Foundation Mathematics Units 1 & 2

Chapter 3 Calculation, ratio and rates

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Theory

Whenever you carry out a calculation — especially on a calculator — it is easy to press a wrong key, misplace a decimal point or read a figure incorrectly. An **estimate** is a quick, rough answer worked out in your head. Comparing your exact answer with an estimate is one of the most useful habits in mathematics: if the two are close, your answer is probably right; if they are wildly different, you know to look for a mistake. Deciding whether an answer makes sense is called checking its **reasonableness**.

Estimating by rounding.

The fastest estimate rounds each number to **one significant figure** (its leading digit) and then does the easy calculation:

$$312 \times 48 \approx 300 \times 50 = 15\,000.$$

The exact value is $14\,976$, so the estimate confirms the answer is about right. Rounding each number to one or two significant figures is enough for a check — you are only after the right *size* of answer, not a precise one.

Compatible (friendly) numbers.

For division, choose nearby numbers that divide easily. To estimate $4128 \div 19$, replace the numbers with the **compatible** pair $4000 \div 20 = 200$. The exact answer, 217.3, is close, so 200 was a good check.

Order of magnitude — "out by a factor of 10".

The most common calculator slip is a misplaced decimal point, which makes an answer 10, 100 or 1000 times too big or too small. An estimate of the **order of magnitude** (the size as a power of 10) catches this at once. Buying 6 items at $\$4.95$ should cost about $6 \times \$5 = \$30$; a till showing $\$3.00$ or $\$300$ is out by a factor of 10 and must be wrong.

Sensible rounding for the context.

An exact calculation often gives more decimal places than the situation needs, so round the **final** answer to suit the context:

- money to the nearest cent (or the nearest 5 cents for a cash payment);
- a count of people, cars or items to a **whole number**;
- **round up** when you must have enough — tins of paint, boxes of tiles, buses to carry a group;
- **round down** when you can only use complete items — how many whole platters the ingredients make.

Judging reasonableness.

Finally, stand back and ask whether the answer makes sense in the real situation. Is it the right **size**? the right **sign** (a length or a mass cannot be negative)? in sensible **units**? A person's height of 17 m or a weekly grocery bill of $\$12$ should be rejected immediately, whatever the calculator says.

Worked examples

Example 1 — scaffolded

A shopper buys 18 cartons of juice at $\$3.85$ each. Estimate the total cost, then find the exact cost and check that it is reasonable.

Working	Comment
Round the price: $\$3.85 \approx \$4$. Estimate $18 \times \$4 = \$72$.	Round to a friendly number for a quick check.
Exact cost $= 18 \times \$3.85 = \$69.30$.	Do the full calculation on the calculator.

Working	Comment
$\$69.30$ is close to the estimate of $\$72$, and a little less (the price was rounded up).	Compare exact with estimate.
$\therefore$ the total $\$69.30$ is reasonable.	The two agree in size, so the answer is trustworthy.

Example 2 — scaffolded

A student calculates the area of a rectangular room measuring 4.2 m by 3.5 m and writes down 147 m². Use an estimate to show the answer is wrong, and give the correct area.

Working	Comment
Estimate: $4.2 \approx 4$ and $3.5 \approx 4$, so area $\approx 4 \times 4 = 16$ m².	Round each measurement to one significant figure.
147 m² is about 10 times larger than 16 m².	The answer is "out by a factor of 10".
The student probably worked out 42×3.5 — a misplaced decimal point.	A factor of 10 almost always means a decimal-point slip.
Correct area $= 4.2 \times 3.5 = 14.7$ m².	This matches the estimate of 16 m².

Example 3 — unscaffolded

A painter is quoting to paint a wall 12.6 m long and 2.7 m high with **two** coats. The paint covers 11 m² per litre. Estimate the paint needed, then calculate it exactly and state how many whole litres must be bought.

Area of one coat $= 12.6 \times 2.7 = 34.02$ m², so two coats cover

$$2 \times 34.02 = 68.04\,\text{m}^2.$$

The paint required is $68.04 \div 11 = 6.185\ldots$ litres. A quick check estimates $13 \times 3 = 39$ m² per coat, about 78 m² for two coats, and $78 \div 11 \approx 7$ litres — the same size, so the calculation is reasonable. Because paint is sold in whole litres and the painter must have enough, round **up**.

