Foundation Mathematics Units 3 & 4

Chapter 3 — Ratio, proportion and percentage

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Contents

Theory

A **ratio** compares two or more quantities *of the same kind*, measured in the same unit. The ratio of flour to sugar "3 parts to 2 parts" is written $3:2$ and read "three to two". A ratio has no units — it tells you the *relative* sizes of the quantities, not their actual amounts.

Simplifying ratios.

A ratio is in **simplest form** when the numbers are whole and share no common factor. Divide every part by the highest common factor.

- Whole numbers: $12:18$ — divide both by 6 — becomes $2:3$.
- Different units: convert to the **same unit first**. 45 minutes $:2$ hours becomes $45:120$ (minutes), which simplifies to $3:8$.
- Fractions or decimals: multiply every part by the same number to clear them. $0.6:0.9$ becomes $6:9$ (times 10), which simplifies to $2:3$.

Equivalent ratios.

Multiplying or dividing every part of a ratio by the same number gives an **equivalent ratio**: $2:3=4:6=20:30$. To find a missing part, find the multiplier: in $3:5=12:\square$, the first part was multiplied by 4 (since $12=3\times4$), so $\square=5\times4=20$.

Dividing a quantity in a ratio.

To divide an amount in a given ratio:

1. Add the parts of the ratio to find the **total number of parts**.
2. Divide the amount by the total to find the **value of one part**.
3. Multiply each ratio number by the value of one part.

For example, to share $\$840$ in the ratio $3:4$: the total is $3+4=7$ parts, one part is $\$840\div7=\$120$, so the shares are $3\times\$120=\$360$ and $4\times\$120=\$480$. (Check: $\$360+\$480=\$840$.) The same method works for three-part ratios such as $2:3:6$.

The unitary method and proportion.

Two quantities are **in proportion** when they increase or decrease at the same rate. The **unitary method** solves these problems by first finding the value of **one** unit. If 6 identical pens cost $\$9$, then one pen costs $\$9\div6=\$1.50$, so 10 pens cost $10\times\$1.50=\$15$.

Scale ratios (maps and plans).

A **scale** such as $1:200$ means 1 unit on the drawing represents 200 of the same units in real life.

- To find a **real** length: multiply the drawing length by the scale number.
- To find a **drawing** length: divide the real length by the scale number.

Always convert the answer to a sensible unit at the end (e.g. cm to m or km).

Worked examples

Example 1 — scaffolded

Write each ratio in simplest form: (a) $12:18$, (b) 45 minutes $:2$ hours, (c) $0.6:0.9$.

Working	Comment
(a) The HCF of 12 and 18 is 6: $12:18=2:3$.	Divide both parts by the highest common factor.
(b) 2 hours $=120$ minutes, so the ratio is $45:120$.	Convert to the same unit before simplifying.
$45:120$ — divide both by 15 — $=3:8$.	HCF of 45 and 120 is 15.
(c) $0.6:0.9=6:9$ (times 10) $=2:3$.	Clear the decimals, then simplify.

Example 2 — scaffolded

A prize of $\$550$ is shared between three clubs in the ratio $2:3:6$. How much does each club receive?

Working	Comment
Total parts $=2+3+6=11$.	Add the parts of the ratio.
One part $=\$550\div 11=\$50$.	Divide the amount by the total number of parts.
Shares $=2\times\$50,3\times\$50,6\times\$50=\$100,\$150,\$300$.	Multiply each ratio number by one part.
Check: $\$100+\$150+\$300=\$550$.	The shares must add back to the total.

Example 3 — unscaffolded

A recipe for 4 people uses 300 g of flour. Using the same recipe, how much flour is needed for 10 people?

Using the unitary method, the flour per person is

$$300\div 4=75\,\text{g per person}.$$

For 10 people this is $10\times 75=750$ g.

∴ 750 g of flour is needed for 10 people.

Example 4 — unscaffolded

Two towns are 8 cm apart on a map with scale $1:25\,000$. Find the real distance between the towns, in kilometres.

The scale $1:25\,000$ means 1 cm on the map represents $25\,000$ cm in reality. The real distance is

$$8\times 25\,000=200\,000\,\text{cm}.$$

Converting: $200\,000\div 100=2000$ m $=2$ km.

∴ the towns are 2 km apart.

Practice questions

Scaffolded

Q1. Write each ratio in simplest form. (a) $20:35$ (b) 750 mL : 2 L (c) $1.5:2.5$

Q2. Find the missing number in each equivalent ratio. (a) $3:5=12:\square$ (b) $2:7=\square:56$

Q3. Divide $\$360$ between two people in the ratio $5:4$. (a) Find the total number of parts. (b) Find the value of one part. (c) Find each person's share.

Q4. Divide 720 g of trail mix in the ratio $2:3:4$ (nuts : seeds : fruit). (a) Find the total number of parts. (b) Find the mass of each ingredient.

Q5. Six identical drink bottles cost $\$21$. (a) Find the cost of one bottle. (b) Find the cost of 10 bottles.

Q6. A house plan is drawn to a scale of $1:150$. (a) A wall is 4 cm long on the plan. Find its real length in metres. (b) A fence is 9 m long in reality. Find its length on the plan, in centimetres.

Unscaffolded

Q7. Green and white paint are mixed in the ratio $3:2$ to make 15 L of paint. How many litres of each colour are used?

Q8. A cake recipe for 6 people uses 240 g of sugar. How much sugar is needed to make the cake for 15 people?

Q9. A business profit of $\$9600$ is shared between three partners in the ratio $5:3:2$. Find each partner's share.

Q10. A two-stroke fuel mix requires petrol and oil in the ratio $25:1$. How many millilitres of oil should be added to 5 L of petrol?

Q11. On a map with scale $1:50\,000$, two landmarks are 12 cm apart. Find the real distance between them, in kilometres.

Q12. A scale drawing of a machine part uses a scale of $1:20$. The real part is 3 m long. How long is the part in the drawing, in centimetres?

Q13. Concrete is mixed from cement, sand and gravel in the ratio $1:2:4$. A job uses 21 buckets of concrete in total. How many buckets of each material are needed?

Q14. Write the ratio $2.4:0.8$ in simplest form.

Q15. Six apples cost $\$4.20$. Using the unitary method, find the cost of 10 apples.

Q16. A cordial is mixed with water in the ratio $1:4$ (cordial : water). If 200 mL of cordial is used, how much water is added, and what is the total volume of the drink?

Q17. An inheritance of $\$525$ is divided between two charities in the ratio $4:3$. Find each charity's share.

Q18. Two towns are 6 cm apart on a map with scale $1:250\,000$. Find the real distance between them, in kilometres.

Answers

Q1. (a) $4:7$. (b) $3:8$. (c) $3:5$. **Q2.** (a) 20. (b) 16. **Q3.** (a) 9 parts. (b) $\$40$. (c) $\$200$ and $\$160$. **Q4.** (a) 9 parts. (b) nuts 160 g, seeds 240 g, fruit 320 g. **Q5.** (a) $\$3.50$. (b) $\$35$. **Q6.** (a) 6 m. (b) 6 cm. **Q7.** 9 L green and 6 L white. **Q8.** 600 g. **Q9.** $\$4800$, $\$2880$ and $\$1920$. **Q10.** 200 mL of oil. **Q11.** 6 km. **Q12.** 15 cm. **Q13.** cement 3, sand 6, gravel 12 buckets. **Q14.** $3:1$. **Q15.** $\$7$. **Q16.** 800 mL of water; total 1000 mL (1 L). **Q17.** $\$300$ and $\$225$. **Q18.** 15 km.

Fully-worked solutions

Q1. (a) HCF of 20 and 35 is 5: $20:35 = 4:7$. (b) 2 L $= 2000$ mL, so $750:2000$; dividing both by 250 gives $3:8$. (c) $1.5:2.5 = 15:25$ (times 10) $= 3:5$ (divide by 5).

Q2. (a) $12 = 3 \times 4$, so multiply the second part by 4: $\square = 5 \times 4 = 20$. (b) $56 = 7 \times 8$, so multiply the first part by 8: $\square = 2 \times 8 = 16$.

Q3. (a) $5 + 4 = 9$ parts. (b) one part $= \$360 \div 9 = \$40$. (c) shares $= 5 \times \$40 = \$200$ and $4 \times \$40 = \$160$ (check $\$200 + \$160 = \$360$).

Q4. (a) $2 + 3 + 4 = 9$ parts. (b) one part $= 720 \div 9 = 80$ g, so nuts $= 2 \times 80 = 160$ g, seeds $= 3 \times 80 = 240$ g, fruit $= 4 \times 80 = 320$ g.

Q5. (a) one bottle $= \$21 \div 6 = \$3.50$. (b) 10 bottles $= 10 \times \$3.50 = \$35$.

Q6. (a) real length $= 4 \times 150 = 600$ cm $= 6$ m. (b) 9 m $= 900$ cm, so plan length $= 900 \div 150 = 6$ cm.

Q7. Total parts $= 3 + 2 = 5$; one part $= 15 \div 5 = 3$ L. Green $= 3 \times 3 = 9$ L, white $= 2 \times 3 = 6$ L.

