# General Mathematics Units 1 & 2

## Chapter 2 — Sequences and recurrence relations



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### Contents

# Chapter 2 Sequences and recurrence relations — 2A Sequences and first-order linear recurrence relations

## Theory

A **sequence** is an ordered list of numbers, written one after another. Each number in the list is a **term** of the sequence, and the terms are labelled in order with subscripts: $t_0, t_1, t_2, t_3, \ldots$ The subscript is the **term number** — it records the position of the term — so a sequence is really a function whose **domain is the non-negative integers**: to each whole number $n = 0, 1, 2, \ldots$ the sequence assigns the term $t_n$. We call $t_0$ the **starting value**. (Some sequences are more naturally numbered from $t_1$ — for example, sales in week 1 of a promotion — and both conventions are used, so always check where the numbering starts.)

**Recursion: defining a sequence by a rule.** In the sequences studied in this course, each term is built from the term before it. Such a sequence can be defined **recursively** by giving two things: a **starting value**, and a **rule** that produces the next term from the current term. Together the starting value and the rule form a **recurrence relation**. For example,

$$t_0 = 5, t_{n+1} = t_n + 3$$

says "start at 5, and add 3 at each step", and generates the terms $5, 8, 11, 14, \ldots$ Both parts are essential: the rule on its own does not pin the sequence down, because starting the same rule at $t_0 = 6$ generates the completely different sequence $6, 9, 12, 15, \ldots$

**First-order linear recurrence relations.** The recurrence relations used in this course are **first-order linear**: *first-order* because the next term is calculated from the current term only (one step back), and *linear* because the current term is changed by one simple operation — either the same constant is **added** to it, or it is **multiplied** by the same constant. So there are two forms to know:

$$t_0 = a, t_{n+1} = t_n + d \text{ (add the constant } d \text{ at every step)},$$

$$t_0 = a, t_{n+1} = R\,t_n \text{ (multiply by the constant } R \text{ at every step)}.$$

In both, $a$ is the starting value. Adding the same amount at every step produces an **arithmetic sequence** (studied in 2B); multiplying by the same factor at every step produces a **geometric sequence** (studied in 2C). The constant $d$ may be negative, in which case the rule subtracts the same amount each step, and the multiplier $R$ may be a fraction or negative. To **generate** the terms of the sequence, apply the rule repeatedly: find $t_1$ from $t_0$, then $t_2$ from $t_1$, and so on. Each application of the rule is one **iteration**, and each new term is computed from the term just found.

**Setting up a recurrence from a description.** To model a situation with a recurrence relation, take the initial amount as the starting value $t_0$, then translate one step of the repeating change into the rule. The question to ask is whether each step changes the quantity by the same **amount** or by the same **factor**. A fixed increase or decrease gives the adding form: a tank that loses 90 litres each week has $t_{n+1} = t_n - 90$. A percentage change gives the multiplying form, because a percentage of the current amount is a fixed fraction of it: a club that keeps $90\%$ of its members each year has $t_{n+1} = 0.9\,t_n$, and a colony growing by $8\%$ each year keeps $108\%$ of its numbers, so $t_{n+1} = 1.08\,t_n$. Always state what $t_n$ stands for and what one step of $n$ represents (a week, a month, a year).

**Tables and graphs.** A sequence can be displayed in a **table**, with one row for the term number $n$ and one row for the term $t_n$, or in a **graph**, by plotting the points $(n, t_n)$ with $n$ on the horizontal axis and $t_n$ on the vertical axis. Because the domain is the non-negative integers, the graph of a sequence is a set of **separate points** — never join them with a line or curve, as there is no term between, say, $n = 2$ and $n = 3$. A graph shows the behaviour of a sequence at a glance.

**Describing the behaviour of a sequence.** A sequence is **increasing** if each term is larger than the one before, **decreasing** if each term is smaller, and **constant** if every term is the same (as happens when nothing is added, or when the multiplier is 1). A sequence **oscillates** if its terms alternately rise and fall, as happens when the multiplier is negative and the terms swing from positive to negative and back. Some sequences **approach a limiting value**: the terms change by less and less, settling closer and closer to a fixed number. You find that value by **iterating far**

**enough and watching where the terms settle** — in a table or on a graph the terms crowd towards one level while the gaps between them shrink almost to nothing. For $t_{n+1}=0.5t_n$, iterating from $t_0=40$ gives $40, 20, 10, 5, 2.5, 1.25, \ldots$ : each term is half the one before, so the terms shrink towards a limiting value of 0 without ever reaching it. The rule $t_{n+1}=-0.5t_n$ from the same start gives $40, -20, 10, -5, 2.5, \ldots$, which oscillates between positive and negative values while also settling towards 0. Always read the behaviour off the terms you have generated, or off their table or graph.

**Using CAS.** A CAS calculator generates the terms of a recurrence quickly using its answer key. Type the starting value and press ENTER; then type the rule with Ans in place of $t_n$ — for $t_{n+1}=t_n+5$ enter $\text{Ans}+5$, and for $t_{n+1}=0.5t_n$ enter $0.5\times\text{Ans}$ — and every further press of ENTER produces the next term. Count the presses carefully so that you know which term number you have reached. The sequence or spreadsheet application of a CAS will also tabulate and plot a recurrence directly. Learn the by-hand iteration below as well, so that you understand what the calculator is doing and can check that its output is reasonable.

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## Worked examples

### *Example 1 — scaffolded*

A sequence is generated by the recurrence relation

$$t_0=6, t_{n+1}=4t_n.$$

Write down the first five terms of the sequence and describe its behaviour.

| Working | Comment |
|---|---|
| Starting value: $t_0=6$. | The recurrence states the first term directly. |
| $t_1=4t_0=4(6)=24$. | Apply the rule $t_{n+1}=4t_n$ with $n=0$: multiply the current term by 4. |
| $t_2=4t_1=4(24)=96$. | Each new term is computed from the term just found. |
| $t_3=4(96)=384$ and $t_4=4(384)=1536$. | Keep iterating the same rule. |
| $6, 24, 96, 384, 1536$ — each term is larger than the one before. | Compare consecutive terms to describe the behaviour. |

So the first five terms are $6, 24, 96, 384, 1536$, and the sequence is increasing.

### *Example 2 — scaffolded*

A new Australian history podcast launches with 6 episodes, and the host uploads 5 more episodes each month. The number of episodes available after $n$ months, $t_n$, satisfies

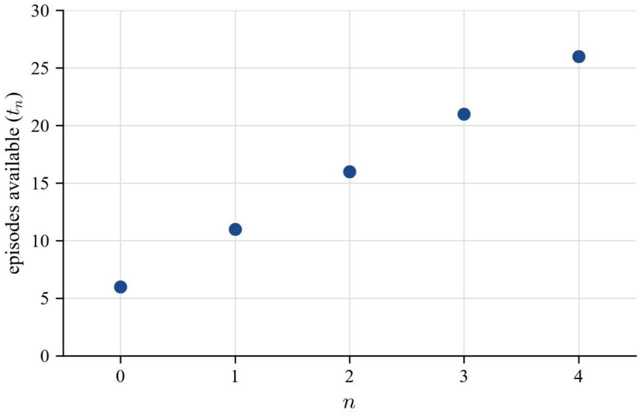
$$t_0=6, t_{n+1}=t_n+5.$$

Display the terms $t_0$ to $t_4$ in a table, plot the sequence, and describe its behaviour.

| Working | Comment |
|---|---|
| $t_1=t_0+5=6+5=11$. | Apply the rule with $n=0$. Here $d=5$, so each iteration adds 5. |
| $t_2=11+5=16$, $t_3=16+5=21$, $t_4=21+5=26$. | Iterate — each new term comes from the term just found. |
| Record the terms in a table of $n$ against $t_n$ (below). | The table pairs each term number with its term. |
| Plot the points $(0,6)$, $(1,11)$, $(2,16)$, $(3,21)$, $(4,26)$ (below). | The graph is a set of separate points, never joined: $n$ takes whole-number values only. |
| Each term is 5 more than the term before. | Comparing consecutive terms gives the behaviour: |

| Working | Comment |
|---|---|
| | increasing. |

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_n$ | 6 | 11 | 16 | 21 | 26 |




So the sequence $6, 11, 16, 21, 26, \ldots$ is increasing: the podcast library grows by 5 episodes every month.

### *Example 3 — unscaffolded*

An online gaming community has 1250 members. Each month, $20\%$ of the current members leave and are not replaced. Write a recurrence relation for the number of members $t_n$ after $n$ months, generate the terms $t_1$ to $t_4$, and describe the behaviour of the sequence.

The starting value is the initial membership, $t_0 = 1250$. Losing $20\%$ of the members leaves $80\%$ of them, and a percentage change is a fixed factor, so the rule multiplies by 0.8: $t_{n+1} = 0.8 t_n$. Iterating: $t_1 = 0.8(1250) = 1000$; $t_2 = 0.8(1000) = 800$; $t_3 = 0.8(800) = 640$; $t_4 = 0.8(640) = 512$. The sequence $1250, 1000, 800, 640, 512, \ldots$ is decreasing, and each month's fall ($250, 200, 160, 128, \ldots$) is smaller than the one before, because $20\%$ is taken of a smaller membership every month.

$\therefore$ $t_0 = 1250$, $t_{n+1} = 0.8 t_n$; the terms are $1000, 800, 640, 512$, and the sequence is decreasing, with each fall smaller than the one before.

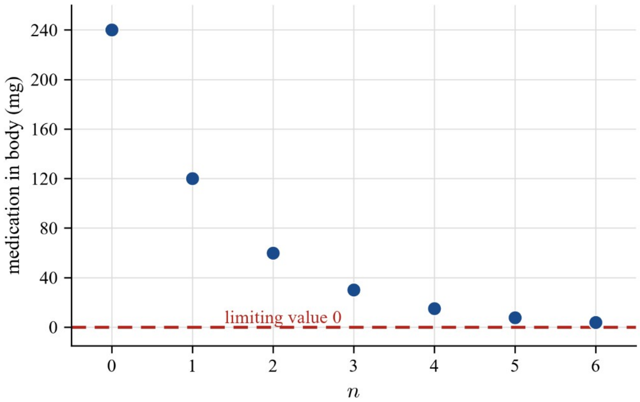
### *Example 4 — unscaffolded*

A patient is given a single 240 mg dose of a medication, and over each 24 hours the body eliminates half of the medication present. Let $t_n$ be the amount of medication (in mg) in the body $n$ days after the dose, so that

$$t_0 = 240, t_{n+1} = 0.5 t_n.$$

Generate the terms up to $t_6$, plot the sequence, and describe its behaviour, including any limiting value.

Iterating the rule: $t_1 = 0.5(240) = 120$; $t_2 = 0.5(120) = 60$; $t_3 = 30$; $t_4 = 15$; $t_5 = 7.5$; $t_6 = 3.75$. The terms $240, 120, 60, 30, 15, 7.5, 3.75, \ldots$ are decreasing, and each fall is half the one before — the drops go $120, 60, 30, 15, 7.5, 3.75, \ldots$, halving every day. So the terms shrink towards 0: continuing the iteration they settle closer and closer to 0 without ever reaching it, and the graph below shows the points flattening out towards this limiting value of 0.




∴ the sequence is decreasing and approaches a limiting value of 0: the model never clears the medication completely, but the amount left becomes negligible after about a week.