$\therefore$ the painter needs 6.19 litres of paint and must buy 7 litres.

Example 4 — unscaffolded

A caterer needs 385 bread rolls, which are sold in packs of 12. An assistant claims "that is about 30 packs". Find how many packs must be bought, and comment on the assistant's estimate.

Using compatible numbers, $385 \div 12 \approx 384 \div 12 = 32$, so about 32 packs are needed. The exact value is

$$385 \div 12 = 32.08\ldots\,\text{packs}.$$

Since a pack cannot be split and the caterer must have at least 385 rolls, round **up** to 33 packs.

$\therefore$ the caterer must buy 33 packs. The assistant's "about 30" has the right order of magnitude but is a little low — 33 packs are actually required to have enough rolls.

Practice questions

Scaffolded

Q1. Estimate each calculation by rounding every number to one significant figure, then find the exact value with a calculator. (a) 412×28. (b) $5980 \div 19$. (c) $38.7 + 61.2 + 49.5$.

Q2. A calculation and a proposed answer are given. Use estimation to decide whether each answer is reasonable. If it is not, give the correct answer. (a) $6 \times \$19.95 = \$11.97$. (b) $2450 \div 48 = 51$. (c) $313 + 289 + 198 = 800$.

Q3. Round each result sensibly for its context. (a) A monthly phone plan works out to $\$47.8333$ — round to the nearest cent. (b) A calculation shows a school group needs 6.4 buses — how many buses must be booked? (c) A caterer can make 12.7 identical platters from the ingredients — how many **complete** platters can be served?

Q4. Estimate each using compatible numbers, then find the exact value. (a) $6127 \div 29$. (b) 197×41. (c) $3964 \div 78$.

Q5. A carpenter needs 23.5 m of timber costing $\$6.80$ per metre. (a) Estimate the cost by rounding. (b) Find the exact cost. (c) State whether your estimate is close to the exact cost.

Q6. A worker is paid $\$28.50$ per hour and works 38 hours in a week. The payroll system displays her gross pay as $\$108.30$. (a) Estimate her weekly pay. (b) State whether $\$108.30$ is reasonable. (c) Give the correct gross pay.

Unscaffolded

Q7. Estimate 592×31 by rounding each number to one significant figure. Then find the exact product and the difference between the estimate and the exact value.

Q8. A café buys 24 boxes of coffee pods at $\$18.90$ each. Estimate the total, then find the exact cost, and comment on whether the estimate was close.

Q9. Use estimation to decide whether each answer is reasonable; if it is not, give the correct answer. (a) $715 \div 5 = 143$. (b) $4.8 \times 5.1 = 244.8$. (c) $9.6 + 8.5 + 11.2 = 29.3$.

Q10. A room measures 5.8 m by 4.3 m. A flooring quote is based on an area of 249 m². Use estimation to show the area figure is wrong, and find the correct area.

Q11. A recipe for 4 people needs 560 g of flour. Estimate the flour needed for 30 people, then calculate it exactly and give the answer in kilograms.

Q12. A car travels 487 km using 38 litres of petrol. Estimate the fuel use in litres per 100 km, then calculate it exactly, correct to one decimal place.

Q13. A fortnightly payslip is for 76 hours at $\$32.40$ per hour, but the printout shows $\$246.24$. Use an estimate to show this is wrong, and give the correct amount.

Q14. A water tank holds 4500 litres and is filled at 38 litres per minute. Estimate the time to fill it, then calculate it exactly to the nearest minute.

Q15. A shopper estimates that 3 shirts at $\$24.95$, 2 pairs of socks at $\$6.50$ and a jacket at $\$79.90$ will cost "about $\$170$". (a) Show how they might have made that estimate. (b) Find the exact total. (c) State whether $\$170$ is a reasonable estimate.

Q16. State whether each value is reasonable. If it is not, give a sensible value and the likely mistake. (a) The mass of an adult man is 720 kg. (b) A brisk 45-minute walk covers about 4 km. (c) A standard drink bottle holds 600 litres of water.

Q17. A farmer has 3860 m of fencing and wants to fence identical paddocks, each needing 240 m of fencing. Estimate, then find how many **complete** paddocks can be fenced.