Q8. Sugar per person $= 240 \div 6 = 40$ g. For 15 people $= 15 \times 40 = 600$ g.

Q9. Total parts $= 5 + 3 + 2 = 10$; one part $= \$9600 \div 10 = \$960$. Shares $= 5 \times \$960 = \$4800$, $3 \times \$960 = \$2880$, $2 \times \$960 = \$1920$.

Q10. The ratio $25:1$ means 25 mL of petrol per 1 mL of oil. 5 L $= 5000$ mL of petrol, so oil $= 5000 \div 25 = 200$ mL.

Q11. Real distance $= 12 \times 50\,000 = 600\,000$ cm $= 6000$ m $= 6$ km.

Q12. 3 m $= 300$ cm, so the drawing length $= 300 \div 20 = 15$ cm.

Q13. Total parts $= 1 + 2 + 4 = 7$; one part $= 21 \div 7 = 3$ buckets. Cement $= 3$, sand $= 2 \times 3 = 6$, gravel $= 4 \times 3 = 12$ buckets.

Q14. $2.4:0.8 = 24:8$ (times 10) $= 3:1$ (divide by 8).

Q15. One apple $= \$4.20 \div 6 = \$0.70$; ten apples $= 10 \times \$0.70 = \$7$.

Q16. Water $= 4 \times 200 = 800$ mL; total volume $= 200 + 800 = 1000$ mL $= 1$ L.

Q17. Total parts $= 4 + 3 = 7$; one part $= \$525 \div 7 = \$75$. Shares $= 4 \times \$75 = \$300$ and $3 \times \$75 = \$225$.

Q18. Real distance $= 6 \times 250\,000 = 1\,500\,000$ cm $= 15\,000$ m $= 15$ km.

Theory

Two quantities are in **direct proportion** when they increase (or decrease) at the same rate — double one and the other doubles, treble one and the other trebles. If you buy petrol at a fixed price per litre, the **cost is directly proportional to the number of litres**: twice the fuel costs twice as much. Direct proportion is one of the most common relationships in everyday measurement, money and work.

The rule $y = kx$.

If y is directly proportional to x (written $y \propto x$), then

$$y = kx,$$

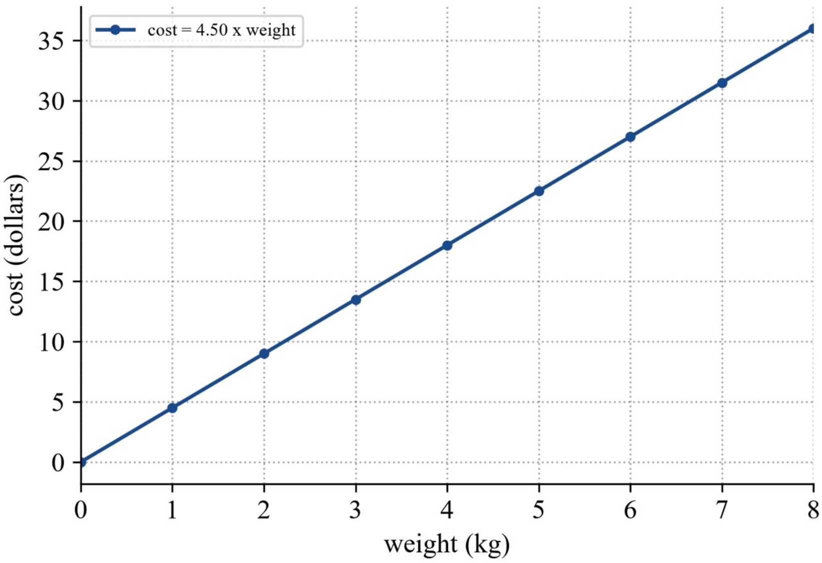
where k is a fixed number called the **constant of proportionality**. Rearranging gives $k = \frac{y}{x}$, so k is just the **value of y for one unit of x** — a price per litre, a wage per hour, a mass per cubic metre.

Recognising direct proportion.

A relationship is direct proportion when **either** of these holds:

- the **ratio $\frac{y}{x}$ is the same** for every pair of values, or
- the **graph is a straight line that passes through the origin** $(0, 0)$.

The graph below shows the cost of apples at $\$4.50$ per kilogram. It is a straight line through the origin, because 0 kg costs $\$0$; its steepness (gradient) is the constant $k = 4.50$.



A straight line that does **not** pass through the origin — for example $y = 2x + 3$ — is **not** direct proportion: there is a fixed amount even when $x = 0$, and the ratio $\frac{y}{x}$ changes.

Finding k and making predictions.

Given one pair of matching values, find $k = \frac{y}{x}$, then use $y = kx$ to predict any other value — either y from a known x, or x from a known y (by dividing).

The unitary method.

The same idea is often written as the **unitary method**: find the value of **one** unit first, then multiply. If 3 hours of work pays $\$75$, then one hour pays $\$75 \div 3 = \$25$ (this is k), so 5 hours pays $5 \times \$25 = \$125$.

Worked examples

Example 1 — scaffolded

A casual worker is paid a fixed hourly rate. She earns $\$60$ for 4 hours. Assuming pay is directly proportional to hours worked, find her pay for 7 hours and the hours she must work to earn $\$180$.

Working	Comment
$k = \dfrac{y}{x} = \dfrac{60}{4} = 15$.	The constant is the pay for one hour: $\$15$/h.
Pay $= k\,x = 15 \times 7 = \$105$.	Use $y = k\,x$ with $x = 7$ hours.
Hours $= \dfrac{y}{k} = \dfrac{180}{15} = 12$ hours.	To find x, divide the pay by k.

Example 2 — scaffolded

Apples are sold by weight at a fixed price per kilogram. 3 kg cost $\$13.50$. Find the cost of 5 kg and of 8 kg.

Working	Comment
$k = \dfrac{13.50}{3} = 4.50$.	Cost of 1 kg is $\$4.50$ — the constant of proportionality.
5 kg: cost $= 4.50 \times 5 = \$22.50$.	$y = k\,x$ with $x = 5$.
8 kg: cost $= 4.50 \times 8 = \$36.00$.	$y = k\,x$ with $x = 8$.

Example 3 — unscaffolded

A car travels at a constant speed, covering 240 km in 3 hours. If distance is directly proportional to time, how far does it travel in 5 hours, and how far in 7 hours?

Since distance $\propto$ time, the constant is the speed:

$$k = \frac{240}{3} = 80\,\text{km/h}.$$

Then distance $= 80 \times \text{time}$, so in 5 hours it travels $80 \times 5 = 400$ km, and in 7 hours it travels $80 \times 7 = 560$ km.

$\therefore$ the car travels 400 km in 5 hours and 560 km in 7 hours.

Example 4 — unscaffolded

The table shows matching values of two quantities x and y.

x	2	4	5	8
y	7	14	17.5	28

Show that y is directly proportional to x, state the constant of proportionality, and find y when $x = 6$.

Checking the ratio $\dfrac{y}{x}$ for each column:

$$\frac{7}{2} = 3.5, \frac{14}{4} = 3.5, \frac{17.5}{5} = 3.5, \frac{28}{8} = 3.5.$$

The ratio is the same every time, so y is directly proportional to x with constant $k=3.5$. Then $y=3.5x$, so when $x=6$, $y=3.5\times 6=21$.

$\therefore\ y\propto x$ with $k=3.5$, and $y=21$ when $x=6$.

Practice questions

Scaffolded

Q1. A worker's pay is directly proportional to the hours worked. She earns $\$95$ for 5 hours. (a) Find the constant of proportionality (the hourly rate). (b) Find her pay for 8 hours. (c) Find the hours she must work to earn $\$228$.

Q2. The cost of fabric is directly proportional to its length. 4 m costs $\$26$. (a) Find the cost per metre. (b) Find the cost of 7 m. (c) Find the length that costs $\$52$.

Q3. Water flows from a tap at a constant rate, filling 18 L in 3 minutes. (a) Find the flow rate (litres per minute). (b) Find the volume in 7 minutes. (c) Find the time to fill 48 L.

Q4. On one day, A$\$1$ is worth 95 yen, and conversions are directly proportional. (a) State the constant of proportionality. (b) Convert A$\$40$ to yen. (c) Convert 1710 yen to Australian dollars.

Q5. A muffin recipe uses flour in direct proportion to the number of muffins: 8 muffins need 200 g of flour. (a) Find the flour needed per muffin. (b) Find the flour for 20 muffins. (c) Find how many muffins can be made with 300 g of flour.

Q6. A direct-variation graph is a straight line through the origin passing through the point $(4,30)$. (a) Find the constant of proportionality k. (b) Find y when $x=10$. (c) Find x when $y=60$.

Unscaffolded

Q7. For each table, decide whether y is directly proportional to x.

x	1	2	3	x	1	2	3	x	2	4	6
y	4	8	12	y	5	7	9	y	5	9	13

Table A (left), Table B (middle), Table C (right).

Q8. Apples cost $\$2.50$ per kg. Find the cost of 6.4 kg.