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## Practice questions

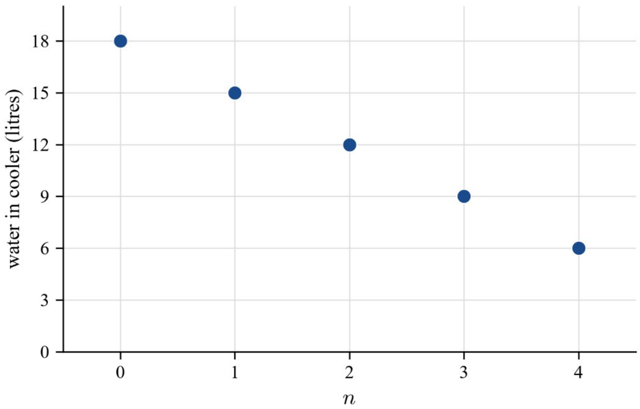
### *Scaffolded*

**Q1.** A sequence is defined by the recurrence relation $t_0 = 4$, $t_{n+1} = t_n + 6$. (a) Generate the terms $t_1$ to $t_4$. (b) Display the terms $t_0$ to $t_4$ in a table of $n$ against $t_n$. (c) Describe the behaviour of the sequence. (d) If the sequence is continued, which term is the first to exceed 50?

**Q2.** A sequence is defined by $t_0 = 48$, $t_{n+1} = 0.5t_n$. (a) Generate the terms $t_1$ to $t_4$. (b) Find $t_5$ and $t_6$. (c) Describe the behaviour of the sequence, including its limiting value.

**Q3.** A sequence is defined by $t_1 = 7$, $t_{n+1} = 3t_n$. (This sequence is numbered from $t_1$.) (a) Generate $t_2$, $t_3$ and $t_4$. (b) Show that 567 is a term of the sequence, and state which term it is. (c) Priya says that the rule $t_{n+1} = 3t_n$ on its own defines the sequence. Explain what is missing from her description.

**Q4.** The graph below shows the volume of water $t_n$, in litres, left in an office water cooler at the start of each hour, for $n = 0$ to $4$.




(a) Write down the starting value $t_0$.
(b) List the first five terms of the sequence.
(c) Write a recurrence relation of the form $t_0 = \ldots$, $t_{n+1} = t_n + d$ for the sequence.
(d) If the pattern continues, find $t_5$.

**Q5.** Write a recurrence relation (a starting value and a rule) for the quantity $t_n$ in each situation. (a) A plantation has 1500 trees, and 90 trees are harvested each year. ($t_n$ is the number of trees after $n$ years.) (b) A bacteria culture contains 300 cells, and the number of cells doubles every hour. ($t_n$ is the number of cells after $n$ hours.) (c) A quoll population in a fenced reserve is estimated at 180 animals, and it falls by $15\%$ each year. ($t_n$ is the number of quolls after $n$ years.)

**Q6.** Consider the two sequences below. (a) For $t_0 = 3$, $t_{n+1} = -2t_n$, generate the terms $t_1$ to $t_4$. (b) Describe the behaviour of the sequence in part (a). (c) For $u_0 = 64$, $u_{n+1} = 0.25u_n$, generate the terms $u_1$ to $u_4$. (d) The sequence in part (c) approaches a limiting value. By generating several more terms, find this limiting value.

### *Unscaffolded*

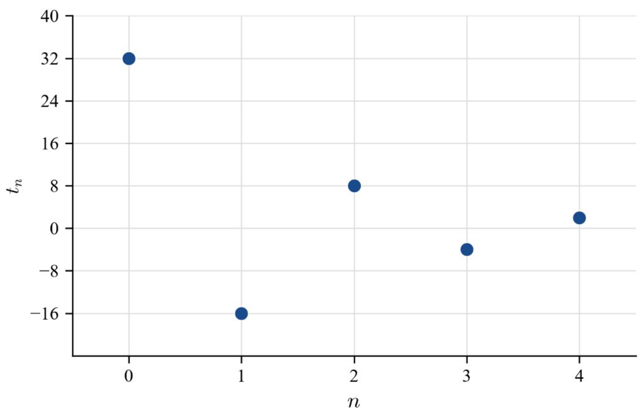
**Q7.** A sequence is defined by $t_0 = 3$, $t_{n+1} = 5t_n$. Write down the first five terms of the sequence and describe its behaviour.

**Q8.** A community garden opens with 26 plots, and the council adds 4 new plots each spring. Write a recurrence relation for the number of plots $t_n$ after $n$ springs, generate the terms $t_1$ to $t_4$, and determine after how many springs the number of plots first exceeds 50.

**Q9.** The table below shows terms of the sequence generated by $t_0 = 2$, $t_{n+1} = 4t_n$, with two entries missing. Find the missing values.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_n$ | 2 | 8 | ? | 128 | ? |

**Q10.** The graph below shows the terms $t_n$ of a sequence, for $n = 0$ to $4$.




Three recurrence relations are proposed for the sequence: A: $t_0 = 32$, $t_{n+1} = t_n - 48$; B: $t_0 = 32$, $t_{n+1} = -0.5t_n$; C: $t_0 = 32$, $t_{n+1} = 0.5t_n$. Determine which recurrence generates the graphed sequence, explaining why each of the other two does not, and describe the behaviour of the sequence.

**Q11.** Three sequences all start at $t_0 = 100$ and are generated by the rules A: $t_{n+1} = t_n - 25$; B: $t_{n+1} = 0.2t_n$; C: $t_{n+1} = -t_n$. Generate the terms $t_1$ to $t_4$ of each sequence and describe the behaviour of each.

**Q12.** A bush-regeneration group has planted 60 trees along a creek, and plans to plant 15 more trees at the end of each month. The recurrence relation $t_0 = 60$, $t_{n+1} = t_n + 15$ models the total number of trees planted after $n$ months. (a) Interpret the numbers 60 and 15 in the context of the model. (b) How many trees have been planted after 4 months? (c) The sequence is graphed as separate points. Explain why the points should not be joined by a line.

**Q13.** Two sequences both start at $t_0 = 50$: sequence A is generated by $t_{n+1} = t_n + 50$ and sequence B by $t_{n+1} = 2t_n$. Generate the terms $t_1$ to $t_4$ of each sequence, describe the behaviour of each, and comment on how the two kinds of growth compare.

**Q14.** The first four terms of a sequence are $8, 24, 72, 216$. A student claims that the sequence is generated by $t_0 = 8$, $t_{n+1} = t_n + 16$. Show that this rule is wrong, write down a correct recurrence relation for the sequence, and find the next term.

**Q15.** A ball is dropped from a height of 16 m, and each bounce reaches half the height from which the ball last fell. Let $t_0 = 16$ be the release height, in metres, and let $t_n$ be the height reached after the $n$th bounce. Write a recurrence relation for the sequence, find the heights reached after each of the first four bounces, determine after which bounce the rebound height is first below 0.5 m, and describe the behaviour of the sequence.

**Q16.** A swimming pool is being drained. The table shows the volume of water $t_n$, in litres, remaining after $n$ hours.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_n$ | 4000 | 3400 | 2800 | 2200 | 1600 |

(a) Write a recurrence relation for the sequence.
(b) If draining continues at the same rate, after how many hours is the volume first below 1000 litres?
(c) Describe the behaviour of the sequence.

**Q17.** A sequence is defined by $t_0 = 250$, $t_{n+1} = -0.4t_n$. Generate the terms $t_1$ to $t_4$, and describe the behaviour of the sequence, including its limiting value.

**Q18.** A sequence satisfies $t_{n+1} = 2t_n$, and $t_2 = 44$. (a) Find $t_1$ and $t_0$ by working backwards. (b) Find $t_3$ and $t_4$.

## Answers

**Q1.** (a) $t_1=10$, $t_2=16$, $t_3=22$, $t_4=28$. (b) $t_n$: 4, 10, 16, 22, 28 for $n=0$ to $4$. (c) Increasing. (d) $t_8=52$. **Q2.** (a) 24, 12, 6, 3. (b) $t_5=1.5$, $t_6=0.75$. (c) Decreasing, approaching the limiting value 0. **Q3.** (a) $t_2=21$, $t_3=63$, $t_4=189$. (b) $t_5=567$, the fifth term. (c) The starting value $t_1=7$ is missing. **Q4.** (a) $t_0=18$. (b) 18, 15, 12, 9, 6. (c) $t_0=18$, $t_{n+1}=t_n-3$. (d) $t_5=3$ litres. **Q5.** (a) $t_0=1500$, $t_{n+1}=t_n-90$. (b) $t_0=300$, $t_{n+1}=2t_n$. (c) $t_0=180$, $t_{n+1}=0.85t_n$. **Q6.** (a) $-6$, 12, $-24$, 48. (b) Oscillating: the terms alternate in sign, and their sizes double each step. (c) 16, 4, 1, 0.25. (d) 0 (the terms settle towards 0). **Q7.** 3, 15, 75, 375, 1875; increasing. **Q8.** $t_0=26$, $t_{n+1}=t_n+4$; $t_1$ to $t_4$: 30, 34, 38, 42; the plots first exceed 50 after 7 springs ($t_7=54$). **Q9.** $t_2=32$ and $t_4=512$. **Q10.** Recurrence B; the sequence oscillates, its terms alternating in sign while their sizes halve, settling towards the limiting value 0. **Q11.** A: 75, 50, 25, 0 — decreasing by a constant 25. B: 20, 4, 0.8, 0.16 — decreasing, settling towards the limiting value 0. C: $-100$, 100, $-100$, 100 — oscillating between 100 and $-100$. **Q12.** (a) 60 trees already planted at the start; 15 trees planted each month. (b) 120 trees. (c) $n$ takes whole-number values only, so there are no terms between the plotted points. **Q13.** A: 100, 150, 200, 250 (increasing by a constant 50); B: 100, 200, 400, 800 (increasing, doubling). Both are increasing and equal at $t_1=100$, but B's steps grow while A's stay the same, so B pulls far ahead. **Q14.** The rule gives $t_2=40$, not 72. A correct recurrence is $t_0=8$, $t_{n+1}=3t_n$; next term 648. **Q15.** $t_0=16$, $t_{n+1}=0.5t_n$; heights 8, 4, 2, 1 m; first below 0.5 m after the 6th bounce ($t_6=0.25$); decreasing, approaching 0. **Q16.** (a) $t_0=4000$, $t_{n+1}=t_n-600$. (b) After 6 hours ($t_6=400$). (c) Decreasing, by 600 litres each hour. **Q17.** $-100$, 40, $-16$, 6.4; oscillating either side of 0 with shrinking sizes, approaching the limiting value 0. **Q18.** (a) $t_1=22$, $t_0=11$. (b) $t_3=88$, $t_4=176$.

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## Fully-worked solutions

**Q1.** (a) Iterating $t_{n+1}=t_n+6$ from $t_0=4$: $t_1=4+6=10$, $t_2=10+6=16$, $t_3=16+6=22$, $t_4=22+6=28$. (b)

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_n$ | 4 | 10 | 16 | 22 | 28 |

(c) Each term is 6 more than the one before, so the sequence is increasing.
(d) Continuing: $t_5=34$, $t_6=40$, $t_7=46$, $t_8=52$. The first term to exceed 50 is $t_8=52$.

**Q2.** (a) Each iteration halves the current term: $t_1=0.5(48)=24$, $t_2=12$, $t_3=6$, $t_4=3$. (b) $t_5=0.5(3)=1.5$ and $t_6=0.5(1.5)=0.75$. (c) The sequence is decreasing, but each fall is half the previous one, and the terms never become negative: iterating further gives $1.5, 0.75, 0.375, 0.1875, \ldots$, which settle closer and closer to 0, so the limiting value is 0.

**Q3.** (a) $t_2=3t_1=3(7)=21$, $t_3=3(21)=63$, $t_4=3(63)=189$. (b) $t_5=3(189)=567$, so 567 is a term of the sequence — the fifth term. (c) A recurrence relation needs both a rule and a starting value. The rule alone generates a different sequence from every different start (from 8, for instance, it gives $8, 24, 72, \ldots$), so without $t_1=7$ the sequence is not determined.