Q18. A tiler must cover a floor of area 46.5 m². Tiles are sold in boxes that each cover 1.8 m² and cost $\$54.90$ per box. (a) Estimate the number of boxes needed. (b) Find the exact number of boxes that must be bought (only whole boxes are sold). (c) Find the total cost and check that it is reasonable.

Answers

Q1. (a) est $12\,000$, exact $11\,536$. (b) est 300, exact 314.7. (c) est 150, exact 149.4. **Q2.** (a) unreasonable — out by a factor of 10; correct $\$119.70$. (b) reasonable (exact 51.04). (c) reasonable (exact 800). **Q3.** (a) $\$47.83$. (b) 7 buses. (c) 12 platters. **Q4.** (a) est 200, exact 211.3. (b) est 8000, exact 8077. (c) est 50, exact 50.8. **Q5.** (a) about $\$168$ ($\$7 \times 24$). (b) $\$159.80$. (c) yes — close. **Q6.** (a) about $\$1200$. (b) not reasonable — out by a factor of 10. (c) $\$1083.00$. **Q7.** est $18\,000$; exact $18\,352$; difference 352. **Q8.** est about $\$450$–$\$500$; exact $\$453.60$; the estimate is close. **Q9.** (a) reasonable (143). (b) unreasonable — out by a factor of 10; correct 24.48. (c) reasonable (29.3). **Q10.** est $6 \times 4 = 24$ m²; 249 is $10 \times$ too big; correct area 24.94 m². **Q11.** est about 4200 g; exact 4200 g $= 4.2$ kg. **Q12.** est 8 L/100 km; exact 7.8 L/100 km. **Q13.** est about $\$2500$; $\$246.24$ is out by a factor of 10; correct $\$2462.40$. **Q14.** est about 120 min; exact 118 min. **Q15.** (a) $3 \times \$25 + 2 \times \$6.50 + \$80 \approx \$168$. (b) $\$167.75$. (c) yes — very close. **Q16.** (a) unreasonable — about 72 kg (factor-of-10 error). (b) reasonable. (c) unreasonable — about 600 mL (wrong unit). **Q17.** est about 16; exact 16 complete paddocks. **Q18.** (a) about 26 boxes. (b) 26 boxes. (c) $\$1427.40$ (estimate $26 \times \$55 = \$1430$ — reasonable).

Fully-worked solutions

Q1. (a) Round to $400 \times 30 = 12\,000$; exact $412 \times 28 = 11\,536$ — the estimate confirms the size. (b) Round to $6000 \div 20 = 300$; exact $5980 \div 19 = 314.7$ (to 1 d.p.). (c) Round to $40 + 60 + 50 = 150$; exact $38.7 + 61.2 + 49.5 = 149.4$.

Q2. (a) Estimate $6 \times \$20 = \$120$, so the answer should be about $\$120$. The value $\$11.97$ is ten times too small (a decimal-point slip); the correct total is $6 \times \$19.95 = \$119.70$. (b) Estimate $2400 \div 50 = 48$, so an answer near 51 is the right size. The exact value is $2450 \div 48 = 51.04$, so 51 is reasonable. (c) Estimate $300 + 300 + 200 = 800$. The exact sum is $313 + 289 + 198 = 800$, so the answer is reasonable (indeed exact).

Q3. (a) To the nearest cent (two decimal places), the third decimal is $3 < 5$, so $\$47.8333 \rightarrow \$47.83$. (b) 6.4 buses is more than 6, and part of a bus cannot be booked, so round **up** to 7 buses. (c) Only complete platters can be served, so round **down**: 12 platters (the leftover 0.7 of a platter is not enough for another full one).

Q4. (a) Compatible numbers $6000 \div 30 = 200$; exact $6127 \div 29 = 211.3$ (to 1 d.p.). (b) $200 \times 40 = 8000$; exact $197 \times 41 = 8077$. (c) $4000 \div 80 = 50$; exact $3964 \div 78 = 50.8$ (to 1 d.p.).

Q5. (a) Round to $\$7$ per metre and 24 m: estimate $\$7 \times 24 = \$168$. (b) Exact cost $= 23.5 \times \$6.80 = \$159.80$. (c) The exact cost $\$159.80$ is within about $\$8$ of the estimate $\$168$, so the estimate is close and the answer is reasonable.