Q9. A cyclist rides at a constant speed, covering 60 km in 4 hours. How far does she ride in 6 hours?

Q10. A casual is paid $\$23.40$ per hour, with pay directly proportional to hours. Find the pay for a 38-hour week.

Q11. Paint covers a wall in direct proportion to its volume: 3 L covers 24 m². How many litres are needed to cover 56 m²?

Q12. On a given day, 1 US dollar is worth 1.52 Australian dollars, and conversions are directly proportional. (a) Convert 250 US dollars to Australian dollars. (b) Convert 760 Australian dollars to US dollars.

Q13. A direct-variation graph passes through the origin and the point $(5,20)$. Find the value of x when $y=52$.

Q14. A phone plan charges $y=4x+5$ dollars for x gigabytes of data. Explain, using both the ratio $\frac{y}{x}$ and the graph, why y is **not** directly proportional to x.

Q15. A pump moves water at a constant rate, delivering 2500 L in 20 minutes. How long does it take to deliver 500 L?

Q16. The cost of printing flyers is directly proportional to the number printed: 150 flyers cost $\$9$. (a) Find the cost of 400 flyers. (b) Find how many flyers can be printed for $\$30$.

Q17. For a spinning gear, the number of teeth that pass a point is directly proportional to the number of turns: 5 turns pass 120 teeth. How many teeth pass in 8 turns?

Q18. The mass of a steel bar is directly proportional to its volume. A bar of volume 0.5 m^3 has a mass of 3925 kg. Find the mass of a steel bar of volume 0.2 m^3.

Answers

Q1. (a) $\$19$/h. (b) $\$152$. (c) 12 hours. **Q2.** (a) $\$6.50$/m. (b) $\$45.50$. (c) 8 m. **Q3.** (a) 6 L/min. (b) 42 L. (c) 8 minutes. **Q4.** (a) $k=95$ (yen per dollar). (b) 3800 yen. (c) A$\$18$. **Q5.** (a) 25 g. (b) 500 g. (c) 12 muffins. **Q6.** (a) $k=7.5$. (b) 75. (c) 8. **Q7.** Table A: yes ($y/x=4$ throughout). Table B: no ($5, 3.5, 3$ — not constant). Table C: no ($2.5, 2.25, \approx 2.17$ — not constant). **Q8.** $\$16.00$. **Q9.** 90 km. **Q10.** $\$889.20$. **Q11.** 7 L. **Q12.** (a) A$\$380$. (b) US $\$500$. **Q13.** 13. **Q14.** The ratio y/x is not constant (e.g. 9 at $x=1$, 6.5 at $x=2$) and the graph cuts the y-axis at 5, not the origin. **Q15.** 4 minutes. **Q16.** (a) $\$24$. (b) 500 flyers. **Q17.** 192 teeth. **Q18.** 1570 kg.

Fully-worked solutions

Q1. (a) $k=\frac{95}{5}=19$, i.e. $\$19$ per hour. (b) Pay $=19\times 8=\$152$. (c) Hours $=\frac{228}{19}=12$.

Q2. (a) $k=\frac{26}{4}=6.5$, i.e. $\$6.50$ per metre. (b) Cost $=6.5\times 7=\$45.50$. (c) Length $=\frac{52}{6.5}=8$ m.

Q3. (a) $k=\frac{18}{3}=6$ L/min. (b) Volume $=6\times 7=42$ L. (c) Time $=\frac{48}{6}=8$ minutes.

Q4. (a) $k=\frac{95}{1}=95$ yen per dollar. (b) $95\times 40=3800$ yen. (c) $\frac{1710}{95}=18$, i.e. A$\$18$.

Q5. (a) $k=\frac{200}{8}=25$ g per muffin. (b) $25\times 20=500$ g. (c) $\frac{300}{25}=12$ muffins.

Q6. (a) $k=\frac{30}{4}=7.5$. (b) $y=7.5\times 10=75$. (c) $x=\frac{60}{7.5}=8$.

Q7. Table A: $\frac{4}{1}=\frac{8}{2}=\frac{12}{3}=4$, constant, so **yes** — directly proportional with $k=4$. Table B: $\frac{5}{1}=5$, $\frac{7}{2}=3.5$, $\frac{9}{3}=3$ — the ratio changes, so **no**. Table C: $\frac{5}{2}=2.5$, $\frac{9}{4}=2.25$, $\frac{13}{6}\approx 2.17$ — not constant, so **no**.

Q8. $k=\$2.50$ per kg, so cost $=2.50\times 6.4=\$16.00$.

Q9. $k=\frac{60}{4}=15$ km/h, so distance $=15\times 6=90$ km.

Q10. Pay $=23.40\times 38=\$889.20$.

Q11. $k=\frac{24}{3}=8$ m² per litre, so litres $=\frac{56}{8}=7$ L.

Q12. (a) $1.52\times 250=\$380$ Australian. (b) $\frac{760}{1.52}=500$ US dollars.

Q13. $k=\frac{20}{5}=4$, so $y=4x$; when $y=52$, $x=\frac{52}{4}=13$.

Q14. For direct proportion the ratio $\frac{y}{x}$ must be constant and the graph must pass through the origin. Here $\frac{y}{x}=\frac{4x+5}{x}$ changes with x (it is 9 when $x=1$ and 6.5 when $x=2$), and at $x=0$ the cost is $y=5$, so the graph cuts the y-axis at $(0,5)$, not the origin. Both tests fail, so y is not directly proportional to x (the fixed $\$5$ is the reason).

Q15. $k=\frac{2500}{20}=125$ L/min, so time $=\frac{500}{125}=4$ minutes.

Q16. $k = \frac{9}{150} = 0.06$, i.e. $\$0.06$ per flyer. (a) Cost $= 0.06 \times 400 = \$24$. (b) Flyers $= \frac{30}{0.06} = 500$.

Q17. $k = \frac{120}{5} = 24$ teeth per turn, so $24 \times 8 = 192$ teeth.

Q18. $k = \frac{3925}{0.5} = 7850$ kg/m³ (the density), so mass $= 7850 \times 0.2 = 1570$ kg.

Theory

In **direct** variation (Section 3D) two quantities grow together: double one and the other doubles. **Inverse variation** (also called *indirect* variation) is the opposite relationship — as one quantity **increases**, the other **decreases** in the same proportion. If you drive a fixed distance, going faster means the trip takes *less* time; if more people share a fixed bill, each person pays *less*.

The rule.

Two quantities x and y are in **inverse variation** if their **product is constant**:

$$x\,y=k\text{, equivalently } y=\frac{k}{x},$$

where k is the **constant of proportionality** (a fixed number). We say "y varies inversely with x" or "y is inversely proportional to x".

Recognising inverse variation.

The test is the **constant product**: if $x\times y$ gives the same value for every pair, the variation is inverse. Compare the two tables below.

x	2	4	6	8
y	12	6	4	3

x	2	4	6	8
y	6	12	18	24

Table A (left), Table B (right).

In **Table A** the product is the same for every pair: $2\times 12=4\times 6=6\times 4=8\times 3=24$, so Table A is **inverse** variation with $k=24$. In **Table B** the product changes ($2\times 6=12$, but $4\times 12=48$); there it is the *ratio* $y/x=3$ that stays constant, so Table B is **direct** variation, $y=3x$. In inverse variation the **product** is fixed; in direct variation the **ratio** is fixed.

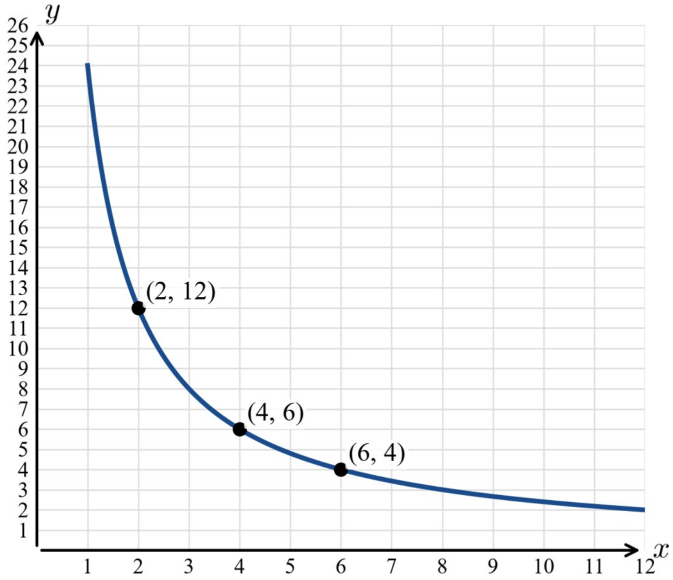
Finding the constant and making predictions.

To solve an inverse-variation problem:

1. Find k by multiplying a known pair: $k=x\times y$.
2. Write the rule $y=\frac{k}{x}$.
3. Substitute to find the missing value — given x, compute $y=\frac{k}{x}$; given y, rearrange to $x=\frac{k}{y}$.

The graph.