**Q4.** (a) The point at $n=0$ is at a height of 18, so $t_0=18$ litres. (b) Reading the points in order: 18, 15, 12, 9, 6. (c) The volume falls by 3 litres each hour ($15-18=-3$, and the same fall each step), so $t_0=18$, $t_{n+1}=t_n-3$. (d) $t_5=t_4-3=6-3=3$ litres.

**Q5.** (a) The starting number is 1500 and each year the same 90 trees are removed — a fixed amount, so the rule adds a constant: $t_0=1500$, $t_{n+1}=t_n-90$. (b) The starting number is 300 and each hour the count doubles — a fixed factor: $t_0=300$, $t_{n+1}=2t_n$. (c) The starting number is 180, and a fall of $15\%$ leaves $85\%$ of the population — a percentage change is a fixed factor, so the rule multiplies by $0.85$: $t_0=180$, $t_{n+1}=0.85t_n$.

**Q6.** (a) $t_1=-2t_0=-2(3)=-6$; $t_2=-2(-6)=12$; $t_3=-2(12)=-24$; $t_4=-2(-24)=48$. (b) The terms $3, -6, 12, -24, 48, \ldots$ alternate in sign — a negative term, then a positive term, and so on — so the sequence oscillates; because the multiplier is $-2$, the size of each term is twice the size of the one before, so the swings grow.

(c) $u_1 = 0.25(64) = 16$; $u_2 = 0.25(16) = 4$; $u_3 = 0.25(4) = 1$; $u_4 = 0.25(1) = 0.25$. (d) Continuing to iterate the rule: $u_5 = 0.0625$, $u_6 = 0.015625$, $u_7 = 0.00390625, \ldots$. Each term is a quarter of the one before, so the terms $64, 16, 4, 1, 0.25, \ldots$ keep falling by less and less and settle towards a limiting value of 0.

**Q7.** $t_1 = 5(3) = 15$, $t_2 = 5(15) = 75$, $t_3 = 5(75) = 375$, $t_4 = 5(375) = 1875$. The first five terms are $3, 15, 75, 375, 1875$. Each term is larger than the one before, so the sequence is increasing — and because each term is five times the previous one, the jumps grow rapidly.

**Q8.** The garden starts with 26 plots and gains 4 each spring, so $t_0 = 26$, $t_{n+1} = t_n + 4$. Iterating: $t_1 = 30$, $t_2 = 34$, $t_3 = 38$, $t_4 = 42$. Continuing the sequence: $t_5 = 46$, $t_6 = 50$, $t_7 = 54$. Now $t_6 = 50$ does not exceed 50, but $t_7 = 54$ does, so the number of plots first exceeds 50 after 7 springs.

**Q9.** The missing second term is $t_2 = 4t_1 = 4(8) = 32$. As a check, the rule then gives $t_3 = 4(32) = 128$, which agrees with the table. The missing fourth term is $t_4 = 4t_3 = 4(128) = 512$.

**Q10.** Test each recurrence from $t_0 = 32$ against the graphed points $32, -16, 8, -4, 2$. Recurrence A gives $32, -16, -64, \ldots$ — it matches at $n = 1$ but gives $-64$ at $n = 2$, where the graph shows 8, so A fails. Recurrence C gives $t_1 = 0.5(32) = 16$, but the graph shows $-16$ at $n = 1$, so C fails. Recurrence B gives $t_1 = -0.5(32) = -16$, $t_2 = -0.5(-16) = 8$, $t_3 = -4$, $t_4 = 2$ — matching every plotted point. So recurrence B generates the sequence. The behaviour is oscillating: the terms alternate in sign because the multiplier is negative, while each term is half the size of the one before, so the sequence settles towards a limiting value of 0.

**Q11.** Sequence A: $t_1 = 100 - 25 = 75$, $t_2 = 50$, $t_3 = 25$, $t_4 = 0$ — the same 25 is subtracted at every step, so A is decreasing by equal amounts. Sequence B: $t_1 = 0.2(100) = 20$, $t_2 = 0.2(20) = 4$, $t_3 = 0.8$, $t_4 = 0.16$ — each term is one-fifth of the one before, so B is decreasing, and continuing $(0.032, 0.0064, \ldots)$ the terms settle towards a limiting value of 0. Sequence C: $t_1 = -100$, $t_2 = 100$, $t_3 = -100$, $t_4 = 100$ — the terms alternate between 100 and $-100$ forever, so C oscillates, and its terms never settle.

**Q12.** (a) The 60 is the starting value $t_0$ — the number of trees already planted at the start of the program — and the 15 is the number of extra trees planted at the end of each month. (b) $t_1 = 75$, $t_2 = 90$, $t_3 = 105$, $t_4 = 120$: after 4 months, 120 trees have been planted. (c) The term number $n$ counts whole months, so the sequence has a value only at $n = 0, 1, 2, \ldots$ — there is no term at, say, $n = 2.5$. Joining the points would wrongly suggest that the model gives values between the months, so the graph must be left as separate (discrete) points.

**Q13.** Sequence A: $t_1 = 50 + 50 = 100$, $t_2 = 150$, $t_3 = 200$, $t_4 = 250$ — the same 50 is added at every step, so A is increasing by equal amounts. Sequence B: $t_1 = 2(50) = 100$, $t_2 = 200$, $t_3 = 400$, $t_4 = 800$ — each term is double the one before, so B is increasing by ever larger amounts. Both sequences start at 50 and agree at $t_1 = 100$, but from there they part company: A's steps stay at 50 ($100 \rightarrow 150 \rightarrow 200$) while B's steps keep doubling ($100 \rightarrow 200 \rightarrow 400$), so B pulls far ahead — adding the same amount each step grows much more slowly than multiplying by the same factor each step.

**Q14.** Test the student's rule by iterating it: $t_0 = 8$ gives $t_1 = 8 + 16 = 24$ (correct so far), but then $t_2 = 24 + 16 = 40$, whereas the third term of the sequence is 72. So the rule is wrong — the terms do not go up by the same amount each step (the increases are $16, 48, 144$). Testing the other form, each term divided by the one before gives $\frac{24}{8} = 3$, $\frac{72}{24} = 3$ and $\frac{216}{72} = 3$ — the same factor every time. So the sequence is generated by $t_0 = 8$, $t_{n+1} = 3t_n$, and the next term is $t_4 = 3(216) = 648$.

**Q15.** Each rebound height is half the previous height, so $t_0 = 16$, $t_{n+1} = 0.5t_n$. The heights after the first four bounces are $t_1 = 8$, $t_2 = 4$, $t_3 = 2$, $t_4 = 1$ (metres). Continuing: $t_5 = 0.5$, which is not below 0.5 m, and $t_6 = 0.25$, which is. So the rebound height is first below 0.5 m after the 6th bounce. The sequence is decreasing, with the heights approaching the limiting value 0.

**Q16.** (a) The volume falls by the same amount each hour: $4000-3400=600$, and each later step also falls by 600. So $t_0=4000$, $t_{n+1}=t_n-600$. (b) Continuing the table: $t_5=1600-600=1000$, which is not below 1000, and $t_6=1000-600=400$, which is. The volume is first below 1000 litres after 6 hours. (c) The sequence is decreasing, falling by a constant 600 litres each hour.

**Q17.** $t_1=-0.4(250)=-100$; $t_2=-0.4(-100)=40$; $t_3=-0.4(40)=-16$; $t_4=-0.4(-16)=6.4$. Continuing the iteration: $t_5=-2.56$, $t_6=1.024$, $t_7=-0.4096,\ldots$. The terms $250,-100,40,-16,6.4,-2.56,1.024,\ldots$ alternate in sign because the multiplier is negative, so the sequence oscillates; each term is $0.4$ times the size of the one before, so the swings shrink and the terms crowd in towards a limiting value of 0.

**Q18.** (a) From $t_2=2t_1$: $44=2t_1$, so $t_1=22$. From $t_1=2t_0$: $22=2t_0$, so $t_0=11$. (Check forwards: $11\rightarrow 2(11)=22\rightarrow 2(22)=44$.) (b) $t_3=2t_2=2(44)=88$ and $t_4=2(88)=176$.

## Theory

In 2A we generated sequences with first-order linear recurrence relations $t_{n+1}=a\,t_n+b$. This section studies the most important special case: sequences that grow or shrink by the **same fixed amount** at every step.

**Arithmetic sequences.** A sequence is **arithmetic** if each term is obtained from the term before it by adding a constant, called the **common difference** $d$. An arithmetic sequence is generated by the recurrence relation

$$t_0=\text{starting value}\,,\,t_{n+1}=t_n+d\,.$$

(This is the rule $t_{n+1}=a\,t_n+b$ with $a=1$ and $b=d$.) For example, $t_0=9$, $t_{n+1}=t_n+4$ generates the arithmetic sequence $9,13,17,21,\ldots$ with $d=4$. The common difference may be negative, in which case the same amount is subtracted at each step: $t_0=20$, $t_{n+1}=t_n-6$ generates $20,14,8,2,\ldots$ with $d=-6$.

**Recognising an arithmetic sequence.** To test whether a list of terms is arithmetic, subtract each term from the term after it. If **every** difference is the same number, the sequence is arithmetic and that number is $d$. The sequence $2,9,16,23,\ldots$ has differences $7,7,7$, so it is arithmetic with $d=7$; the sequence $1,2,4,8,\ldots$ has differences $1,2,4$, so it is not arithmetic. Check every gap — a single matching pair is not enough.

**The explicit rule for the $n$-th term.** Iterating a recurrence produces the terms one at a time, which is slow if the term wanted is far along the sequence. An arithmetic sequence also has an **explicit rule** that jumps straight to any term. Starting from $t_0$, the term $t_n$ is reached after exactly $n$ additions of $d$, so

$$t_n=t_0+n\,d\,.$$

For the sequence $9,13,17,21,\ldots$ above, $t_n=9+4\,n$, so $t_{20}=9+4(20)=89$ — found immediately, without computing the nineteen terms in between.

**Numbering from $t_1$: the $a+(n-1)d$ form.** Many questions (and many exam papers) call the first term $a$ and number the terms from $t_1$, so that $t_1=a$. Counting from $t_1$, the term $t_n$ has had $d$ added only $n-1$ times, so the rule becomes

$$t_n=a+(n-1)d\,.$$

The two forms describe exactly the same sequence — only the labels on the terms differ. With first term 9 and $d=4$, the $t_1$-numbered rule gives $t_3=9+2(4)=17$, the same value that the $t_0$-numbered rule calls $t_2$. Before using either rule, always check whether the sequence in the question starts at $t_0$ or at $t_1$.

**Using the explicit rule.** Three standard tasks all come from the one rule.

- *Finding a term:* substitute the term number. If $t_n=25+8\,n$, then $t_{10}=105$.
- *Finding which term has a given value:* set the rule equal to the value and solve for $n$. If $t_0+n\,d=v$ then $n=\dfrac{v-t_0}{d}$; the value $v$ is a term of the sequence only if this $n$ is a whole number ($n=0,1,2,\ldots$). If $n$ is not a whole number, $v$ is never a term.
- *Finding $d$ or the starting value from two given terms:* between $t_m$ and $t_k$ there are $k-m$ steps, so $t_k-t_m=(k-m)d$. For instance, if $t_4=26$ and $t_9=41$, then $5\,d=15$, so $d=3$, and $t_0=t_4-4\,d=14$.