Q6. (a) Round to $\$30 \times 40 = \$1200$. (b) The displayed $\$108.30$ is about ten times smaller than $\$1200$, so it is not reasonable — the decimal point is in the wrong place. (c) Correct gross pay $= 38 \times \$28.50 = \$1083.00$, which matches the estimate.

Q7. Rounding to one significant figure: $600 \times 30 = 18\,000$. The exact product is $592 \times 31 = 18\,352$, so the difference is $18\,352 - 18\,000 = 352$.

Q8. Estimate $24 \times \$19 \approx \$456$ (or $25 \times \$20 = \$500$). The exact cost is $24 \times \$18.90 = \$453.60$, which is very close to the estimate, so it is reasonable.

Q9. (a) Estimate $700 \div 5 = 140$; exact $715 \div 5 = 143$ — reasonable. (b) Estimate $5 \times 5 = 25$, so the answer should be about 25. The value 244.8 is ten times too large; the correct product is $4.8 \times 5.1 = 24.48$. (c) Estimate $10 + 9 + 11 = 30$; exact $9.6 + 8.5 + 11.2 = 29.3$ — reasonable.

Q10. Estimate the area as $6 \times 4 = 24$ m². The quoted 249 m² is about ten times too large (a decimal-point error — perhaps 58×4.3 was used). The correct area is $5.8 \times 4.3 = 24.94$ m², which matches the estimate.

Q11. One person needs $560 \div 4 = 140$ g of flour. For 30 people: $140 \times 30 = 4200$ g. An estimate of $560 \div 4 = 140$, then $140 \times 30 = 4200$, confirms this. In kilograms, $4200 \div 1000 = 4.2$ kg.

Q12. Fuel use per 100 km $= \frac{38}{487} \times 100$. Estimate $\frac{40}{500} \times 100 = 8$ L/100 km. The exact value is $\frac{38}{487} \times 100 = 7.803\ldots = 7.8$ L/100 km (to 1 d.p.), close to the estimate.

Q13. Estimate $80 \times \$32 = \$2560$, so the pay should be around $\$2500$. The printed $\$246.24$ is about ten times too small. The correct amount is $76 \times \$32.40 = \$2462.40$.

Q14. Estimate $4500 \div 40 \approx 112$, so a bit under 120 minutes. The exact time is $4500 \div 38 = 118.42\ldots$ minutes, which is 118 minutes to the nearest minute (about 2 hours).

Q15. (a) Rounding each price: $3 \times \$25 = \$75$, $2 \times \$6.50 = \$13$ and the jacket about $\$80$, giving $75 + 13 + 80 = \$168 \approx \$170$. (b) Exact total $= 3 \times \$24.95 + 2 \times \$6.50 + \$79.90 = \$74.85 + \$13.00 + \$79.90 = \$167.75$. (c) The exact total $\$167.75$ is within a few dollars of $\$170$, so the estimate is reasonable.

Q16. (a) Not reasonable: an adult man has a mass of roughly 70–90 kg, so 720 kg is a factor-of-10 error — a sensible value is about 72 kg. (b) Reasonable: a brisk walk is about 5 km/h, so in 45 minutes ($3/4$ h) a person covers roughly 4 km. (c) Not reasonable: 600 litres would fill a large tank. A drink bottle holds about 600 **millilitres** — the unit is wrong.

Q17. Estimate $3860 \div 240 \approx 4000 \div 240 \approx 16.7$. The exact value is $3860 \div 240 = 16.08\ldots$ Since only complete paddocks count, round **down** to 16 paddocks (the remaining fencing is not enough for a 17th).

Q18. (a) Estimate $46.5 \div 1.8 \approx 46 \div 1.8 \approx 25–26$ boxes. (b) Exact $46.5 \div 1.8 = 25.83\ldots$; only whole boxes are sold and the whole floor must be covered, so round **up** to 26 boxes. (c) Total cost $= 26 \times \$54.90 = \$1427.40$. An estimate of $26 \times \$55 = \$1430$ is almost the same, so the cost is reasonable.

Theory

Most real problems need more than one calculation. To solve them confidently you must be able to add, subtract, multiply and divide with the four kinds of number you meet every day: **whole numbers**, **integers** (which include negatives), **fractions** and **decimals**.

