5. Priya buys a $3000 electric bike with a personal loan: 24 monthly repayments of $139, plus a $60 establishment fee. (a) Find the total of the repayments. (b) Find the total cost of the bike, including the fee. (c) How much more than the $3000 price does Priya pay in interest and fees altogether?

Q6. A refrigerator is priced at $1200. Three ways of paying are available. **Option A (cash):** the store gives a 2.5% discount for payment in full. **Option B (credit card):** simple interest at 21% per annum; the balance and interest are paid after 6 months. **Option C (BNPL):** four fortnightly instalments of $300, all paid on time, but the scheme charges a $6 account-keeping fee with each instalment. (a) Find the total cost under Option A. (b) Find the total cost under Option B. (c) Find the total cost under Option C. (d) Which option is cheapest? State how much it saves compared with each of the other two options.

Unscaffolded

Q7. A regional bus fare is $5.00 today. If fares rise with inflation at 2.5% per annum, find the expected fare after three years, to the nearest cent.

Q8. Tom keeps $1000 in cash at home for five years, during which inflation averages 4% per annum. Find the expected cost in five years of goods that cost $1000 today, and the spending power of Tom's $1000 after the five years in today's terms, both to the nearest cent.

Q9. A music-streaming subscription costs $14.00 per month today. If the price rises with inflation at 6% per annum, find the expected monthly price after two years and after four years, each to the nearest cent.

Q10. A jar of coffee cost $8.50 one year ago and costs $8.84 today. Find the annual inflation rate suggested by this price rise. Assuming the same rate continues, find the expected price of the jar three years from now, to the nearest cent.

Q11. Hannah buys a $1500 washing machine on a credit card charging simple interest at 20% per annum, and clears the balance and the interest after 10 months. Find the interest charged and her total cost, and state how much more she paid than the cash price.

Q12. Noah buys a $480 gaming console bundle with a buy-now-pay-later scheme: four fortnightly instalments, no interest, but a $15 fee for each late instalment. Noah pays two instalments late. Find his total cost, how much more this is than paying cash, and the extra as a percentage of the price.

Q13. For each shopper, state which payment option (cash, debit card, credit card, personal loan or BNPL) is most suitable, and give a brief reason. (a) Ruby has the full amount saved, and the seller offers a 5% discount for paying up front. (b) Marcus is shopping online and has more than enough money in his everyday bank account. (c) Lily wants the item today, has no savings yet, but is confident she can meet four equal fortnightly payments; the scheme charges no interest. (d) Sam needs a $4000 purchase and will need about two years to repay it.

Q14. A $2000 treadmill can be bought two ways. **Personal loan:** 12 monthly repayments of $178, plus a $45 establishment fee. **Credit card:** simple interest at 22% per annum, with the balance and interest paid after 9 months. Find the total cost of each option, state which is cheaper and by how much, and find the total of the interest and fees paid under the cheaper option.

Q15. A family's weekly grocery bill is $250 today. If grocery prices rise with inflation at 3.5% per annum, find the expected weekly bill in ten years' time, to the nearest cent, and how much more per week that is than today.

Q16. A retiree receives a fixed payment of $800 per fortnight, and the payment will not change for five years. Inflation averages 3% per annum over that time. Find the spending power of the $800 payment at the end of the five years, in today's terms, to the nearest cent, and the spending power lost. Explain what your answers mean for the retiree.

Q17. A lounge suite is priced at $3600. Four ways of paying are available. **Cash:** a 4% discount for payment in full. **Credit card:** simple interest at 21% per annum, with the balance and interest paid after one year. **Personal loan:** 24 monthly repayments of $164, plus a $90 establishment fee. **Store instalment plan:** 12 monthly instalments of $300, plus a $10 account fee each month. Find the total cost of each option, rank the options from cheapest to dearest, and state how much the cheapest option saves compared with the dearest.

Q18. Piper wants a $1250 camera and is weighing up two plans. **Plan A:** buy it today on a credit card charging simple interest at 20% per annum, and pay off the balance and the interest after 6 months. **Plan B:** save for a year and then pay cash — but by then the price will have risen with inflation at 4% per annum. Find the total cost of each plan, state which is cheaper and by how much, and comment on the trade-off between the two plans.