**Behaviour of an arithmetic sequence.** Because the same $d$ is added at every step, an arithmetic sequence is **increasing** if $d>0$, **decreasing** if $d<0$, and **constant** if $d=0$. When the points $(n,t_n)$ are plotted, each step across by 1 moves up (or down) by the same amount $d$, so the points of an arithmetic sequence always lie on a **straight line**, with $d$ as its slope — though the graph is still a set of separate points, never joined, because $n$ takes whole-number values only. Arithmetic sequences model practical situations of **linear growth** or **linear decay**, in which a quantity changes by a fixed amount in each time period — a road crew sealing the same length of road each day, or a candle burning down by the same amount each hour.

**Using CAS.** An arithmetic sequence can be generated on a CAS calculator with the answer key: type the starting value, press ENTER, then type $\text{Ans}+d$ and press ENTER repeatedly. For distant terms the explicit rule is faster: define $t(n)=t_0+nd$ as a function, and read any term, or a whole table of terms, directly. The CAS solve command will solve $t_0+nd=v$ for $n$ in one step — but you must still check that the solution is a whole number before declaring that $v$ is a term of the sequence. Learn the by-hand methods below so that you understand what the calculator is doing and can check that its output is reasonable.

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## Worked examples

### *Example 1 — scaffolded*

Consider the two sequences

$$\text{A}: 8, 13, 18, 23, \ldots \quad \text{B}: 3, 6, 12, 24, \ldots$$

Decide whether each sequence is arithmetic. For the arithmetic sequence, state the common difference, write a recurrence relation using $t_0$, and find the next two terms.

| Working | Comment |
|---|---|
| Sequence A: $13-8=5$, $18-13=5$, $23-18=5$. | Subtract each term from the term after it. |
| Every difference is 5, so A is arithmetic with $d=5$. | A constant difference between consecutive terms is exactly what makes a sequence arithmetic. |
| Sequence B: $6-3=3$, $12-6=6$, $24-12=12$. | Check every gap — one matching pair is not enough. |
| The differences $3, 6, 12$ are not constant, so B is not arithmetic. | (Each term of B is double the one before — a different kind of rule.) |
| Recurrence for A: $t_0=8$, $t_{n+1}=t_n+5$. | A recurrence relation needs both the starting value and the rule. |
| Next two terms of A: $23+5=28$ and $28+5=33$. | Keep adding the common difference. |

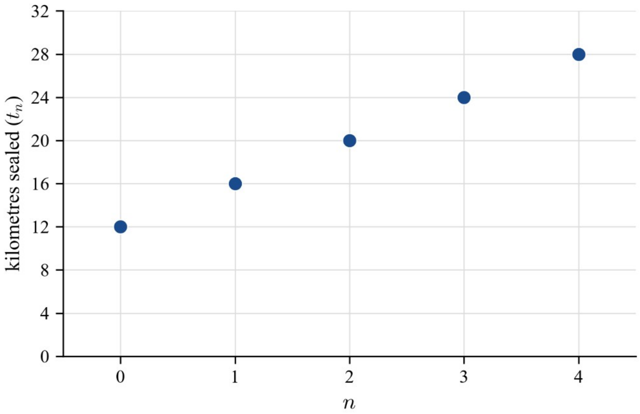
So sequence A is arithmetic with common difference 5 and continues $28, 33$; sequence B is not arithmetic.

### *Example 2 — scaffolded*

An outback shire's works crew is sealing a long gravel road. At the start of the month, 12 km of the road are already sealed, and the crew seals a further 4 km on each working day. Let $t_n$ be the total length sealed, in kilometres, after $n$ working days. Write a recurrence relation for $t_n$, display $t_0$ to $t_4$ in a table, find the explicit rule, use it to find $t_{10}$, and describe the behaviour of the sequence.

| Working | Comment |
|---|---|
| Starting value $t_0=12$; each day adds the same $d=4$. | A fixed increase each day makes the sequence arithmetic. |
| Recurrence: $t_0=12$, $t_{n+1}=t_n+4$. | Starting value plus rule. |
| $t_1=16$, $t_2=20$, $t_3=24$, $t_4=28$ (table below). | Iterate: each term is 4 more than the last. |
| Explicit rule: after $n$ days, 4 has been added $n$ times, so $t_n=12+4n$. | The explicit rule jumps straight to any term. |
| $t_{10}=12+4(10)=52$. | No need to iterate ten times. |
| The plotted points (below) lie on a straight line going upwards. | $d=4>0$, so the sequence is increasing; the line rises 4 for each step of 1. |

| $n$ | 0 | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- | --- |
| $t_n$ | 12 | 16 | 20 | 24 | 28 |




So $t_0 = 12$, $t_{n+1} = t_n + 4$, with explicit rule $t_n = 12 + 4n$; after 10 working days $t_{10} = 52$ km are sealed, and the sequence is increasing — linear growth of 4 km per day.

### *Example 3 — unscaffolded*

An arithmetic sequence is numbered from $t_1$. Its first term is $t_1 = 9$ and its common difference is 6. Find a rule for the $n$-th term and use it to find $t_{20}$. Then determine whether each of 141 and 100 is a term of the sequence.

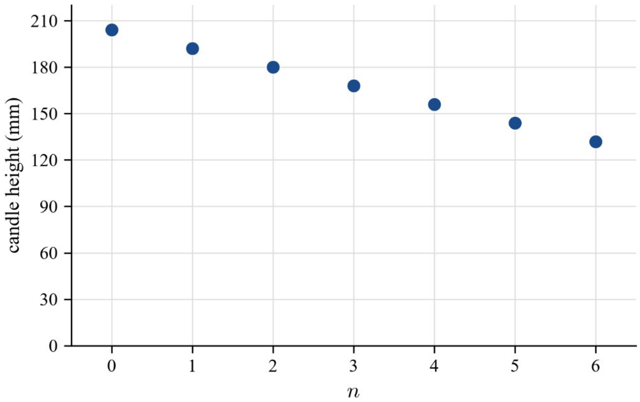
The sequence is numbered from $t_1$, so the rule is $t_n = a + (n-1)d$ with $a = 9$ and $d = 6$: $t_n = 9 + (n-1) \times 6$. The 20th term is $t_{20} = 9 + 19 \times 6 = 9 + 114 = 123$. For 141, solve $9 + (n-1) \times 6 = 141$: then $(n-1) \times 6 = 132$, so $n - 1 = 22$ and $n = 23$, a whole number — so 141 is a term, the 23rd. For 100, $9 + (n-1) \times 6 = 100$ gives $(n-1) \times 6 = 91$, so $n - 1 = \frac{91}{6} \approx 15.17$, which is not a whole number — so the sequence skips over 100 ($t_{16} = 99$ and $t_{17} = 105$) and 100 is not a term.

$\therefore$ $t_n = 9 + (n-1) \times 6$, $t_{20} = 123$; 141 is the 23rd term, but 100 is not a term of the sequence.

### *Example 4 — unscaffolded*

A dinner candle burns down steadily. Two hours after it is lit, its height is 180 mm; five hours after it is lit, its height is 144 mm. Let $t_n$ be the height of the candle, in millimetres, $n$ hours after lighting. Find the common difference and the original height, write a recurrence relation and the explicit rule, describe the behaviour of the sequence, and find after how many whole hours the candle is first shorter than 60 mm.

From $t_2 = 180$ to $t_5 = 144$ is $5 - 2 = 3$ steps, so $3d = 144 - 180 = -36$ and $d = -12$: the candle burns down 12 mm every hour. Working back two steps from $t_2$, the original height is $t_0 = 180 + 2 \times 12 = 204$ mm. The recurrence relation is $t_0 = 204$, $t_{n+1} = t_n - 12$, and the explicit rule is $t_n = 204 - 12n$. Since $d = -12 < 0$ the sequence is decreasing — linear decay — and the plotted points below lie on a straight line sloping downwards. For the height to be below 60 mm we need $204 - 12n < 60$, that is, $12n > 144$, so $n > 12$. After 12 hours the height is $t_{12} = 204 - 144 = 60$ mm exactly — not yet shorter than 60 mm — so the candle is first shorter than 60 mm after 13 hours, when $t_{13} = 48$ mm.




$\therefore\ d=-12$, $t_0=204$; $t_{n+1}=t_n-12$ with $t_n=204-12n$; the sequence is decreasing, and the candle is first shorter than 60 mm after 13 hours.

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## Practice questions

### *Scaffolded*

**Q1.** For each sequence, find the differences between consecutive terms and decide whether the sequence is arithmetic. (a) $5,9,13,17,\ldots$ (b) $20,17,14,11,\ldots$ (c) $2,4,8,16,\ldots$ (d) For each arithmetic sequence above, state the common difference and write down the next two terms.

**Q2.** A sequence is defined by the recurrence relation $t_0=8$, $t_{n+1}=t_n+6$. (a) Generate the terms $t_1$ to $t_4$. (b) Explain why the sequence is arithmetic, and state the common difference. (c) Write the explicit rule for $t_n$. (d) Use the rule to find $t_{12}$. (e) Check that the rule gives the same value of $t_4$ as part (a).

**Q3.** A street library is set up with 40 books, and volunteers add 6 new books each week. Let $t_n$ be the number of books after $n$ weeks. (a) Write a recurrence relation for $t_n$. (b) Write the explicit rule for $t_n$. (c) How many books are there after 10 weeks? (d) In which week does the number of books first exceed 100?

**Q4.** In an outdoor amphitheatre, row 1 (the front row) has 22 seats, and each row behind has 4 more seats than the row in front. Let $t_n$ be the number of seats in row $n$, so the sequence is numbered from $t_1=22$. (a) State the common difference. (b) Write down $t_2$ and $t_3$. (c) Write the explicit rule for $t_n$. (d) How many seats are in row 15? (e) Which row has exactly 62 seats?

**Q5.** An arithmetic sequence numbered from $t_0$ has $t_0=50$ and $t_4=30$. (a) Find the common difference. (b) Write the recurrence relation and generate $t_1$ to $t_3$. (c) Write the explicit rule for $t_n$. (d) Find $t_9$. (e) Describe the behaviour of the sequence.

**Q6.** Three sequences are defined by the recurrence relations below. (a) For $t_0=3$, $t_{n+1}=t_n+7$, generate $t_1$ to $t_3$ and describe the behaviour of the sequence. (b) For $t_0=12$, $t_{n+1}=t_n-0.5$, generate $t_1$ to $t_4$ and describe the behaviour. (c) For $t_0=9$, $t_{n+1}=t_n$, write down $t_1$ to $t_3$, state the common difference, and describe the behaviour. (d) The points $(n,t_n)$ of any arithmetic sequence are plotted on a graph. Describe how the points are arranged, state what feature of the graph the common difference $d$ gives, and explain why the points are not joined.

### *Unscaffolded*

**Q7.** An arithmetic sequence has $t_0=4$ and common difference 9. Write down the first five terms, write the explicit rule for $t_n$, and use it to find $t_{20}$.

**Q8.** Show that the sequence $87,79,71,63,\ldots$ is arithmetic, and write its recurrence relation using $t_0$ and its explicit rule. Find $t_{10}$, and determine whether 15 is a term of the sequence — and if so, which term.

**Q9.** Mia is training for a charity bike ride along the Great Ocean Road. She cycles 15 km in week 1 of her training program and increases the distance by 7 km each week, so the weekly distances form an arithmetic sequence numbered from $t_1 = 15$. Write the explicit rule for $t_n$, find how far Mia cycles in week 8, and find the first week in which she cycles more than 90 km.

**Q10.** An arithmetic sequence numbered from $t_0$ has $t_3 = 29$ and $t_7 = 53$. Find the common difference and $t_0$, write the explicit rule, and find $t_{15}$.