The graph of $y=\frac{k}{x}$ is a **curve**, not a straight line. As x gets larger the curve falls towards the horizontal axis (but never reaches it); as x gets smaller y grows very large. This curved shape is the visual signature of inverse variation — quite different from the straight line through the origin of direct variation.



The curve above is $y=\frac{24}{x}$; the marked points $(2,12)$, $(4,6)$ and $(6,4)$ all have product 24.

In context.

Inverse variation appears whenever a **fixed total** is shared or covered:

- **speed and time** for a fixed distance: $\text{time}=\frac{\text{distance}}{\text{speed}}$;
- **workers and time** for a fixed job: more workers, fewer days (a constant number of "worker-days");
- **people and cost** for a fixed bill: $\text{cost each}=\frac{\text{total}}{\text{number of people}}$.

Always check the relationship makes sense: in inverse variation, when one quantity is multiplied by a number, the other is **divided** by the same number.

Worked examples

Example 1 — scaffolded

A delivery van must travel 240 km. The time t (hours) for the trip depends on the average speed v (km/h). Show that t varies inversely with v, and find the time when the van averages 80 km/h.

Working	Comment
$t=\frac{\text{distance}}{\text{speed}}=\frac{240}{v}$.	Time = distance ÷ speed; the distance 240 is fixed.
This has the form $t=\frac{k}{v}$ with $k=240$, so t varies inversely with v.	A constant k over v is inverse variation.
Check: at $v=60$, $t=\frac{240}{60}=4$; at $v=120$, $t=\frac{240}{120}=2$. Product $vt=240$ each time.	Doubling the speed halves the time — the inverse pattern.
At $v=80$: $t=\frac{240}{80}=3$ hours.	Substitute $v=80$ into the rule.

Example 2 — scaffolded

It takes 6 landscapers 12 days to build a garden. Assuming everyone works at the same rate, find how long 8 landscapers would take.

Working	Comment
Total work $=6\times 12=72$ worker-days, so $k=72$.	The number of worker-days to finish the job is fixed.
The rule is $d=\frac{72}{n}$, where n is the number of landscapers and d the days.	More workers ⇒ fewer days: inverse variation.
With $n=8$: $d=\frac{72}{8}=9$ days.	Substitute $n=8$.
Check: $8\times 9=72$ worker-days, as required.	The product returns the constant.

Example 3 — unscaffolded

The quantity y varies inversely with x, and $y=15$ when $x=4$. Find the constant of proportionality, then find y when $x=10$, and x when $y=5$.

Since y varies inversely with x, $y=\frac{k}{x}$ with $k=x\,y$. Using the given pair,

$$k=4\times 15=60\text{, so } y=\frac{60}{x}.$$

When $x=10$, $y=\frac{60}{10}=6$. When $y=5$, rearrange to $x=\frac{60}{y}=\frac{60}{5}=12$.

$\therefore k=60$, with $y=6$ when $x=10$, and $x=12$ when $y=5$.

Example 4 — unscaffolded

A community group wins a prize of $\$1800$, to be shared equally among the members who attend the presentation. Find the amount each person receives if 5, 6 or 9 members attend, and explain how the share varies with the number attending.

Let p be the amount each person receives and n the number attending. The prize is fixed, so $p=\frac{1800}{n}$ (a constant 1800 divided by n) — inverse variation.

$$n=5:p=\frac{1800}{5}=\$360\,;n=6:p=\frac{1800}{6}=\$300\,;n=9:p=\frac{1800}{9}=\$200.$$

$\therefore$ the share varies **inversely** with the number attending — the more members who attend, the less each receives, with the product ($n\times p$) always equal to $\$1800$.

Practice questions

Scaffolded

Q1. The quantity y varies inversely with x. Some values are shown.

x	2	3	4	6	9
y	18				

(a) Find the constant of proportionality k.

(b) Write the rule connecting y and x.

(c) Copy and complete the table.

Q2. A cyclist rides a fixed route of 360 km over several days. The time t (hours) depends on the average speed v (km/h). (a) Write a rule for t in terms of v. (b) Find the time at an average speed of 90 km/h, and at 60 km/h. (c) Find the average speed needed to complete the route in 5 hours.

Q3. A building job takes 4 workers 15 days. (a) Find the number of worker-days for the job. (b) Write a rule for the number of days d in terms of the number of workers n. (c) Find how long the job would take with 5 workers, and with 6 workers.

Q4. For each set of values below, state whether the variation is **direct**, **inverse** or **neither**, giving a reason. (a) x: 1, 2, 3, 4 and y: 5, 10, 15, 20. (b) x: 1, 2, 3, 6 and y: 12, 6, 4, 2. (c) x: 1, 2, 3, 4 and y: 2, 5, 10, 17.

Q5. The quantity y varies inversely with x, and $y=8$ when $x=3$. (a) Find k. (b) Find y when $x=6$. (c) Find x when $y=2$.

Q6. A restaurant bill of $\$480$ is split equally among a group of friends. (a) Write a rule for the amount a each pays in terms of the number of friends n. (b) Find the amount each pays if there are 4, 6 or 8 friends. (c) Explain what happens to the amount each pays as more friends join.

Unscaffolded

Q7. p varies inversely with q, and $p=12$ when $q=5$. Find k, then find p when $q=4$, and q when $p=10$.

Q8. A truck must cover 500 km. Write a rule for the travel time t (hours) in terms of the average speed v (km/h), and find the time at 100 km/h and at 80 km/h.

Q9. Eight people can paint a fence in 6 days. Assuming the same work rate, find how long 12 people would take.

Q10. For a fixed amount of gas at constant temperature, the pressure P and volume V are inversely related (Boyle's law), $PV=k$. When $V=3$ litres, $P=200$ kPa. Find k, then find the pressure when the volume is increased to 5 litres.

Q11. A prize pool of $\$720$ is shared equally among the winners. Find the amount each winner receives if there are 9 winners, and if there are 12 winners.

Q12. As x is doubled, y is halved, and the pair $x=3$, $y=20$ fits the relationship. State whether the variation is direct or inverse, find k, and write the rule connecting x and y.

Q13. y varies inversely with x, and $y=2.5$ when $x=8$. Find y when $x=4$, and y when $x=10$.

Q14. A circuit runs at a fixed voltage of 240 volts. The current I (amps) is inversely proportional to the resistance R (ohms), with $I=\frac{240}{R}$. Find the current when $R=60$ ohms and when $R=48$ ohms.

Q15. Three pumps can empty a tank in 8 hours. Working at the same rate, find how long 4 pumps would take.

Q16. The number of days d to complete an order varies inversely with the number of machines m used, and 5 machines complete the order in 12 days. Find the number of days needed with 10 machines and with 6 machines.

Q17. The time to fill a swimming pool varies inversely with the number of hoses used. With 2 hoses the pool fills in 9 hours. Find how long it takes with 3 hoses, and explain why using 6 hoses does not fill it in "no time".

Q18. Explain the difference between direct and inverse variation in terms of what stays constant (the ratio versus the product), and give one everyday example of each.

Answers

Q1. (a) $k=36$. (b) $y=\frac{36}{x}$. (c) $y=12, 9, 6, 4$. **Q2.** (a) $t=\frac{360}{v}$. (b) 4 h; 6 h. (c) 72 km/h. **Q3.** (a) 60 worker-days. (b) $d=\frac{60}{n}$. (c) 12 days; 10 days. **Q4.** (a) direct (constant ratio $y/x=5$). (b) inverse (constant product $xy=12$). (c) neither (ratio and product both change). **Q5.** (a) $k=24$. (b) $y=4$. (c) $x=12$. **Q6.** (a) $a=\frac{480}{n}$. (b) $\$120$; $\$80$; $\$60$. (c) each pays less as n increases. **Q7.** $k=60$; $p=15$; $q=6$. **Q8.** $t=\frac{500}{v}$; 5 h; 6.25 h. **Q9.** 4 days. **Q10.** $k=600$; $P=120$ kPa. **Q11.** $\$80$; $\$60$. **Q12.** inverse; $k=60$; $y=\frac{60}{x}$. **Q13.** $y=5$; $y=2$. **Q14.** 4 amps; 5 amps. **Q15.** 6 hours. **Q16.** 6 days; 10 days. **Q17.** 6 hours; using more hoses keeps dividing the same fixed filling-work, so the time keeps falling but never reaches zero. **Q18.** In direct variation the **ratio** y/x is constant ($y=kx$, a line through the origin); in inverse variation the **product** xy is constant ($y=k/x$, a curve). Examples: direct — pay for hours worked; inverse — time to travel a fixed distance at different speeds.

Fully-worked solutions

Q1. (a) The product is constant: $k=2\times 18=36$. (b) $y=\frac{36}{x}$. (c) $x=3: y=\frac{36}{3}=12$; $x=4: y=9$; $x=6: y=6$; $x=9: y=4$.

Q2. (a) $t=\frac{\text{distance}}{\text{speed}}=\frac{360}{v}$. (b) $v=90: t=\frac{360}{90}=4$ h; $v=60: t=\frac{360}{60}=6$ h. (c) $t=5$ gives $v=\frac{360}{5}=72$ km/h.