Answers

Q1. (a) \$16.64. (b) \$17.31. (c) \$20.25. (d) \$4.25 more. **Q2.** (a) \$562.75. (b) \$444.24. (c) \$55.76 of spending power. **Q3.** (a) $I = \$106.40$. (b) \$946.40. (c) \$106.40 more. **Q4.** (a) \$90. (b) \$360 — the same as the cash price. (c) \$384. (d) $6.67\,\%$. **Q5.** (a) \$3336. (b) \$3396. (c) \$396. **Q6.** (a) \$1170. (b) \$1326. (c) \$1224. (d) Option A (cash); it saves \$54 compared with Option C and \$156 compared with Option B. **Q7.** \$5.38. **Q8.** Goods will cost \$1216.65; the \$1000 has the spending power of \$821.93 in today's terms. **Q9.** \$15.73 after two years; \$17.67 after four years. **Q10.** $4\,\%$ per annum; expected price \$9.94. **Q11.** Interest \$250; total cost \$1750; \$250 more than cash. **Q12.** Total \$510; \$30 more than cash; $6.25\,\%$ of the price. **Q13.** (a) Cash — the discount lowers the price. (b) Debit card — his own money, no interest or fees. (c) BNPL — no interest if every instalment is on time (late fees are the risk). (d) Personal loan — a fixed term of equal repayments at a lower rate than a credit card. **Q14.** Loan \$2181; card \$2330; the loan is cheaper by \$149; interest and fees on the loan total \$181. **Q15.** \$352.65 per week — \$102.65 more than today. **Q16.** Spending power \$690.09; a loss of \$109.91 — the fixed \$800 buys about \$110 less worth of goods than it does today. **Q17.** Cash \$3456; card \$4356; loan \$4026; instalment plan \$3720. Cheapest to dearest: cash, plan, loan, card; cash saves \$900 compared with the card. **Q18.** Plan A \$1375; Plan B \$1300; Plan B is cheaper by \$75, but Piper must wait a year for the camera, whereas Plan A delivers it today at a higher total cost.

Fully-worked solutions

Q1. (a) One year at $4\,\%$: $16.00 \times 1.04 = \$16.64$. (b) $V_2 = 16.00 \times 1.04^2 = 16.00 \times 1.0816 = 17.3056$, so \$17.31. (c) $V_6 = 16.00 \times 1.04^6 = 16.00 \times 1.265319\ldots = 20.2451\ldots$, so \$20.25. (d) $20.25 - 16.00 = \$4.25$ more than today's price.

Q2. (a) The factor is $1.03^4 = 1.12550881$, so the item will cost $500 \times 1.12550881 = 562.754\ldots$, i.e. \$562.75. (b) Spending power $= 500 \div 1.12550881 = 444.243\ldots$, i.e. \$444.24 in today's terms. (c) $500 - 444.24 = \$55.76$ of spending power has been lost.

Q3. (a) Eight months is $t = \dfrac{8}{12}$ of a year, so $I = 840 \times \dfrac{19}{100} \times \dfrac{8}{12} = 159.60 \times \dfrac{2}{3} = \$106.40$. (b) Total cost $= 840 + 106.40 = \$946.40$. (c) The interest is the extra: Liam paid \$106.40 more than the \$840 cash price.

Q4. (a) $360 \div 4 = \$90$ per instalment. (b) On time, the total is $4 \times 90 = \$360$ — exactly the cash price, since the scheme charges no interest. (c) Two late fees add $2 \times 12 = \$24$, so the total is $360 + 24 = \$384$. (d) $\dfrac{24}{360} \times 100 = 6.666\ldots \approx 6.67\,\%$ of the price.

Q5. (a) $24 \times 139 = \$3336$. (b) $3336 + 60 = \$3396$. (c) $3396 - 3000 = \$396$ in interest and fees altogether.

Q6. (a) Option A: a $2.5\,\%$ discount leaves $97.5\,\%$ of the price: $1200 \times 0.975 = \$1170$. (b) Option B: $I = 1200 \times \dfrac{21}{100} \times \dfrac{6}{12} = \$126$, so the total is $1200 + 126 = \$1326$. (c) Option C: the instalments total $4 \times 300 = \$1200$ and the fees total $4 \times 6 = \$24$, so the total is $1200 + 24 = \$1224$. (d) Option A is cheapest at \$1170. It saves $1224 - 1170 = \$54$ compared with Option C, and $1326 - 1170 = \$156$ compared with Option B.

Q7. $V_3 = 5.00 \times 1.025^3 = 5.00 \times 1.076890625 = 5.3844\ldots$, so the expected fare is \$5.38.