**Q11.** A grain silo on a Wimmera farm holds 200 tonnes of wheat. An auger empties the silo at a steady 8 tonnes per hour. Let $t_n$ be the number of tonnes left after $n$ hours. Write a recurrence relation and the explicit rule for $t_n$. After how many hours do exactly 104 tonnes remain? Explain why there is no whole number of hours after which exactly 90 tonnes remain. After how many whole hours does the load first fall below 40 tonnes?

**Q12.** The organisers of a regional food festival release 960 tickets, which sell steadily at 120 tickets per day. The number of tickets still available after $n$ days is modelled by $t_0 = 960$, $t_{n+1} = t_n - 120$. (a) Interpret the numbers 960 and 120 in the context of the model. (b) Write the explicit rule for $t_n$. (c) How many tickets remain after 5 days? (d) After how many days does the festival sell out?

**Q13.** A grandstand at a country racecourse has rows of seats. The front row has 10 seats, and each row behind has 3 more seats than the row in front. Ava counts the front row as row 1 and writes the rule $t_n = 10 + (n-1) \times 3$ for the number of seats in row $n$. Ben counts the front row as row 0 and writes the rule $t_n = 10 + 3n$. Show that both rules give 31 seats for the eighth row from the front, and explain why the two rules look different but describe the same grandstand.

**Q14.** An arithmetic sequence numbered from $t_0$ has $t_5 = 32$ and $t_{12} = 67$. Find the common difference and $t_0$, and determine which term of the sequence equals 92.

**Q15.** After a cold front, the snow depth at a Victorian alpine resort is 42 cm. During the mild week that follows, the depth falls by 3.5 cm each day. Let $t_n$ be the depth, in centimetres, after $n$ days. Write a recurrence relation for $t_n$, find the depths after each of the first four days, and write the explicit rule. On which day does the depth first fall below 10 cm? After how many days is the snow gone completely?

**Q16.** For each arithmetic sequence, state the common difference, describe the behaviour of the sequence, and write the explicit rule using $t_0$. (a) $3, 3, 3, 3, \ldots$ (b) $-8, -3, 2, 7, \ldots$ (c) $5, 2.5, 0, -2.5, \ldots$

**Q17.** On game day at a suburban football ground, two overflow carparks open at the same time. Carpark A already holds 14 cars and fills at 6 cars per minute; carpark B already holds 50 cars and fills at 2 cars per minute. Let $t_n$ and $u_n$ be the numbers of cars in carparks A and B respectively, $n$ minutes after opening. Write an explicit rule for each carpark, find after how many minutes the two carparks hold the same number of cars (and how many cars each then holds), and determine which carpark holds more cars 12 minutes after opening.

**Q18.** The numbers $7, x, y, 28$ are the first four terms of an arithmetic sequence. (a) Find the common difference, and hence the values of $x$ and $y$. (b) Taking $t_0 = 7$, write the explicit rule for the sequence and use it to find $t_{12}$.

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## Answers

**Q1.** (a) Differences $4, 4, 4$ — arithmetic. (b) Differences $-3, -3, -3$ — arithmetic. (c) Differences $2, 4, 8$ — not arithmetic. (d) Sequence (a): $d=4$, next terms 21, 25; sequence (b): $d=-3$, next terms 8, 5. **Q2.** (a) $t_1=14$, $t_2=20$, $t_3=26$, $t_4=32$. (b) The same 6 is added at each step; $d=6$. (c) $t_n=8+6n$. (d) $t_{12}=80$. (e) $t_4=8+6(4)=32$, which agrees. **Q3.** (a) $t_0=40$, $t_{n+1}=t_n+6$. (b) $t_n=40+6n$. (c) $t_{10}=100$ books. (d) Week 11 ($t_{11}=106$). **Q4.** (a) $d=4$. (b) $t_2=26$, $t_3=30$. (c) $t_n=22+(n-1)\times 4$. (d) 78 seats. (e) Row 11. **Q5.** (a) $d=-5$. (b) $t_0=50$, $t_{n+1}=t_n-5$; terms 45, 40, 35. (c) $t_n=50-5n$. (d) $t_9=5$. (e) Decreasing ($d<0$). **Q6.** (a) 10, 17, 24; increasing. (b) 11.5, 11, 10.5, 10; decreasing. (c) 9, 9, 9; $d=0$; constant. (d) The points lie on a straight line; $d$ is the slope (the rise or fall per step of 1); the points are not joined because $n$ takes whole-number values only. **Q7.** 4, 13, 22, 31, 40; $t_n=4+9n$; $t_{20}=184$. **Q8.** Differences all $-8$, so arithmetic; $t_0=87$, $t_{n+1}=t_n-8$; $t_n=87-8n$; $t_{10}=7$; 15 is a term, $t_9$. **Q9.** $t_n=15+(n-1)\times 7$; week 8: 64 km; first over 90 km in week 12 ($t_{12}=92$). **Q10.** $d=6$, $t_0=11$; $t_n=11+6n$; $t_{15}=101$. **Q11.** $t_0=200$, $t_{n+1}=t_n-8$; $t_n=200-8n$; 104 tonnes after 12 hours; 90 tonnes would need $n=13.75$, not a whole number; first below 40 tonnes after 21 hours ($t_{21}=32$). **Q12.** (a) 960 is the number of tickets available at the start; 120 is the number sold each day. (b) $t_n=960-120n$. (c) 360 tickets. (d) After 8 days. **Q13.** Ava: the eighth row is $n=8$, giving $10+7\times 3=31$; Ben: the eighth row is $n=7$, giving $10+3(7)=31$. The rules agree because both add $d=3$ exactly 7 times to the front row's 10 seats — they just label the front row differently. **Q14.** $d=5$, $t_0=7$; $92=t_{17}$. **Q15.** $t_0=42$, $t_{n+1}=t_n-3.5$; depths 38.5, 35, 31.5, 28 cm; $t_n=42-3.5n$; first below 10 cm on day 10 ($t_{10}=7$); gone after 12 days. **Q16.** (a) $d=0$; constant; $t_n=3$. (b) $d=5$; increasing; $t_n=-8+5n$. (c) $d=-2.5$; decreasing; $t_n=5-2.5n$. **Q17.** $t_n=14+6n$, $u_n=50+2n$; equal after 9 minutes, each holding 68 cars; after 12 minutes carpark A holds more (86 cars to 74). **Q18.** (a) $d=7$; $x=14$, $y=21$. (b) $t_n=7+7n$; $t_{12}=91$.

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## Fully-worked solutions

**Q1.** (a) The differences are $9-5=4$, $13-9=4$, $17-13=4$. All equal, so the sequence is arithmetic. (b) The differences are $17-20=-3$, $14-17=-3$, $11-14=-3$. All equal, so the sequence is arithmetic (with a negative common difference). (c) The differences are $4-2=2$, $8-4=4$, $16-8=8$. These are not constant, so the sequence is not arithmetic. (d) Sequence (a) has $d=4$; the next two terms are $17+4=21$ and $21+4=25$. Sequence (b) has $d=-3$; the next two terms are $11-3=8$ and $8-3=5$.

**Q2.** (a) $t_1=8+6=14$, $t_2=14+6=20$, $t_3=20+6=26$, $t_4=26+6=32$. (b) The rule adds the same number, 6, at every step, so consecutive terms always differ by 6: the sequence is arithmetic with $d=6$. (c) Starting from $t_0=8$, the term $t_n$ has had 6 added $n$ times: $t_n=8+6n$. (d) $t_{12}=8+6(12)=8+72=80$. (e) The rule gives $t_4=8+6(4)=32$, the same value found by iterating in part (a).

**Q3.** (a) The starting value is 40 and each week adds 6: $t_0=40$, $t_{n+1}=t_n+6$. (b) $t_n=40+6n$. (c) $t_{10}=40+6(10)=100$ books. (d) Solve $40+6n>100$: $6n>60$, so $n>10$. After 10 weeks there are exactly 100 books, which does not exceed 100, so the number of books first exceeds 100 in week 11, when $t_{11}=106$.

**Q4.** (a) Each row has 4 more seats than the row in front, so $d=4$. (b) $t_2=22+4=26$ and $t_3=26+4=30$. (c) The sequence is numbered from $t_1=22$, so $t_n=a+(n-1)d=22+(n-1)\times 4$. (d) $t_{15}=22+14\times 4=22+56=78$ seats. (e) Solve $22+(n-1)\times 4=62$: $(n-1)\times 4=40$, so $n-1=10$ and $n=11$. Row 11 has exactly 62 seats.

**Q5.** (a) From $t_0$ to $t_4$ is 4 steps, so $t_4-t_0=4d$: $30-50=4d$, giving $4d=-20$ and $d=-5$. (b) $t_0=50$, $t_{n+1}=t_n-5$. Iterating: $t_1=45$, $t_2=40$, $t_3=35$. (c) $t_n=50-5n$. (d) $t_9=50-5(9)=50-45=5$. (e) The common difference is negative ($d=-5<0$), so the sequence is decreasing — each term is 5 less than the one before.

**Q6.** (a) $t_1=3+7=10$, $t_2=17$, $t_3=24$. The common difference 7 is positive, so the sequence is increasing. (b) $t_1=12-0.5=11.5$, $t_2=11$, $t_3=10.5$, $t_4=10$. The common difference $-0.5$ is negative, so the sequence is decreasing. (c) The rule adds nothing, so $t_1=t_2=t_3=9$. The common difference is $d=0$ and the sequence is constant: every term is 9. (d) Each step across by 1 changes the term by the same amount $d$, so the plotted points all lie on a straight line, and $d$ is the slope of that line — the amount the points rise (if $d>0$) or fall (if $d<0$) per step. The points are not joined because the term number $n$ takes whole-number values only: there is no term between, say, $n=2$ and $n=3$.

**Q7.** Iterating $t_{n+1}=t_n+9$ from $t_0=4$: the first five terms are $4,13,22,31,40$. The explicit rule is $t_n=4+9n$, so $t_{20}=4+9(20)=4+180=184$. (Check: the rule gives $t_4=4+36=40$, matching the iteration.)

**Q8.** The differences are $79-87=-8$, $71-79=-8$, $63-71=-8$ — constant — so the sequence is arithmetic with $d=-8$. The recurrence relation is $t_0=87$, $t_{n+1}=t_n-8$, and the explicit rule is $t_n=87-8n$. Then $t_{10}=87-8(10)=87-80=7$. For 15, solve $87-8n=15$: $8n=72$, so $n=9$, a whole number. So 15 is a term of the sequence: $t_9=15$.

**Q9.** The sequence is numbered from $t_1=15$ with $d=7$, so $t_n=a+(n-1)d=15+(n-1)\times 7$. In week 8, Mia cycles $t_8=15+7\times 7=15+49=64$ km. For more than 90 km, solve $15+(n-1)\times 7>90$: $(n-1)\times 7>75$, so $n-1>\frac{75}{7}\approx 10.71$, giving $n\geq 12$. Checking: $t_{11}=15+70=85$ km (not more than 90) and $t_{12}=15+77=92$ km. So week 12 is the first week in which Mia cycles more than 90 km.

**Q10.** From $t_3$ to $t_7$ is $7-3=4$ steps, so $4d=53-29=24$ and $d=6$. Working back three steps from $t_3$: $t_0=29-3\times 6=11$. The explicit rule is $t_n=11+6n$, so $t_{15}=11+6(15)=11+90=101$. (Check: $t_7=11+42=53$, as given.)