Q3. (a) $4\times 15=60$ worker-days. (b) $d=\frac{60}{n}$. (c) $n=5: d=\frac{60}{5}=12$ days; $n=6: d=\frac{60}{6}=10$ days.

Q4. (a) $\frac{y}{x}=5$ for every pair (product changes), so **direct**. (b) $xy=12$ for every pair ($1\times 12, 2\times 6, 3\times 4, 6\times 2$), so **inverse**. (c) Neither the ratio nor the product is constant ($1\times 2=2$ but $2\times 5=10$), so **neither**.

Q5. (a) $k=3\times 8=24$. (b) $y=\frac{24}{6}=4$. (c) $x=\frac{24}{y}=\frac{24}{2}=12$.

Q6. (a) $a=\frac{480}{n}$. (b) $n=4: a=\frac{480}{4}=\$120$; $n=6: a=\$80$; $n=8: a=\$60$. (c) As n increases, a decreases — the fixed bill is split more ways, so each pays less.

Q7. $k=5\times 12=60$, so $p=\frac{60}{q}$. At $q=4$, $p=\frac{60}{4}=15$; when $p=10$, $q=\frac{60}{10}=6$.

Q8. $t=\frac{500}{v}$. At $v=100$, $t=5$ h; at $v=80$, $t=\frac{500}{80}=6.25$ h.

Q9. Total work $=8\times 6=48$ person-days, so $d=\frac{48}{n}$. With $n=12$, $d=\frac{48}{12}=4$ days.

Q10. $k=PV=3\times 200=600$. At $V=5$, $P=\frac{600}{5}=120$ kPa.

Q11. Share $=\frac{720}{n}$. With 9 winners, $\frac{720}{9}=\$80$; with 12 winners, $\frac{720}{12}=\$60$.

Q12. Halving y when x doubles is the inverse pattern, so the variation is **inverse**. Then $k = x\,y = 3 \times 20 = 60$ and $y = \frac{60}{x}$.

Q13. $k = 8 \times 2.5 = 20$, so $y = \frac{20}{x}$. At $x = 4$, $y = 5$; at $x = 10$, $y = 2$.

Q14. $I = \frac{240}{R}$. At $R = 60$, $I = 4$ amps; at $R = 48$, $I = \frac{240}{48} = 5$ amps.

Q15. Total work $= 3 \times 8 = 24$ pump-hours, so time $= \frac{24}{\text{pumps}}$. With 4 pumps, $\frac{24}{4} = 6$ hours.

Q16. $k = 5 \times 12 = 60$, so $d = \frac{60}{m}$. With $m = 10$, $d = 6$ days; with $m = 6$, $d = 10$ days.

Q17. Total filling work $= 2 \times 9 = 18$ hose-hours, so time $= \frac{18}{\text{hoses}}$. With 3 hoses, $\frac{18}{3} = 6$ hours. With 6 hoses the time is $\frac{18}{6} = 3$ hours — still a real time, because the same fixed amount of work is only being divided among more hoses, never removed.

Q18. In **direct** variation the ratio $\frac{y}{x} = k$ stays constant, so $y = k\,x$ and the graph is a straight line through the origin (e.g. pay for hours worked at a fixed rate). In **inverse** variation the product $x\,y = k$ stays constant, so $y = \frac{k}{x}$ and the graph is a falling curve (e.g. the time to travel a fixed distance at different speeds).

Theory

A **percentage** is a number written as a fraction of 100; the symbol $\%$ means "out of 100". Percentages let us compare and communicate proportions in a way everyone understands — a discount, a test result, a rate of tax. This section covers the everyday percentage calculations; percentage **change** (increase and decrease) follows in 3G.

Percentages, fractions and decimals.

The three forms describe the same proportion, and you switch between them freely:

- **percentage → fraction:** write it over 100 and simplify, e.g. $35\% = \dfrac{35}{100} = \dfrac{7}{20}$.
- **percentage → decimal:** divide by 100 (move the point two places left), e.g. $35\% = 0.35$.
- **fraction/decimal → percentage:** multiply by 100, e.g. $\dfrac{3}{8} = 0.375 = 37.5\%$.

Finding a percentage of a quantity.

To find a percentage of an amount, convert the percentage to a fraction or decimal and multiply:

$$p\,\%\,\text{of}\;A = \frac{p}{100} \times A\,.$$

For example, 15% of $\$240$ is $\dfrac{15}{100} \times 240 = \$36$.

Expressing one quantity as a percentage of another.

To write a "part" as a percentage of a "whole", form the fraction and multiply by 100:

$$\frac{\text{part}}{\text{whole}} \times 100\,\%\,.$$

So 18 marks out of 40 is $\dfrac{18}{40} \times 100\% = 45\%$. **The two quantities must be in the same units.**

Finding the whole from a percentage part (reverse).

Sometimes you know how much a percentage is *worth* and need the whole. Divide the known amount by the percentage (as a fraction or decimal):

$$\text{whole} = \frac{\text{part}}{p\,/\,100}\,.$$

If 40% of a number is 30, the number is $\dfrac{30}{0.4} = 75$.

GST — Goods and Services Tax.

In Australia, **GST is** 10% and is added to the price of most goods and services.

- To **add** GST, multiply the pre-GST price by 1.1 (that is, $100\% + 10\%$): a $\$120$ service costs $120 \times 1.1 = \$132$.
- To find the **GST included** in a total that already has GST, divide the total by 11 (because the total is 110% of the price, and $\dfrac{10}{110} = \dfrac{1}{11}$): an $\$88$ total contains $\dfrac{88}{11} = \$8$ of GST, so the pre-GST price was $\$80$.

Percentages over 100%.

A percentage can exceed $100\,\%$ when the "part" is larger than the "whole" — for example this year's sales might be $120\,\%$ of last year's. Then $120\,\%$ of $\$80\,000$ is $1.2\times 80\,000=\$96\,000$, and 90 compared with 60 is $\dfrac{90}{60}\times 100\,\%=150\,\%$.

Reasonableness. Always sanity-check: a percentage *of* a quantity (under $100\,\%$) is smaller than the quantity; a reverse answer (the whole) is larger than the part.

Worked examples

Example 1 — scaffolded

A jacket is priced at $\$240$. (a) A member receives $15\,\%$ off — find the value of the discount. (b) A customer pays an $\$18$ deposit on a separate $\$40$ item — express the deposit as a percentage of that item's price.

Working	Comment
(a) Discount $=\dfrac{15}{100}\times 240$.	"$15\,\%$ of $\$240$": convert and multiply.
$=0.15\times 240=\$36$.	The discount is worth $\$36$.
(b) As a fraction: $\dfrac{18}{40}$.	Deposit over price (same units, dollars).
$\dfrac{18}{40}\times 100\,\%=45\,\%$.	Multiply by 100 to get a percentage.

Example 2 — scaffolded

(a) A plumber quotes $\$120$ for labour, plus $10\,\%$ GST. Find the total charge.
(b) A café receipt shows a total of $\$88$, which includes GST. Find the amount of GST in the total.

Working	Comment
(a) Total $=120\times 1.1$.	Adding $10\,\%$ GST multiplies by 1.1.
$=\$132$.	The customer pays $\$132$.
(b) GST $=\dfrac{88}{11}$.	A GST-inclusive total is divided by 11.
$=\$8$ (so the pre-GST price was $\$80$).	Check: $80+8=88$.

Example 3 — unscaffolded

An $\$84$ deposit is $35\,\%$ of the price of a phone. Find the full price of the phone.

The deposit is $35\,\%$ of the price, so the price is the deposit divided by 0.35:

$$\text{price}=\frac{84}{0.35}=\$240\,.$$

Checking, $35\,\%$ of $\$240$ is $0.35\times 240=\$84$, as required.

$\therefore$ the full price of the phone is $\$240$.

Example 4 — unscaffolded

A sales representative earns $3\,\%$ commission on the value of everything she sells. In one month she sold $\$45\,000$ worth of equipment. Find her commission.

The commission is $3\,\%$ of $\$45\,000$:

$$\frac{3}{100}\times 45\,000 = 0.03\times 45\,000 = \$1350.$$

$\therefore$ her commission for the month is $\$1350$.

Practice questions

Scaffolded

Q1. Convert between forms. (a) Write 16% as a simplified fraction and as a decimal. (b) Write $\frac{5}{8}$ as a percentage. (c) Write 0.6 as a percentage.

Q2. Find: (a) 20% of $\$150$; (b) 8% of 250 g; (c) 75% of 64 students.

Q3. Express each as a percentage. (a) 12 out of 50; (b) 27 out of 60; (c) $\$15$ out of $\$75$.

Q4. GST is 10%. (a) Add GST to a pre-GST price of $\$60$. (b) Add GST to a pre-GST price of $\$245$. (c) Find the GST included in a GST-inclusive total of $\$99$.

Q5. Find the whole in each case. (a) 70% of a number is 56. Find the number. (b) 25% of a class is 7 students. Find the class size. (c) A $\$150$ deposit is 20% of a price. Find the price.