Q8. The five-year factor is $1.04^5 = 1.2166529024$. Goods costing \$1000 today will cost $1000 \times 1.2166529024 = 1216.652\ldots$, i.e. \$1216.65. The spending power of the cash is $1000 \div 1.2166529024 = 821.927\ldots$, i.e. \$821.93 in today's terms — keeping the money at home has cost Tom nearly \$180 of buying power.

Q9. After two years: $14.00 \times 1.06^2 = 14.00 \times 1.1236 = 15.7304$, so \$15.73. After four years: $14.00 \times 1.06^4 = 14.00 \times 1.26247696 = 17.6746\ldots$, so \$17.67.

Q10. The rise is $8.84-8.50=\$0.34$, so the rate is $\frac{0.34}{8.50}\times 100=4\%$ per annum. Continuing at 4% for three more years: $V_3=8.84\times 1.04^3=8.84\times 1.124864=9.9437\ldots$, so the expected price is $9.94.

Q11. Ten months is $t=\frac{10}{12}$ of a year, so $I=1500\times\frac{20}{100}\times\frac{10}{12}=300\times\frac{5}{6}=\$250$. The total cost is $1500+250=\$1750$, which is $250 more than the cash price.

Q12. Each instalment is $480\div 4=\$120$, and on time the instalments would total the cash price of $480. The late fees add $2\times 15=\$30$, so Noah's total is $480+30=\$510$ — $30 more than paying cash, which is $\frac{30}{480}\times 100=6.25\%$ of the price.

Q13. (a) **Cash** — Ruby has the money, and the 5% discount makes the up-front price the cheapest possible total cost. (b) **Debit card** — it spends Marcus's own money directly, so there is no interest and no debt, and it works for online payment where notes and coins cannot. (c) **BNPL** — with no interest and all instalments paid on time the total equals the cash price, so it suits Lily provided she is sure to pay on time; late fees are the risk. (d) **Personal loan** — two years is far longer than any interest-free period, and carrying $4000 on a credit card at around 20% p.a. would be expensive; a personal loan offers a lower rate and fixed repayments that clear the debt by the end of the term.

Q14. Personal loan: the repayments total $12\times 178=\$2136$, plus the $45 fee gives $2136+45=\$2181$. Credit card: $I=2000\times\frac{22}{100}\times\frac{9}{12}=\$330$, so the total is $2000+330=\$2330$. The loan is cheaper by $2330-2181=\$149$. Under the loan, the interest and fees total $2181-2000=\$181$.

Q15. $V_{10}=250\times 1.035^{10}=250\times 1.410599\ldots=352.649\ldots$, so the expected weekly bill is $352.65 — an increase of $352.65-250.00=\$102.65$ per week.

Q16. The five-year factor is $1.03^5=1.1592740743$, so the spending power of the payment is $800\div 1.1592740743=690.087\ldots$, i.e. $690.09 in today's terms — a loss of $800-690.09=\$109.91$. Although the retiree still receives $800 each fortnight, by the end of the five years it buys only what about $690 buys today: roughly $110 worth of each fortnight's goods and services has been eaten by inflation, even though the dollar amount never changed.

Q17. Cash: $3600\times 0.96=\$3456$. Credit card: $I=3600\times\frac{21}{100}\times 1=\$756$, so the total is $3600+756=\$4356$. Personal loan: $24\times 164=\$3936$ in repayments, plus the $90 fee gives $4026. Store instalment plan: $12\times 300=\$3600$ in instalments plus $12\times 10=\$120$ in fees gives $3720.

Option	Total cost
Cash (4% discount)	$3456
Store instalment plan	$3720
Personal loan	$4026
Credit card	$4356

From cheapest to dearest: cash, instalment plan, personal loan, credit card. The cheapest option (cash) saves $4356-3456=\$900$ compared with the dearest (the credit card).

Q18. Plan A: $I=1250\times\frac{20}{100}\times\frac{6}{12}=\$125$, so the total cost is $1250+125=\$1375$. Plan B: after one year of 4% inflation the price is $1250\times 1.04=\$1300$, paid in cash with no interest or fees. Plan B is cheaper by $1375-1300=\$75$. The trade-off: Plan A delivers the camera today but the convenience costs $125 in interest, while Plan B costs less in total — inflation adds only $50 to the price — but Piper must wait a full year before she has the camera.