**Q11.** The starting value is 200 tonnes and each hour removes 8 tonnes: $t_0=200$, $t_{n+1}=t_n-8$, with explicit rule $t_n=200-8n$. For 104 tonnes, solve $200-8n=104$: $8n=96$, so $n=12$ — exactly 104 tonnes remain after 12 hours. For 90 tonnes, $200-8n=90$ gives $8n=110$, so $n=13.75$, which is not a whole number: the load passes from $t_{13}=96$ tonnes to $t_{14}=88$ tonnes, skipping 90, so there is no whole number of hours at which exactly 90 tonnes remain. For the load to be below 40 tonnes, solve $200-8n<40$: $8n>160$, so $n>20$. After 20 hours exactly 40 tonnes remain (not below 40), so the load first falls below 40 tonnes after 21 hours, when $t_{21}=200-168=32$ tonnes.

**Q12.** (a) The 960 is the starting value $t_0$ — the number of tickets available when sales open — and the 120 is the fixed number of tickets sold each day, so the count falls by 120 per day ($d=-120$). (b) $t_n=960-120n$. (c) $t_5=960-120(5)=960-600=360$ tickets. (d) Sold out means $t_n=0$: $960-120n=0$, so $n=\frac{960}{120}=8$. The festival sells out after 8 days.

**Q13.** In Ava's numbering the front row is row 1, so the eighth row from the front is row $n=8$, and her rule gives $t_8=10+(8-1)\times 3=10+21=31$ seats. In Ben's numbering the front row is row 0, so the eighth row from the front is row $n=7$, and his rule gives $t_7=10+3(7)=10+21=31$ seats — the same answer. The eighth row from the front is 7 rows behind the front row, so in both rules the common difference 3 has been added exactly 7 times to the front row's 10 seats. The rules look different only because Ava's counter starts at 1 while Ben's starts at 0; they describe the same grandstand.

**Q14.** From $t_5$ to $t_{12}$ is $12-5=7$ steps, so $7d=67-32=35$ and $d=5$. Working back five steps from $t_5$: $t_0=32-5\times 5=7$, so $t_n=7+5n$. For 92, solve $7+5n=92$: $5n=85$, so $n=17$, a whole number. So 92 is the term $t_{17}$. (Check: $t_{12}=7+60=67$, as given.)

**Q15.** The starting depth is 42 cm and each day removes 3.5 cm: $t_0=42$, $t_{n+1}=t_n-3.5$. Iterating: $t_1=38.5$, $t_2=35$, $t_3=31.5$, $t_4=28$ cm. The explicit rule is $t_n=42-3.5n$. For the depth to be below 10 cm, solve $42-3.5n<10$: $3.5n>32$, so $n>\frac{32}{3.5}\approx 9.14$. Checking: $t_9=42-31.5=10.5$ cm (not below 10) and $t_{10}=42-35=7$ cm. So the depth first falls below 10 cm on day 10. The snow is gone when $t_n=0$: $42-3.5n=0$, so $n=\frac{42}{3.5}=12$ — after 12 days.

**Q16.** (a) The differences are all $0$, so $d=0$ and the sequence is constant. The explicit rule is $t_n=3+0\times n$, that is, $t_n=3$. (b) The differences are $-3-(-8)=5$, $2-(-3)=5$, $7-2=5$, so $d=5$ and the sequence is increasing. With $t_0=-8$, the rule is $t_n=-8+5n$. (Check: $t_3=-8+15=7$, as listed.) (c) The differences are $2.5-5=-2.5$, $0-2.5=-2.5$, $-2.5-0=-2.5$, so $d=-2.5$ and the sequence is decreasing. With $t_0=5$, the rule is $t_n=5-2.5n$. (Check: $t_3=5-7.5=-2.5$, as listed.)

**Q17.** Carpark A starts at 14 and gains 6 per minute: $t_n=14+6n$. Carpark B starts at 50 and gains 2 per minute: $u_n=50+2n$. The carparks hold equal numbers when $t_n=u_n$: $14+6n=50+2n$, so $4n=36$ and $n=9$. After 9 minutes each holds $t_9=14+54=68$ cars (and $u_9=50+18=68$, confirming the answer). After 12 minutes, $t_{12}=14+72=86$ and $u_{12}=50+24=74$, so carpark A holds more cars — its faster fill rate has overtaken carpark B's head start.

**Q18.** (a) From 7 to 28 is 3 steps of the sequence, so $3d=28-7=21$ and $d=7$. Then $x=7+7=14$ and $y=14+7=21$. (Check: $21+7=28$, as given.) (b) With $t_0=7$ and $d=7$, the explicit rule is $t_n=7+7n$, so $t_{12}=7+7(12)=7+84=91$.

### Theory

In 2A we generated sequences by applying a rule over and over. A **geometric sequence** is generated by the simplest multiplying rule of all: each new term is found by **multiplying the current term by the same constant**. That constant is called the **common ratio**, written $R$, and the recurrence relation of a geometric sequence is

$$t_0 = a, t_{n+1} = R t_n,$$

where $a$ is the starting value. For example, $t_0 = 10$, $t_{n+1} = 2t_n$ generates $10, 20, 40, 80, 160, \ldots$ — each term is double the term before. A geometric sequence is the special case of a first-order linear recurrence in which the current term is multiplied and nothing is then added.

**Recognising a geometric sequence.** In a geometric sequence, dividing any term by the term before it always gives the same answer — the common ratio:

$$\frac{t_1}{t_0} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \ldots = R.$$

So to test whether a list of numbers could be geometric, compute the **ratio of every pair of consecutive terms**: the list is geometric exactly when all of the ratios are equal, and that common value is $R$. The list $5, 15, 45, 135$ has ratios $\frac{15}{5} = 3$, $\frac{45}{15} = 3$ and $\frac{135}{45} = 3$, so it is geometric with $R = 3$. The list $7, 14, 21, 28$ is **not** geometric: $\frac{14}{7} = 2$ but $\frac{21}{14} = 1.5$ — it grows by equal amounts, not by equal factors.

**The explicit rule.** Iterating the recurrence, $t_1 = t_0 R$, then $t_2 = (t_0 R) R = t_0 R^2$, then $t_3 = t_0 R^3$: reaching term $t_n$ from $t_0$ takes $n$ multiplications by $R$, so

$$t_n = t_0 R^n.$$

This **explicit rule** jumps straight to any term of the sequence without generating all the terms before it. **Watch the numbering.** Many exam questions number a sequence from $t_1$, with first term $a$. Reaching $t_n$ from $t_1$ takes only $n-1$ multiplications, so with that numbering the rule is

$$t_n = a R^{n-1}.$$

The two forms describe the same list of numbers: in both, the power of $R$ equals the number of steps taken from the first term. Always check where the numbering starts before you use (or write) an explicit rule. The rule also works in reverse: two terms that are $k$ positions apart differ by a factor of $R^k$, so given two terms we can find $R$, and then recover the first term by dividing. For instance, if $t_0 = 6$ and $t_3 = 750$, then $750 = 6R^3$, so $R^3 = 125$ and $R = 5$.

**Behaviour and the value of $R$.** For a positive starting value, the behaviour of a geometric sequence is decided entirely by its common ratio.

- If $R > 1$, every term is larger than the one before: the sequence shows **geometric growth** (for example $10, 20, 40, 80, \ldots$).
- If $0 < R < 1$, every term is the same fixed fraction of the one before: the sequence shows **geometric decay**, and the terms approach a **limiting value of 0** — they get closer and closer to 0 but never reach it.
- If $R = 1$, every term equals the starting value: the sequence is **constant**.
- If $R < 0$, the terms **oscillate**, alternating in sign at every step — positive, negative, positive, … Their sizes shrink towards 0 when $-1 < R < 0$ and grow when $R < -1$.

**Repeated percentage change.** Many quantities change by the same percentage in each time period, and such quantities follow geometric sequences. An increase of $p\%$ per step leaves $(100 + p)\%$ of the quantity, giving

$R = 1 + \frac{p}{100}$; a decrease of $p\,\%$ leaves $(100 - p)\%$, giving $R = 1 - \frac{p}{100}$. A possum population growing by $20\,\%$ each year has $R = 1.2$; a filter that removes $30\,\%$ of a pollutant on each pass keeps $70\,\%$ of it, so $R = 0.7$. Note that a percentage **decrease** gives a ratio between 0 and 1 — decay — not a negative ratio.

**Using CAS.** A CAS calculator generates a geometric sequence with its answer key: type the starting value and press ENTER; then type $\text{Ans} \times R$, and every further press of ENTER multiplies by $R$ and produces the next term. For a distant term it is far quicker to evaluate the explicit rule directly with the power key — for example, entering $64 \times 1.5^6$ returns $t_6$ of Example 2 in a single calculation. Learn the by-hand methods below so that you understand what the calculator is doing and can check that its output is reasonable.

## Worked examples

### *Example 1 — scaffolded*

Show that the sequence

$$11, 33, 99, 297, \ldots$$

is geometric, state the common ratio, write down a recurrence relation for the sequence, and find the next two terms.

| Working | Comment |
|---|---|
| $\frac{33}{11} = 3$, $\frac{99}{33} = 3$, $\frac{297}{99} = 3$. | Divide each term by the term before it. |
| The ratios are all equal, so the sequence is geometric with common ratio $R = 3$. | A constant ratio between consecutive terms is exactly what makes a sequence geometric. |
| Recurrence relation: $t_0 = 11$, $t_{n+1} = 3t_n$. | A recurrence relation needs both a starting value and a rule. |
| $t_4 = 3 \times 297 = 891$. | Multiply the last known term by $R$. |
| $t_5 = 3 \times 891 = 2673$. | Multiply by $R$ again. |

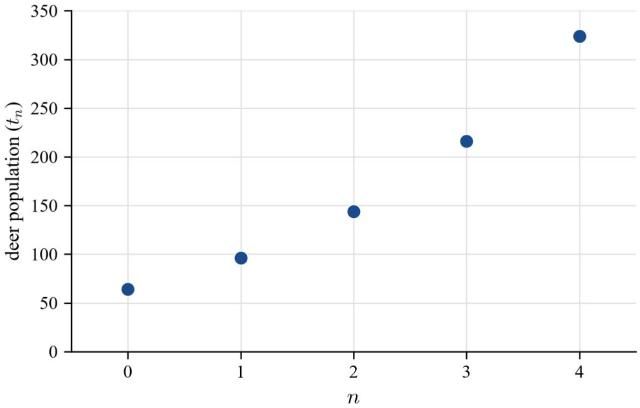
So the sequence is geometric with $R = 3$, satisfying $t_0 = 11$, $t_{n+1} = 3t_n$, and the next two terms are 891 and 2673.

### *Example 2 — scaffolded*

In a conservation park in the Victorian high country, the feral deer population is currently 64 animals, and while it remains unmanaged the population increases by $50\,\%$ each year. Let $t_n$ be the number of deer after $n$ years. Write a recurrence relation for the sequence, generate the terms $t_1$ to $t_4$ and display them in a table, plot the sequence, and use the explicit rule to predict the number of deer after 6 years.

| Working | Comment |
|---|---|
| An increase of $50\,\%$ leaves $150\,\%$ of the population, so $R = 1 + \frac{50}{100} = 1.5$. | A repeated percentage increase becomes a common ratio greater than 1. |
| $t_0 = 64$, $t_{n+1} = 1.5t_n$. | Starting value and rule together form the recurrence relation. |
| $t_1 = 1.5 \times 64 = 96$; $t_2 = 1.5 \times 96 = 144$. | Iterate: multiply by $R$ at every step. |
| $t_3 = 1.5 \times 144 = 216$; $t_4 = 1.5 \times 216 = 324$. | Each new term comes from the term just found. |
| Explicit rule: $t_n = 64 \times 1.5^n$. | Reaching $t_n$ from $t_0$ takes $n$ multiplications by 1.5. |
| $t_6 = 64 \times 1.5^6 = 64 \times 11.390625 = 729$. | The explicit rule jumps straight to term 6 — no need to generate $t_5$ first. |

| $n$ | 0 | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- | --- |
| $t_n$ | 64 | 96 | 144 | 216 | 324 |




So the sequence $64, 96, 144, 216, 324, \ldots$ shows geometric growth ($R = 1.5 > 1$), and the model predicts 729 deer after 6 years.