Q6. (a) A shop's sales this year are 115% of last year's $\$60\,000$. Find this year's sales. (b) Express 63 as a percentage of 45.

Unscaffolded

Q7. Find 12% of $\$1500$.

Q8. What percentage is 45 out of 180?

Q9. Write 0.045 as a percentage, and write 220% as a decimal.

Q10. A store takes 30% off a $\$180$ coat. Find the amount of the discount.

Q11. An electrician charges $\$420$ for a job, plus 10% GST. Find the total charge.

Q12. A restaurant bill of $\$154$ includes 10% GST. How much GST is in the bill?

Q13. 65% of a company's 240 employees work full-time. How many employees is that?

Q14. A student scored 51 out of 60 on a test. What percentage did the student score?

Q15. The $\$2600$ deposit on a car is 8% of its price. Find the full price of the car.

Q16. A real-estate agent earns 4% commission on sales. Last month she sold $\$62\,500$ worth of property. Find her commission.

Q17. This year's festival attendance is 135% of last year's 2000 people. Find this year's attendance.

Q18. A phone plan's data allowance has grown to 250% of the old 12 GB. Find the new allowance.

Answers

Q1. (a) $\frac{4}{25}$, 0.16. (b) 62.5 %. (c) 60 %. **Q2.** (a) \$30. (b) 20 g. (c) 48 students. **Q3.** (a) 24 %. (b) 45 %. (c) 20 %. **Q4.** (a) \$66. (b) \$269.50. (c) \$9. **Q5.** (a) 80. (b) 28 students. (c) \$750. **Q6.** (a) \$69 000. (b) 140 %. **Q7.** \$180. **Q8.** 25 %. **Q9.** 4.5 %; 2.2. **Q10.** \$54. **Q11.** \$462. **Q12.** \$14. **Q13.** 156 employees. **Q14.** 85 %. **Q15.** \$32 500. **Q16.** \$2500. **Q17.** 2700 people. **Q18.** 30 GB.

Fully-worked solutions

Q1. (a) $16\% = \frac{16}{100} = \frac{4}{25}$; dividing by 100, $16\% = 0.16$. (b) $\frac{5}{8} = 5 \div 8 = 0.625 = 62.5\%$. (c) $0.6 \times 100\% = 60\%$.

Q2. (a) $\frac{20}{100} \times 150 = 0.2 \times 150 = \$30$. (b) $\frac{8}{100} \times 250 = 0.08 \times 250 = 20$ g. (c) $\frac{75}{100} \times 64 = 0.75 \times 64 = 48$ students.

Q3. (a) $\frac{12}{50} \times 100\% = 24\%$. (b) $\frac{27}{60} \times 100\% = 45\%$. (c) $\frac{15}{75} \times 100\% = 20\%$.

Q4. (a) $60 \times 1.1 = \$66$. (b) $245 \times 1.1 = \$269.50$. (c) GST $= \frac{99}{11} = \$9$ (the pre-GST price was \$90).

Q5. (a) The number $= \frac{56}{0.7} = 80$. (b) The class size $= \frac{7}{0.25} = 28$ students. (c) The price $= \frac{150}{0.2} = \$750$.

Q6. (a) 115 % of \$60 000 $= 1.15 \times 60\,000 = \$69\,000$. (b) $\frac{63}{45} \times 100\% = 1.4 \times 100\% = 140\%$.

Q7. $\frac{12}{100} \times 1500 = 0.12 \times 1500 = \$180$.

Q8. $\frac{45}{180} \times 100\% = 0.25 \times 100\% = 25\%$.

Q9. $0.045 \times 100\% = 4.5\%$; and $220\% = \frac{220}{100} = 2.2$.

Q10. Discount $= \frac{30}{100} \times 180 = 0.3 \times 180 = \$54$.

Q11. Total $= 420 \times 1.1 = \$462$.

Q12. GST $= \frac{154}{11} = \$14$ (the pre-GST charge was \$140).

Q13. $\frac{65}{100} \times 240 = 0.65 \times 240 = 156$ employees.

Q14. $\frac{51}{60} \times 100\% = 0.85 \times 100\% = 85\%$.

Q15. The price $= \frac{2600}{0.08} = \$32\,500$.

Q16. Commission $= \frac{4}{100} \times 62\,500 = 0.04 \times 62\,500 = \$2500$.

Q17. 135 % of $2000 = 1.35 \times 2000 = 2700$ people.

Q18. $250\,\%$ of $12 = 2.5 \times 12 = 30$ GB.

Theory

Prices, wages, populations and values are constantly rising and falling, and the change is almost always described as a **percentage** rather than a raw amount — a “$50 rise” means very different things on a $200 item and a $5000 item, but a “$25\%$ rise” is clear in both cases. This section covers applying a percentage change, finding the percentage change from two values, and combining several changes.

Applying a percentage increase or decrease — the multiplier method.

To increase a quantity by $p\%$, you keep the original 100% and add $p\%$, giving $\left(100+p\right)\%$ of the original; to decrease, you keep $\left(100-p\right)\%$. The quickest way is to multiply by a **multiplier**:

- a 15% **increase** multiplies by $1+\dfrac{15}{100}=1.15$;
- a 20% **decrease** multiplies by $1-\dfrac{20}{100}=0.80$.

So a $\$80$ jacket after a 15% rise costs $80\times 1.15=\$92$, and after a 20% discount a $\$80$ jacket costs $80\times 0.80=\$64$. The multiplier method is faster and sets up **successive** changes neatly (see below).

Finding the percentage change.

Given the **initial** (original) value and the **final** (new) value, the percentage change is the change as a percentage of the *original* amount. Both forms are on the Foundation Mathematics formula sheet:

$$\text{percentage increase}=\frac{\text{final}-\text{initial}}{\text{initial}}\times 100\%,$$

$$\text{percentage decrease}=\frac{\text{initial}-\text{final}}{\text{initial}}\times 100\%.$$

Always divide by the **initial** value, not the final one. For example, a share rising from $\$2.50$ to $\$3.10$ has increased by $\dfrac{3.10-2.50}{2.50}\times 100\%=24\%$.

Mark-up, discount, profit and loss.

- A **mark-up** is a percentage increase a seller adds to the cost price to set the selling price. A **discount** is a percentage decrease off a marked price.
- **Profit** or **loss** is the selling price minus the cost price, and the **percentage profit/loss** is taken as a percentage of the **cost** price:

$$\text{percentage profit}=\frac{\text{profit}}{\text{cost price}}\times 100\%.$$

Successive percentage changes.

When two percentage changes are applied one after the other, you **cannot** just add the percentages. Apply the multipliers in turn. A 10% rise followed by a 10% fall gives

$$\text{original}\times 1.10\times 0.90=\text{original}\times 0.99,$$

a net **decrease** of 1% — not 0% — because the second change acts on the larger, already-increased amount.

Worked examples

Example 1 — scaffolded

A weekly rent of $\$420$ is increased by 6%. Find the increase and the new rent.

Working	Comment
Increase $=6\%$ of $\$420=\frac{6}{100}\times 420=\$25.20$.	A percentage increase is that percentage of the original.
New rent $=420+25.20=\$445.20$.	Add the increase to the original.
Check with the multiplier: $420\times 1.06=\$445.20$.	A 6% rise multiplies by 1.06.

Example 2 — scaffolded

A company's monthly water bill rose from $\$145$ to $\$178$. Find the percentage increase, correct to two decimal places.

Working	Comment
Change $=178-145=\$33$.	Final minus initial.
Percentage increase $=\frac{33}{145}\times 100\%$.	Divide the change by the **initial** value.
$=22.7586\ldots\%\approx 22.76\%$.	Round at the final step.

Example 3 — unscaffolded

A shop buys a bike for $\$200$ and marks it up by 25%. In a sale, 10% is then taken off the marked price. Find the sale price and the overall percentage change from the original $\$200$.

Applying the multipliers in turn,

$$200\times 1.25=\$250\,(\text{marked price}),\,250\times 0.90=\$225\,(\text{sale price}).$$

The overall change from $\$200$ to $\$225$ is

$$\frac{225-200}{200}\times 100\%=\frac{25}{200}\times 100\%=12.5\%.$$

∴ the sale price is $\$225$, an overall increase of 12.5% — **not** the $25-10=15\%$ that adding the percentages would suggest.

Example 4 — unscaffolded

A trader buys a phone for $\$60$ and sells it for $\$75$. Find the percentage profit.

The profit is $75-60=\$15$. As a percentage of the **cost** price,

$$\text{percentage profit}=\frac{15}{60}\times 100\%=25\%.$$

∴ the trader makes a 25% profit.

Practice questions

Scaffolded

Q1. A $\$1200$ laptop is discounted by 15%. (a) Find the discount in dollars. (b) Find the sale price.

Q2. The population of a town grew from 8000 to 9400. (a) Find the increase. (b) Find the percentage increase.

Q3. A car bought for $\$25\,000$ is later valued at $\$18\,000$. (a) Find the decrease in value. (b) Find the percentage decrease.

Q4. A wholesaler's cost price for a jacket is $\$45$. She marks it up by $60\,\%$. (a) Find the mark-up in dollars. (b) Find the selling price.