Integers — working with negatives.

Integers are the whole numbers together with their negatives: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ They describe temperatures, bank balances, and heights above and below sea level.

- Adding a negative moves **left** on the number line; subtracting a negative moves **right**: $5+(-3)=2$ but $5-(-3)=8$.
- When multiplying or dividing, **same signs give a positive**, **different signs give a negative**: $(-4)\times(-6)=24$, while $(-4)\times 6=-24$ and $-24\div 6=-4$.

Fractions.

- To **add or subtract**, rewrite each fraction with a common denominator, then add or subtract the numerators: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.
- To **multiply**, multiply the numerators and multiply the denominators, then simplify: $\frac{2}{3}\times\frac{3}{4}=\frac{6}{12}=\frac{1}{2}$.
- To **divide**, multiply by the **reciprocal** (the flipped fraction) of the divisor: $\frac{3}{5}\div\frac{2}{5}=\frac{3}{5}\times\frac{5}{2}=\frac{3}{2}$.
- Change a **mixed number** to an **improper fraction** before calculating: $2\frac{1}{4}=\frac{9}{4}$.

Decimals.

- To **add or subtract**, line up the decimal points and fill gaps with zeros.
- To **multiply**, multiply as whole numbers, then place the point so the number of decimal places in the answer equals the total number of decimal places in the two factors: $4.2\times 0.5=2.1$ (one place + one place = two places in $42\times 5=210$).
- To **divide**, use your calculator, or make the divisor a whole number by shifting both decimal points the same number of places.

Using a scientific calculator.

The fraction key enters exact fractions, the $(-)$ key enters a negative number, and the bracket keys keep a multi-step calculation in the correct order. Enter money and measurements as decimals.

Choosing a method. Do simple facts mentally or by hand; reach for the calculator when the numbers are messy — but always make a quick estimate first so you can tell whether the result is reasonable.

Worked examples

Example 1 — scaffolded

At 6 pm the temperature on a farm was 7 °C. It fell by 11 °C overnight, then rose 3 °C by 9 am. Find the temperature at 9 am.

Working	Comment
Start at 7 °C. Falling by 11 means subtract: $7-11=-4$.	“Fell” ⇒ subtract. The result is below zero.
Now rising by 3 means add: $-4+3=-1$.	“Rose” ⇒ add. Adding 3 to -4 moves right to -1.
The temperature at 9 am is -1 °C.	State the answer with its unit.

Example 2 — scaffolded

A timber plank is $3\frac{1}{2}$ m long. A piece $\frac{3}{4}$ m long is sawn off. (a) Find the length remaining. (b) How many complete $\frac{2}{5}$ m pieces can be cut from the remaining length?

Working	Comment
(a) Write as improper fractions: $3\frac{1}{2}=\frac{7}{2}$. Common denominator 4: $\frac{7}{2}=\frac{14}{4}$.	Convert the mixed number, then match denominators.
$\frac{14}{4}-\frac{3}{4}=\frac{11}{4}=2\frac{3}{4}$ m.	Subtract numerators; convert back to a mixed number.
(b) Divide by the reciprocal: $\frac{11}{4}\div\frac{2}{5}=\frac{11}{4}\times\frac{5}{2}=\frac{55}{8}=6\frac{7}{8}$.	Dividing by $\frac{2}{5}$ means multiplying by $\frac{5}{2}$.
Only whole pieces count, so 6 complete pieces can be cut.	$6\frac{7}{8}$ rounds **down** — the part-piece is too short.

Example 3 — unscaffolded

Mia buys 2.5 kg of apples at $\$4.80$ per kilogram and 1.2 kg of grapes at $\$6.50$ per kilogram. She pays with a $\$50$ note. Find her change.

$$2.5\times4.80=12.00,\ 1.2\times6.50=7.80.$$

Total cost $=12.00+7.80=19.80$.

Change $=50-19.80=30.20$.

∴ Mia receives $\$30.20$ change.

Example 4 — unscaffolded

A drum holds 5 L of cleaning concentrate. Each batch of cleaner uses $\frac{3}{8}$ L of concentrate. Find how many complete batches can be made, and how much concentrate is left over.

$$5\div\frac{3}{8}=5\times\frac{8}{3}=\frac{40}{3}=13\frac{1}{3}.$$

So 13 complete batches can be made. These use $13\times\frac{3}{8}=\frac{39}{8}=4.875$ L.