### *Example 3 — unscaffolded*

The terms of a geometric sequence are numbered from $t_1$, all terms are positive, and $t_2 = 24$ and $t_5 = 192$. Find the common ratio and the first term, write down the explicit rule, and find $t_8$.

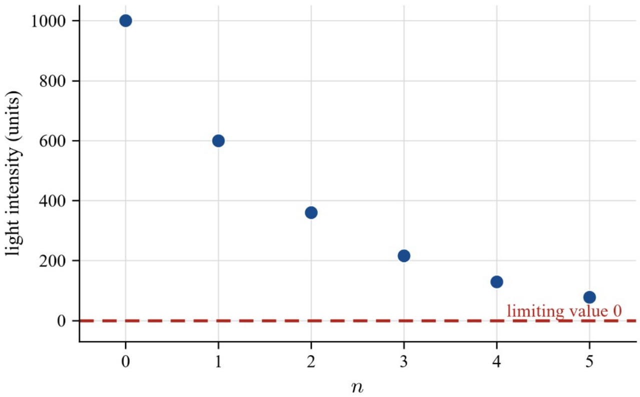
From $t_2$ to $t_5$ is $5 - 2 = 3$ steps, so $t_5 = t_2 \times R^3$. Then $192 = 24\,R^3$, so $R^3 = \dfrac{192}{24} = 8$ and $R = 2$. The first term follows by stepping back once from $t_2$: $t_1 = \dfrac{t_2}{R} = \dfrac{24}{2} = 12$, so $a = 12$. Because this sequence is numbered from $t_1$, the explicit rule uses the power $n - 1$: $t_n = 12 \times 2^{n-1}$. As a check, $t_2 = 12 \times 2^1 = 24$ and $t_5 = 12 \times 2^4 = 192$, as given. Finally $t_8 = 12 \times 2^7 = 12 \times 128 = 1536$.

$\therefore$ $R = 2$, the first term is $t_1 = 12$, the explicit rule is $t_n = 12 \times 2^{n-1}$, and $t_8 = 1536$.

### *Example 4 — unscaffolded*

As sunlight passes down through the water of an estuary, each metre of depth absorbs $40\,\%$ of the light entering it. The light intensity at the surface is 1000 units; let $t_n$ be the intensity at a depth of $n$ metres. Write a recurrence relation and an explicit rule for the sequence, generate the terms to $t_5$, plot the sequence, describe its behaviour, and find the first whole-number depth at which the intensity falls below 100 units.

Each metre absorbs $40\,\%$ of the light, so $60\,\%$ remains: $R = 1 - \dfrac{40}{100} = 0.6$, and the recurrence relation is $t_0 = 1000$, $t_{n+1} = 0.6\,t_n$. Iterating: $t_1 = 0.6 \times 1000 = 600$; $t_2 = 360$; $t_3 = 216$; $t_4 = 129.6$; $t_5 = 77.76$. The explicit rule is $t_n = 1000 \times 0.6^n$. Because $0 < R < 1$, the sequence shows geometric decay: the terms approach a limiting value of 0, each fall being smaller than the one before, and the graph below shows the points flattening towards the horizontal axis. For the depth: $t_4 = 129.6$ is not below 100, but $t_5 = 77.76$ is, so the intensity first falls below 100 units at a depth of 5 metres.




$\therefore\ t_0 = 1000$, $t_{n+1} = 0.6\,t_n$, with explicit rule $t_n = 1000 \times 0.6^n$; the sequence decays towards a limiting value of 0, and the intensity first drops below 100 units at a depth of 5 metres.

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## Practice questions

### *Scaffolded*

**Q1.** A sequence is defined by the recurrence relation $t_0 = 7$, $t_{n+1} = 4\,t_n$. (a) State the common ratio. (b) Generate the terms $t_1$ to $t_4$. (c) Write down the explicit rule for $t_n$. (d) Use the explicit rule to find $t_6$ without generating $t_5$. (e) Describe the behaviour of the sequence.

**Q2.** A sequence is defined by $t_0 = 972$, $t_{n+1} = \dfrac{1}{3}t_n$. (a) Generate the terms $t_1$ to $t_4$. (b) Write down the explicit rule. (c) Use the explicit rule to find $t_7$. (d) Describe the behaviour of the sequence, including its limiting value.

**Q3.** For each list of numbers below, decide whether it could be the successive terms of a geometric sequence; if it could, state the common ratio. (a) $13, 26, 52, 104$ (b) $20, 15, 10, 5$ (c) $64, 32, 16, 8$ (d) $2, 8, 18, 32$ (e) For the sequence in part (c), write down a recurrence relation and the explicit rule.

**Q4.** A revegetation project plants 625 native grass tussocks on a creek bank, and the number of tussocks (the original plants plus self-seeded ones) increases by $20\%$ each year. (a) Explain why the common ratio of the sequence of yearly counts is $R = 1.2$. (b) Write a recurrence relation for the number of tussocks $t_n$ after $n$ years. (c) Generate the terms $t_1$ to $t_4$. (d) Write down the explicit rule. (e) Predict the number of tussocks after 6 years, to the nearest whole number.

**Q5.** A geometric sequence with all terms positive is numbered from $t_0$, and $t_1 = 60$ and $t_3 = 540$. (a) Find the common ratio $R$. (b) Find the starting value $t_0$. (c) Write down the explicit rule. (d) Find $t_5$.

**Q6.** Four sequences are defined by the recurrence relations below. For each of parts (a)–(d), generate the terms $t_1$ to $t_3$ and describe the behaviour of the sequence as growth, decay, constant or oscillating. (a) $t_0 = 16$, $t_{n+1} = 1.5\,t_n$ (b) $t_0 = 250$, $t_{n+1} = 0.2\,t_n$ (c) $t_0 = 7$, $t_{n+1} = t_n$ (d) $t_0 = 5$, $t_{n+1} = -2\,t_n$ (e) One of the four sequences approaches a limiting value. Which one, and what is the limiting value?

### *Unscaffolded*

**Q7.** A sequence is defined by $t_0 = 15$, $t_{n+1} = 2\,t_n$. Write down the first six terms of the sequence and describe its behaviour.

**Q8.** Show that the sequence $10\,000, 3000, 900, 270, 81, \ldots$ is geometric, write down a recurrence relation and the explicit rule (numbering the terms from $t_0$), and find $t_7$.

**Q9.** A geometric sequence is numbered from $t_1$; its first term is $t_1 = 9$ and its common ratio is $R = 2$. Explain why the explicit rule is $t_n = 9 \times 2^{n-1}$ and not $9 \times 2^n$, and find $t_{10}$.

**Q10.** Deepa needs the value of $t_{15}$ for the sequence defined by $t_0 = 17$, $t_{n+1} = 2t_n$. Describe two methods she could use, state which method is more efficient here and why, and find $t_{15}$.

**Q11.** A tank at a water-treatment plant holds 5 kg of a dissolved chemical. Each time the tank is flushed with clean water, $60\%$ of the chemical is washed out and $40\%$ is left behind. Let $t_0 = 5$ be the starting mass of the chemical, in kilograms, and let $t_n$ be the mass remaining after the $n$th flush. Write a recurrence relation and an explicit rule for the sequence, find the mass remaining after each of the first four flushes, determine after which flush the mass first falls below 0.1 kg, and describe the behaviour of the sequence.

**Q12.** After a control program begins, the number of cane toads in a Queensland border wetland is modelled by the explicit rule $t_n = 12800 \times 0.75^n$, where $t_n$ is the number of toads $n$ years after the program begins. (a) Interpret the numbers 12800 and 0.75 in the context of the model. (b) Write the model as a recurrence relation. (c) How many toads does the model predict after 3 years? (d) Describe the long-run behaviour of the sequence, and interpret it in context.

**Q13.** A geometric sequence numbered from $t_0$ has all terms positive, with $t_1 = 36$ and $t_4 = 4.5$. Find the common ratio and the starting value $t_0$, write down the explicit rule, and find $t_6$.

**Q14.** A sequence is defined by $t_0 = 81$, $t_{n+1} = -\frac{1}{3}t_n$. Generate the terms $t_1$ to $t_4$ and describe the behaviour of the sequence.

**Q15.** On an island off the South Australian coast, a feral rabbit population is currently 30 and triples every year. Write a recurrence relation and an explicit rule for the population $t_n$ after $n$ years, generate the terms $t_1$ to $t_4$, and determine after how many years the population first exceeds 2000.

**Q16.** Two sequences are defined by A: $t_0 = 1$, $t_{n+1} = 2t_n$ and B: $u_0 = 1024$, $u_{n+1} = 0.5u_n$. Generate the terms $t_0$ to $t_5$ and $u_0$ to $u_5$, describe the behaviour of each sequence, and find the value of $n$ for which $t_n = u_n$, stating the common value.

**Q17.** The table below shows some terms of a geometric sequence whose terms are all positive, with two entries missing. Find the common ratio and the missing terms.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_n$ | 9 | ? | 144 | ? | 2304 |

**Q18.** A geometric sequence is numbered from $t_1$, with explicit rule $t_n = 8 \times 3^{n-1}$. (a) Write down the first three terms of the sequence. (b) The same list of numbers is now renumbered from $t_0$. Write down the explicit rule in the form $t_n = t_0 R^n$. (c) Use each rule to find the 8th number in the list, and explain why the two rules label it differently.

## Answers

**Q1.** (a) $R=4$. (b) $t_1=28$, $t_2=112$, $t_3=448$, $t_4=1792$. (c) $t_n=7\times 4^n$. (d) $t_6=28672$. (e) Geometric growth (increasing), since $R=4>1$. **Q2.** (a) 324, 108, 36, 12. (b) $t_n=972\times\left(\frac{1}{3}\right)^n$. (c) $t_7=\frac{4}{9}$. (d) Geometric decay, approaching the limiting value 0. **Q3.** (a) Geometric, $R=2$. (b) Not geometric. (c) Geometric, $R=0.5$. (d) Not geometric. (e) $t_0=64$, $t_{n+1}=0.5t_n$; $t_n=64\times 0.5^n$. **Q4.** (a) Each year's count is $120\%$ of the previous count. (b) $t_0=625$, $t_{n+1}=1.2t_n$. (c) 750, 900, 1080, 1296. (d) $t_n=625\times 1.2^n$. (e) About 1866 tussocks. **Q5.** (a) $R=3$. (b) $t_0=20$. (c) $t_n=20\times 3^n$. (d) $t_5=4860$. **Q6.** (a) 24, 36, 54 — growth. (b) 50, 10, 2 — decay. (c) 7, 7, 7 — constant. (d) $-10$, 20, $-40$ — oscillating. (e) The sequence in (b); limiting value 0. **Q7.** 15, 30, 60, 120, 240, 480; geometric growth — each term is double the one before ($R=2>1$). **Q8.** Ratios all equal 0.3, so geometric; $t_0=10\,000$, $t_{n+1}=0.3t_n$; $t_n=10\,000\times 0.3^n$; $t_7=2.187$. **Q9.** Reaching $t_n$ from $t_1$ takes $n-1$ multiplications by 2, so the power is $n-1$; $t_{10}=9\times 2^9=4608$. **Q10.** Iterate the recurrence 15 times, or evaluate the explicit rule $t_{15}=17\times 2^{15}$; the explicit rule is more efficient (one calculation); $t_{15}=557056$. **Q11.** $t_0=5$, $t_{n+1}=0.4t_n$; $t_n=5\times 0.4^n$; masses 2, 0.8, 0.32, 0.128 kg; first below 0.1 kg after the 5th flush ($t_5=0.0512$); decay, approaching the limiting value 0. **Q12.** (a) 12800 is the number of toads when the program begins; 0.75 means $75\%$ of the toads remain each year (a $25\%$ fall). (b) $t_0=12800$, $t_{n+1}=0.75t_n$. (c) 5400 toads. (d) Decay approaching the limiting value 0 — the model predicts the toads are effectively eliminated in the long run. **Q13.** $R=0.5$, $t_0=72$; $t_n=72\times 0.5^n$; $t_6=1.125$. **Q14.** $-27$, 9, $-3$, 1; oscillating — the signs alternate, and the sizes shrink towards 0 (since $-1<R<0$). **Q15.** $t_0=30$, $t_{n+1}=3t_n$; $t_n=30\times 3^n$; 90, 270, 810, 2430; the population first exceeds 2000 after 4 years. **Q16.** A: 1, 2, 4, 8, 16, 32 (growth); B: 1024, 512, 256, 128, 64, 32 (decay towards 0); $t_5=u_5=32$, so $n=5$ with common value 32. **Q17.** $R=4$; $t_1=36$ and $t_3=576$. **Q18.** (a) 8, 24, 72. (b) $t_n=8\times 3^n$ (with $t_0=8$). (c) Both give $8\times 3^7=17496$; it is labelled $t_8$ in the numbered-from-$t_1$ form and $t_7$ in the numbered-from-$t_0$ form.