Q5. An investment of $\$500$ rises by $10\,\%$ in the first year, then falls by $10\,\%$ in the second year. (a) Find its value after the first year. (b) Find its value after the second year. (c) Find the overall percentage change over the two years.

Q6. A restaurant bill is $\$96$ and a $10\,\%$ service charge is added. (a) Find the service charge. (b) Find the total to pay. (c) Write down the single multiplier that takes the bill straight to the total.

Unscaffolded

Q7. A salary of $\$58\,000$ is increased by $3.5\,\%$. Find the new salary.

Q8. A jacket marked $\$89.95$ is reduced by $20\,\%$ in a sale. Find the sale price.

Q9. A household's quarterly electricity bill rose from $\$310$ to $\$425$. Find the percentage increase, correct to two decimal places.

Q10. Attendance at a festival fell from 3400 to 2890. Find the percentage decrease.

Q11. A retailer buys stock at a wholesale price of $\$250$ and marks it up by $40\,\%$. Find the retail price.

Q12. A coat originally priced at $\$120$ is on sale for $\$90$. Find the percentage discount.

Q13. A price of $\$80$ is increased by $25\,\%$, and the new price is later decreased by $20\,\%$. Find the final price and the overall percentage change.

Q14. A dealer buys a car for $\$1500$ and sells it for $\$1875$. Find the percentage profit.

Q15. A shop buys a fridge for $\$600$ and, unable to sell it, clears it for $\$480$. Find the percentage loss.

Q16. Two shares change in price over a week: share A from $\$40$ to $\$50$, and share B from $\$200$ to $\$230$. Which share had the larger percentage increase, and by how many percentage points?

Q17. A coat costs $\$160$. A store takes $30\,\%$ off, then offers a further $10\,\%$ off the reduced price at the register. Find the final price paid.

Q18. A house valued at $\$650\,000$ increases in value by $8\,\%$ over a year. Find the increase and the new value.

Answers

Q1. (a) $\$180$. (b) $\$1020$. **Q2.** (a) 1400. (b) 17.5%. **Q3.** (a) $\$7000$. (b) 28%. **Q4.** (a) $\$27$. (b) $\$72$. **Q5.** (a) $\$550$. (b) $\$495$. (c) a 1% decrease. **Q6.** (a) $\$9.60$. (b) $\$105.60$. (c) $\times 1.10$. **Q7.** $\$60\,030$. **Q8.** $\$71.96$. **Q9.** 37.10%. **Q10.** 15%. **Q11.** $\$350$. **Q12.** 25%. **Q13.** $\$80$; overall change 0% (no change). **Q14.** 25%. **Q15.** 20%. **Q16.** Share A (25%) beat share B (15%) by 10 percentage points. **Q17.** $\$100.80$. **Q18.** increase $\$52\,000$; new value $\$702\,000$.

Fully-worked solutions

Q1. (a) Discount $= 15\%$ of $\$1200 = 0.15 \times 1200 = \$180$. (b) Sale price $= 1200 - 180 = \$1020$ (or $1200 \times 0.85 = \$1020$).

Q2. (a) Increase $= 9400 - 8000 = 1400$. (b) Percentage increase $= \dfrac{1400}{8000} \times 100\% = 17.5\%$.

Q3. (a) Decrease $= 25\,000 - 18\,000 = \$7000$. (b) Percentage decrease $= \dfrac{7000}{25\,000} \times 100\% = 28\%$.

Q4. (a) Mark-up $= 60\%$ of $\$45 = 0.60 \times 45 = \$27$. (b) Selling price $= 45 + 27 = \$72$ (or $45 \times 1.60 = \$72$).

Q5. (a) After the first year: $500 \times 1.10 = \$550$. (b) After the second year: $550 \times 0.90 = \$495$. (c) Overall change $= \dfrac{495 - 500}{500} \times 100\% = \dfrac{-5}{500} \times 100\% = -1\%$, a 1% decrease.

Q6. (a) Service charge $= 10\%$ of $\$96 = 0.10 \times 96 = \$9.60$. (b) Total $= 96 + 9.60 = \$105.60$. (c) Adding 10% multiplies by 1.10, so $96 \times 1.10 = \$105.60$.

Q7. New salary $= 58\,000 \times 1.035 = \$60\,030$.

Q8. Sale price $= 89.95 \times (1 - 0.20) = 89.95 \times 0.80 = \$71.96$.

Q9. Change $= 425 - 310 = \$115$, so percentage increase $= \dfrac{115}{310} \times 100\% = 37.0967\ldots\% \approx 37.10\%$.

Q10. Decrease $= 3400 - 2890 = 510$, so percentage decrease $= \dfrac{510}{3400} \times 100\% = 15\%$.

Q11. Retail price $= 250 \times 1.40 = \$350$.

Q12. Discount $= 120 - 90 = \$30$, so percentage discount $= \dfrac{30}{120} \times 100\% = 25\%$.

Q13. $80 \times 1.25 = \$100$, then $100 \times 0.80 = \$80$. The final price equals the original $\$80$, so the overall change is $\dfrac{80 - 80}{80} \times 100\% = 0\%$. (A 25% rise and a 20% fall cancel because $1.25 \times 0.80 = 1$.)

Q14. Profit $= 1875 - 1500 = \$375$, so percentage profit $= \dfrac{375}{1500} \times 100\% = 25\%$.

Q15. Loss $= 600 - 480 = \$120$, so percentage loss $= \dfrac{120}{600} \times 100\% = 20\%$.

Q16. Share A: $\dfrac{50 - 40}{40} \times 100\% = 25\%$. Share B: $\dfrac{230 - 200}{200} \times 100\% = 15\%$. Share A had the larger percentage increase, by $25 - 15 = 10$ percentage points (even though share B rose by more in dollars — $\$30$ against share A's $\$10$).

Q17. $160 \times 0.70 = \$112$ after the first reduction, then $112 \times 0.90 = \$100.80$ at the register.

Q18. Increase $= 8\,\%$ of $\$650\,000 = 0.08 \times 650\,000 = \$52\,000$. New value $= 650\,000 + 52\,000 = \$702\,000$ (or $650\,000 \times 1.08 = \$702\,000$).

Theory

Whenever we **estimate** a quantity or **measure** it with an instrument, the value we get is rarely exactly right. **Percentage error** tells us how big that mistake is, measured as a percentage of the true value — so it works for prices, crowds, forecasts and measurements alike, and lets us compare how accurate two very different estimates are.

Absolute error.

The **absolute error** is simply the size of the gap between the value we got and the true value:

$$\text{absolute error} = |\text{measured} - \text{actual}|.$$

The vertical bars mean "take the positive size of the difference", so it does not matter whether the estimate was too high or too low. Absolute error is measured in the **same units** as the quantity (dollars, people, metres, …).

Percentage error.

An absolute error of $10 sounds small on a $500 car service but huge on a $12 lunch. To judge an error fairly we compare it with the size of the true value. The **percentage error** is the absolute error written as a percentage of the actual value (this formula is on the Foundation Mathematics formula sheet):

$$\text{percentage error} = \frac{|\text{measured} - \text{actual}|}{\text{actual}} \times 100\,\%.$$

Because the difference is always taken as a positive size, percentage error is never negative.

Why percentage error is useful.

Percentage error puts every error on the same scale — "out of 100" — no matter what is being measured. This lets you:

- decide whether an estimate is **good** (a small percentage error, say under 5%, is usually close enough for everyday purposes); and
- **compare** the accuracy of two estimates or two instruments, even when they measure quantities of very different sizes.

A smaller percentage error means a more accurate estimate.

Worked model.

A stallholder estimates a day's takings as $1200, but the till shows the actual takings were $1275. The absolute error is $|1200 - 1275| = \$75$, and the percentage error is

$$\frac{75}{1275} \times 100\,\% = 5.88\,\% \left(\text{to 2 d.p.}\right).$$

The estimate was low by $75, which is about 5.9% of the true takings — a reasonably close estimate.

Worked examples

Example 1 — scaffolded

A tradesperson estimates that a job will cost $450 in materials, but the actual cost turns out to be $480. Find the absolute error and the percentage error (to two decimal places).

Working	Comment
Absolute error $= |450 - 480| = \$30$.	The size of the gap, in dollars.

Working	Comment
The actual value is \$480, so percentage error $= \frac{30}{480} \times 100\%$.	Divide the absolute error by the **actual** value.
$= 6.25\%$.	$30 \div 480 \times 100 = 6.25$.

Example 2 — scaffolded

A festival organiser estimated the crowd at 5000 people. Ticket scans later showed the actual attendance was 4600. Find the percentage error of the estimate, correct to two decimal places.

Working	Comment
Absolute error $= \lvert 5000 - 4600 \rvert = 400$ people.	The estimate was 400 too high.
Percentage error $= \frac{400}{4600} \times 100\%$.	Compare the error with the actual attendance.
$= 8.70\%$.	$400 \div 4600 \times 100 = 8.6956\ldots \approx 8.70$.