Concentrate left over $=5-4.875=0.125$ L.

∴ 13 complete batches can be made, with 0.125 L $\left(\frac{1}{8}\text{L}\right)$ of concentrate left over.

Practice questions

Scaffolded

Q1. Recorded temperature changes (in °C) give these calculations. Find each result. (a) $-6+10$ (b) $3-8$ (c) $-4-7$

Q2. Evaluate, watching the signs. (a) $(-5)\times 4$ (b) $(-3)\times(-7)$ (c) $-30\div 5$

Q3. Add or subtract these fractions, giving each answer in simplest form. (a) $\frac{1}{3}+\frac{1}{5}$ (b) $\frac{3}{4}-\frac{1}{6}$ (c) $\frac{2}{5}+\frac{3}{10}$

Q4. Multiply or divide these fractions, giving each answer in simplest form. (a) $\frac{2}{5}\times\frac{5}{6}$ (b) $\frac{5}{6}\div\frac{2}{3}$ (c) $1\frac{1}{2}\times\frac{2}{3}$

Q5. Evaluate these decimal calculations. (a) $3.6+12.45$ (b) $20-7.85$ (c) 2.4×0.3

Q6. A casual worker is paid $\$22.50$ per hour. Find the pay for (a) 6 hours, (b) half an hour, (c) $6\frac{1}{2}$ hours.

Unscaffolded

Q7. At the summit of a mountain the temperature is -8 °C. At the base it is 11 °C warmer. Find the temperature at the base.

Q8. A bank account is overdrawn, showing a balance of $-\$47$. A deposit of $\$130$ is made. Find the new balance.

Q9. A submarine is at a depth of -120 m. It descends a further 45 m, then rises 60 m. Find its final depth.

Q10. A bushwalker covers $2\frac{3}{4}$ km before lunch and $1\frac{5}{8}$ km after lunch. Find the total distance walked.

Q11. A 5 kg bag of flour has $1\frac{2}{3}$ kg used in baking. Find the mass of flour remaining.

Q12. A recipe uses $\frac{3}{4}$ cup of sugar. Ravi makes only $\frac{2}{3}$ of the recipe. Find how much sugar he needs.

Q13. A ribbon $4\frac{1}{2}$ m long is cut into pieces each $\frac{3}{4}$ m long. Find how many pieces are obtained.

Q14. Diesel costs $\$1.85$ per litre. Find the cost of 46.5 litres, to the nearest cent.

Q15. A 1.5 kg block of cheese costs $\$18.75$. Find the cost per kilogram.

Q16. Sam buys 3 shirts at $\$24.95$ each and a pair of jeans for $\$59.90$. He uses a $\$150$ gift voucher. Find how much of the voucher is left.

Q17. A tank holds 240 L and is $\frac{5}{8}$ full. Then 30 L is drained out. Find the volume of water remaining.

Q18. Over four nights the overnight low temperatures were -3 °C, -5 °C, 2 °C and -6 °C. Find the mean (average) low temperature.

Answers

Q1. (a) 4 (b) -5 (c) -11 **Q2.** (a) -20 (b) 21 (c) -6 **Q3.** (a) $\frac{8}{15}$ (b) $\frac{7}{12}$ (c) $\frac{7}{10}$ **Q4.** (a) $\frac{1}{3}$ (b) $1\frac{1}{4}$ (c) 1 **Q5.** (a) 16.05 (b) 12.15 (c) 0.72 **Q6.** (a) $\$135.00$ (b) $\$11.25$ (c) $\$146.25$ **Q7.** 3 °C **Q8.** $\$83$ **Q9.** -105 m **Q10.** $4\frac{3}{8}$ km **Q11.** $3\frac{1}{3}$ kg **Q12.** $\frac{1}{2}$ cup **Q13.** 6 pieces **Q14.** $\$86.03$ **Q15.** $\$12.50$ per kg **Q16.** $\$15.25$ **Q17.** 120 L **Q18.** -3 °C

Fully-worked solutions

Q1. (a) $-6+10=4$ (adding 10 moves right past zero). (b) $3-8=-5$. (c) $-4-7=-11$ (both negative, move further left).