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## Fully-worked solutions

**Q1.** (a) The rule multiplies by 4 at every step, so the common ratio is $R=4$. (b) $t_1=4\times 7=28$, $t_2=4\times 28=112$, $t_3=4\times 112=448$, $t_4=4\times 448=1792$. (c) Reaching $t_n$ from $t_0=7$ takes $n$ multiplications by 4, so $t_n=7\times 4^n$. (d) $t_6=7\times 4^6=7\times 4096=28672$. (e) Every term is 4 times the one before, so the sequence shows geometric growth (it is increasing), because $R=4>1$.

**Q2.** (a) $t_1=\frac{1}{3}\times 972=324$, $t_2=108$, $t_3=36$, $t_4=12$. (b) $t_n=972\times\left(\frac{1}{3}\right)^n$. (c) $t_7=972\times\left(\frac{1}{3}\right)^7=\frac{972}{2187}=\frac{4}{9}$. (d) Because $0<R<1$, the sequence shows geometric decay: each term is a third of the one before, every fall is smaller than the last, and the terms approach the limiting value 0 without ever reaching it.

**Q3.** (a) $\frac{26}{13}=2$, $\frac{52}{26}=2$, $\frac{104}{52}=2$: the ratio is constant, so the list is geometric with $R=2$. (b) $\frac{15}{20}=0.75$ but $\frac{10}{15}=\frac{2}{3}$: the ratios are not constant, so the list is not geometric. (It falls by equal amounts, 5 each time, not by equal factors.) (c) $\frac{32}{64}=0.5$, $\frac{16}{32}=0.5$, $\frac{8}{16}=0.5$: geometric with $R=0.5$. (d) $\frac{8}{2}=4$ but $\frac{18}{8}=2.25$: the ratios are not constant, so the list is not geometric. (e) With $t_0=64$ and $R=0.5$: the recurrence relation is $t_0=64$, $t_{n+1}=0.5t_n$, and the explicit rule is $t_n=64\times 0.5^n$.

**Q4.** (a) An increase of $20\%$ leaves $100\%+20\%=120\%$ of the tussocks, and $120\%$ as a decimal is 1.2, so each year's count is 1.2 times the previous count: $R=1.2$. (b) $t_0=625$, $t_{n+1}=1.2t_n$. (c) $t_1=1.2\times 625=750$; $t_2=1.2\times 750=900$; $t_3=1.2\times 900=1080$; $t_4=1.2\times 1080=1296$. (d) $t_n=625\times 1.2^n$. (e) $t_6=625\times 1.2^6=625\times 2.985984=1866.24$, so the model predicts about 1866 tussocks after 6 years.

**Q5.** (a) From $t_1$ to $t_3$ is 2 steps, so $t_3 = t_1 \times R^2$. Then $540 = 60R^2$, so $R^2 = 9$ and, because all terms are positive, $R = 3$. (b) Stepping back one place, $t_0 = \frac{t_1}{R} = \frac{60}{3} = 20$. (c) $t_n = 20 \times 3^n$. (d) $t_5 = 20 \times 3^5 = 20 \times 243 = 4860$.

**Q6.** (a) $t_1 = 1.5 \times 16 = 24$, $t_2 = 36$, $t_3 = 54$: every term is larger than the one before, so the sequence shows growth ( $R = 1.5 > 1$). (b) $t_1 = 0.2 \times 250 = 50$, $t_2 = 10$, $t_3 = 2$: every term is a fifth of the one before, so the sequence shows decay ($0 < R < 1$). (c) $t_1 = 7$, $t_2 = 7$, $t_3 = 7$: the sequence is constant ($R = 1$). (d) $t_1 = -2 \times 5 = -10$, $t_2 = -2 \times (-10) = 20$, $t_3 = -2 \times 20 = -40$: the signs alternate, so the sequence oscillates ($R = -2 < 0$), with the sizes of the terms growing. (e) The decaying sequence in (b): because $0 < R < 1$, its terms approach the limiting value 0.

**Q7.** $t_1 = 2 \times 15 = 30$, $t_2 = 60$, $t_3 = 120$, $t_4 = 240$, $t_5 = 480$, so the first six terms are $15, 30, 60, 120, 240, 480$. Each term is double the one before ($R = 2 > 1$), so the sequence shows geometric growth.

**Q8.** The consecutive ratios are $\frac{3000}{10\,000} = 0.3$, $\frac{900}{3000} = 0.3$, $\frac{270}{900} = 0.3$ and $\frac{81}{270} = 0.3$. The ratio is constant, so the sequence is geometric with $R = 0.3$. Numbering from $t_0 = 10\,000$, the recurrence relation is $t_0 = 10\,000$, $t_{n+1} = 0.3\,t_n$, and the explicit rule is $t_n = 10\,000 \times 0.3^n$. Then $t_7 = 10\,000 \times 0.3^7 = 10\,000 \times 0.0002187 = 2.187$.

**Q9.** The explicit rule raises $R$ to the number of multiplications needed to reach $t_n$ from the first term. Here the first term is $t_1$, and going from $t_1$ to $t_n$ takes $n - 1$ steps — for example, from $t_1$ to $t_3$ takes 2 multiplications, not 3. So the power is $n - 1$, giving $t_n = 9 \times 2^{n-1}$; the rule $9 \times 2^n$ would be correct only if the numbering started at $t_0$. Then $t_{10} = 9 \times 2^9 = 9 \times 512 = 4608$.

**Q10.** Method 1: iterate the recurrence — start at 17 and double 15 times (on CAS, type 17, ENTER, then $2 \times \text{Ans}$ and press ENTER 15 times). Method 2: evaluate the explicit rule $t_n = 17 \times 2^n$ at $n = 15$ in a single calculation. The explicit rule is more efficient for a distant term such as $t_{15}$: it needs one calculation, and there is no risk of losing count of the presses. Either way, $t_{15} = 17 \times 2^{15} = 17 \times 32768 = 557056$.

**Q11.** Each flush leaves $40\,\%$ of the chemical, so $R = 0.4$ and the recurrence relation is $t_0 = 5$, $t_{n+1} = 0.4\,t_n$, with explicit rule $t_n = 5 \times 0.4^n$. The masses remaining after the first four flushes are $t_1 = 0.4 \times 5 = 2$ kg, $t_2 = 0.8$ kg, $t_3 = 0.32$ kg and $t_4 = 0.128$ kg. Continuing, $t_4 = 0.128$ is not below 0.1, but $t_5 = 0.4 \times 0.128 = 0.0512$ is, so the mass is first below 0.1 kg after the 5th flush. Because $0 < R < 1$, the sequence shows geometric decay: the mass of the chemical approaches the limiting value 0.

**Q12.** (a) The 12800 is $t_0$, the number of toads in the wetland when the program begins. The 0.75 is the common ratio: each year the number of toads is $75\,\%$ of the previous year's number — that is, the population falls by $25\,\%$ every year. (b) $t_0 = 12800$, $t_{n+1} = 0.75\,t_n$. (c) $t_3 = 12800 \times 0.75^3 = 12800 \times 0.421875 = 5400$ toads. (d) Because $0 < R < 1$, the sequence decays towards the limiting value 0: the model predicts that the number of toads keeps falling, by less each year, so the toads are effectively eliminated from the wetland in the long run.

**Q13.** From $t_1$ to $t_4$ is 3 steps, so $t_4 = t_1 \times R^3$. Then $4.5 = 36R^3$, so $R^3 = \frac{4.5}{36} = 0.125$ and $R = 0.5$. Stepping back one place, $t_0 = \frac{t_1}{R} = \frac{36}{0.5} = 72$, so the explicit rule is $t_n = 72 \times 0.5^n$. Then $t_6 = 72 \times 0.5^6 = \frac{72}{64} = 1.125$.

**Q14.** $t_1 = -\frac{1}{3} \times 81 = -27$; $t_2 = -\frac{1}{3} \times (-27) = 9$; $t_3 = -\frac{1}{3} \times 9 = -3$; $t_4 = -\frac{1}{3} \times (-3) = 1$. Because $R$ is negative, the terms alternate in sign — the sequence oscillates — and because $-1 < R < 0$, the sizes of the terms ($81, 27, 9, 3, 1, \ldots$ ) shrink towards 0.

**Q15.** Tripling every year means $R = 3$, so $t_0 = 30$, $t_{n+1} = 3\,t_n$, with explicit rule $t_n = 30 \times 3^n$. Iterating: $t_1 = 90$, $t_2 = 270$, $t_3 = 810$, $t_4 = 2430$. Now $t_3 = 810$ does not exceed 2000, but $t_4 = 2430$ does, so the population first exceeds 2000 after 4 years.

**Q16.** Sequence A doubles from 1: $t_0$ to $t_5$ are $1, 2, 4, 8, 16, 32$ — geometric growth ($R=2>1$). Sequence B halves from 1024: $u_0$ to $u_5$ are $1024, 512, 256, 128, 64, 32$ — geometric decay ($0<R<1$), approaching the limiting value 0. Comparing the lists term by term, the sequences first agree at $n=5$, where $t_5=u_5=32$. So the common value is 32, at $n=5$.

**Q17.** From $t_0=9$ to $t_2=144$ is 2 steps, so $144=9R^2$, giving $R^2=16$ and, since all terms are positive, $R=4$. The missing terms are $t_1=4\times9=36$ and $t_3=4\times144=576$. Check with the last column: $t_4=4\times576=2304$, which agrees with the table.

**Q18.** (a) $t_1=8\times3^0=8$, $t_2=8\times3^1=24$, $t_3=8\times3^2=72$. (b) Renumbered from $t_0$, the first term is $t_0=8$ and the common ratio is still 3, so the rule is $t_n=8\times3^n$. (c) With numbering from $t_1$, the 8th number in the list is $t_8=8\times3^7=17496$; with numbering from $t_0$, the 8th number is $t_7=8\times3^7=17496$. The value is the same because in both cases it lies 7 steps (7 multiplications by 3) after the first term — the two rules simply give the same number different labels, $t_8$ in one numbering and $t_7$ in the other.