Example 3 — unscaffolded

Two weather services predicted a stadium's water use for an event. Service A predicted 1200 kL when the actual use was 1150 kL; Service B predicted 800 kL when the actual use was 760 kL. Which service made the more accurate prediction?

For Service A the absolute error is $\lvert 1200 - 1150 \rvert = 50$ kL, so its percentage error is

$$\frac{50}{1150} \times 100\% = 4.35\% \left(\text{to 2 d.p.}\right).$$

For Service B the absolute error is $\lvert 800 - 760 \rvert = 40$ kL, so its percentage error is

$$\frac{40}{760} \times 100\% = 5.26\% \left(\text{to 2 d.p.}\right).$$

Although Service B's absolute error (40 kL) is smaller than Service A's (50 kL), Service B was predicting a smaller quantity, so its **percentage** error is larger.

∴ Service A was more accurate, with a percentage error of 4.35% against Service B's 5.26%.

Example 4 — unscaffolded

A steel rod is measured as 2.03 m long. Its true length is 2.00 m. Find the percentage error of the measurement.

The absolute error is $\lvert 2.03 - 2.00 \rvert = 0.03$ m. As a percentage of the actual length,

$$\text{percentage error} = \frac{0.03}{2.00} \times 100\% = 1.5\%.$$

∴ the measurement has a percentage error of 1.5%.

Practice questions

Scaffolded

Q1. A shopper estimates a grocery bill at \$240. The actual bill is \$250. (a) Find the absolute error. (b) Find the percentage error.

Q2. A forecast gave a maximum temperature of 32 °C, but the actual maximum was 35 °C. (a) Find the absolute error. (b) Find the percentage error, correct to two decimal places.

Q3. A promoter estimated a concert crowd at 18 000, but the actual crowd was 20 000. (a) Find the absolute error. (b) Find the percentage error.

Q4. A bag of sugar is labelled 500 g. A scale measures its mass as 498 g. (a) Find the absolute error. (b) Find the percentage error.

Q5. Two estimates are checked against their true values. Estimate A guessed 95 when the actual value was 100; Estimate B guessed 340 when the actual value was 350. (a) Find the percentage error of Estimate A. (b) Find the percentage error of Estimate B, correct to two decimal places. (c) State which estimate was more accurate.

Q6. A caterer estimates the cost of an event at $1350. The actual cost is $1420. (a) Find the absolute error. (b) Find the percentage error, correct to two decimal places.

Unscaffolded

Q7. A driver estimates a trip as 480 km. The actual distance is 500 km. Find the percentage error.

Q8. A café predicted 850 customers for the week but actually served 800. Find the percentage error of the prediction, correct to two decimal places.

Q9. A student measures a desk as 14.7 cm wide on a scale drawing; the true width on the drawing is 15 cm. Find the percentage error.

Q10. A forecast predicted 40 mm of rain, but 52 mm fell. Find the percentage error, correct to two decimal places.

Q11. Two players estimate a crowd. Player A guesses 60 when the true number is 64; Player B guesses 145 when the true number is 150. By finding each percentage error (to two decimal places), state who estimated more accurately.

Q12. A manager estimates monthly sales at $8500. Actual sales are $8000. Find the percentage error of the estimate.

Q13. A measuring jug reads 245 mL when it actually contains 250 mL. Find the percentage error.

Q14. A council estimates a town's population at 12 500. A census finds 13 000. Find the percentage error, correct to two decimal places.

Q15. Two kitchen scales are tested with known masses. Scale P reads 2.06 kg for a true 2.00 kg mass; Scale Q reads 5.10 kg for a true 5.00 kg mass. By finding each percentage error, state which scale is more accurate.

Q16. A student estimates that a task will take 45 minutes, but it actually takes 50 minutes. Find the percentage error of the estimate.

Q17. A machine is set to cut lengths of 100 cm. One length is measured at 99.5 cm. Find the percentage error.

Q18. Two forecasters predict the day's maximum temperature. Forecaster 1 predicts 26 °C (actual 24 °C); Forecaster 2 predicts 31 °C (actual 30 °C). By finding each percentage error (to two decimal places), state which forecast was more accurate, and comment on whether the smaller absolute error also gave the smaller percentage error.

Answers

Q1. (a) \$10. (b) 4%. **Q2.** (a) 3 °C. (b) 8.57%. **Q3.** (a) 2000. (b) 10%. **Q4.** (a) 2 g. (b) 0.4%. **Q5.** (a) 5%. (b) 2.86%. (c) Estimate B (smaller percentage error). **Q6.** (a) \$70. (b) 4.93%. **Q7.** 4%. **Q8.** 6.25%. **Q9.** 2%. **Q10.** 23.08%. **Q11.** A: 6.25%; B: 3.33%; Player B estimated more accurately. **Q12.** 6.25%. **Q13.** 2%. **Q14.** 3.85%. **Q15.** P: 3%; Q: 2%; Scale Q is more accurate. **Q16.** 10%. **Q17.** 0.5%. **Q18.** Forecaster 1: 8.33%; Forecaster 2: 3.33%; Forecaster 2 was more accurate. Here the smaller absolute error (1 °C vs 2 °C) did also give the smaller percentage error.

Fully-worked solutions

Q1. (a) Absolute error $=|240-250|=\$10$. (b) Percentage error $=\dfrac{10}{250}\times 100\%=4\%$.

Q2. (a) Absolute error $=|32-35|=3$ °C. (b) Percentage error $=\dfrac{3}{35}\times 100\%=8.57\%$ (to 2 d.p.).

Q3. (a) Absolute error $=|18\,000-20\,000|=2000$. (b) Percentage error $=\dfrac{2000}{20\,000}\times 100\%=10\%$.

Q4. (a) Absolute error $=|498-500|=2$ g. (b) Percentage error $=\dfrac{2}{500}\times 100\%=0.4\%$.

Q5. (a) Estimate A: $\dfrac{|95-100|}{100}\times 100\%=\dfrac{5}{100}\times 100\%=5\%$. (b) Estimate B: $\dfrac{|340-350|}{350}\times 100\%=\dfrac{10}{350}\times 100\%=2.86\%$ (to 2 d.p.). (c) Estimate B has the smaller percentage error, so it is more accurate.

Q6. (a) Absolute error $=|1350-1420|=\$70$. (b) Percentage error $=\dfrac{70}{1420}\times 100\%=4.93\%$ (to 2 d.p.).

Q7. Percentage error $=\dfrac{|480-500|}{500}\times 100\%=\dfrac{20}{500}\times 100\%=4\%$.

Q8. Percentage error $=\dfrac{|850-800|}{800}\times 100\%=\dfrac{50}{800}\times 100\%=6.25\%$.

Q9. Percentage error $=\dfrac{|14.7-15|}{15}\times 100\%=\dfrac{0.3}{15}\times 100\%=2\%$.

Q10. Percentage error $=\dfrac{|40-52|}{52}\times 100\%=\dfrac{12}{52}\times 100\%=23.08\%$ (to 2 d.p.).

Q11. Player A: $\dfrac{|60-64|}{64}\times 100\%=\dfrac{4}{64}\times 100\%=6.25\%$. Player B: $\dfrac{|145-150|}{150}\times 100\%=\dfrac{5}{150}\times 100\%=3.33\%$ (to 2 d.p.). Player B has the smaller percentage error, so Player B estimated more accurately.

Q12. Percentage error $=\dfrac{|8500-8000|}{8000}\times 100\%=\dfrac{500}{8000}\times 100\%=6.25\%$.

Q13. Percentage error $=\dfrac{|245-250|}{250}\times 100\%=\dfrac{5}{250}\times 100\%=2\%$.

Q14. Percentage error $=\dfrac{|12\,500-13\,000|}{13\,000}\times 100\%=\dfrac{500}{13\,000}\times 100\%=3.85\%$ (to 2 d.p.).

Q15. Scale P: $\frac{|2.06-2.00|}{2.00}\times 100\,\% = \frac{0.06}{2.00}\times 100\,\% = 3\,\%$. Scale Q: $\frac{|5.10-5.00|}{5.00}\times 100\,\% = \frac{0.10}{5.00}\times 100\,\% = 2\,\%$. Scale Q has the smaller percentage error, so Scale Q is more accurate.

Q16. Percentage error $= \frac{|45-50|}{50}\times 100\,\% = \frac{5}{50}\times 100\,\% = 10\,\%$.

Q17. Percentage error $= \frac{|99.5-100|}{100}\times 100\,\% = \frac{0.5}{100}\times 100\,\% = 0.5\,\%$.

Q18. Forecaster 1: $\frac{|26-24|}{24}\times 100\,\% = \frac{2}{24}\times 100\,\% = 8.33\,\%$ (to 2 d.p.). Forecaster 2: $\frac{|31-30|}{30}\times 100\,\% = \frac{1}{30}\times 100\,\% = 3.33\,\%$ (to 2 d.p.). Forecaster 2 has the smaller percentage error and is more accurate. In this case Forecaster 2 also had the smaller absolute error (1 °C versus 2 °C), so here the smaller absolute error did give the smaller percentage error — though that is not always guaranteed when the actual values differ.