Q2. (a) Different signs ⇒ negative: $(-5)\times 4=-20$. (b) Same signs ⇒ positive: $(-3)\times(-7)=21$. (c) Different signs ⇒ negative: $-30\div 5=-6$.

Q3. (a) $\frac{1}{3}+\frac{1}{5}=\frac{5}{15}+\frac{3}{15}=\frac{8}{15}$. (b) $\frac{3}{4}-\frac{1}{6}=\frac{9}{12}-\frac{2}{12}=\frac{7}{12}$. (c) $\frac{2}{5}+\frac{3}{10}=\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$.

Q4. (a) $\frac{2}{5}\times\frac{5}{6}=\frac{10}{30}=\frac{1}{3}$. (b) $\frac{5}{6}\div\frac{2}{3}=\frac{5}{6}\times\frac{3}{2}=\frac{15}{12}=\frac{5}{4}=1\frac{1}{4}$. (c) $1\frac{1}{2}\times\frac{2}{3}=\frac{3}{2}\times\frac{2}{3}=\frac{6}{6}=1$.

Q5. (a) $3.6+12.45=16.05$. (b) $20-7.85=12.15$. (c) $2.4\times 0.3=0.72$.

Q6. (a) $22.50\times 6=135.00$, so $\$135.00$. (b) $22.50\times 0.5=11.25$, so $\$11.25$. (c) $22.50\times 6.5=146.25$, so $\$146.25$.

Q7. “Warmer” means add: $-8+11=3$, so the base is 3 °C.

Q8. A deposit adds to the balance: $-47+130=83$, so the new balance is $\$83$.

Q9. Descending subtracts, rising adds: $-120-45+60=-165+60=-105$, so the final depth is -105 m (105 m below the surface).

Q10. $2\frac{3}{4}+1\frac{5}{8}=\frac{11}{4}+\frac{13}{8}=\frac{22}{8}+\frac{13}{8}=\frac{35}{8}=4\frac{3}{8}$ km.

Q11. $5-1\frac{2}{3}=\frac{15}{3}-\frac{5}{3}=\frac{10}{3}=3\frac{1}{3}$ kg.

Q12. $\frac{3}{4}\times\frac{2}{3}=\frac{6}{12}=\frac{1}{2}$ cup.

Q13. $4\frac{1}{2}\div\frac{3}{4}=\frac{9}{2}\times\frac{4}{3}=\frac{36}{6}=6$ pieces.

Q14. $1.85\times 46.5=86.025$, which rounds to $\$86.03$ (nearest cent).

Q15. $18.75\div 1.5=12.5$, so the cheese costs $\$12.50$ per kilogram.

Q16. Shirts: $3\times 24.95=74.85$. Add the jeans: $74.85+59.90=134.75$. Voucher remaining: $150-134.75=15.25$, so $\$15.25$ is left.

Q17. Water at the start: $\frac{5}{8}\times 240=150$ L. After draining: $150-30=120$ L.

Q18. Sum of the four lows: $-3+(-5)+2+(-6)=-12$. Mean $=-12\div 4=-3$, so the mean low temperature is -3 °C.

Theory

When a calculation contains more than one operation, the answer depends on the **order** in which we carry them out. For example, $2+3\times 4$ could be read as "add, then multiply" ($5\times 4=20$) or "multiply, then add" ($2+12=14$). To make sure everyone — and every calculator — gets the **same** answer, mathematicians agree on a fixed **order of operations**.

The order of operations.

Work through a calculation in this order:

1. **Brackets** — do anything inside brackets first (innermost brackets first if they are nested).
2. **Indices** — then powers and roots, such as 3^2 or $\sqrt{16}$.
3. **Division and multiplication** — then these two, working **left to right**.
4. **Addition and subtraction** — finally these two, working **left to right**.

A useful memory aid is **BIDMAS**: **B**rackets, **I**ndices, **D**ivision/**M**ultiplication, **A**ddition/**S**ubtraction. The correct value of $2+3\times 4$ is therefore $2+12=14$: multiplication is done before addition.

Left to right for equal priority.

Division and multiplication have the **same** priority, so when both appear you work from left to right — you do **not** always do multiplication first. The same applies to addition and subtraction. For example,

$$20-8+3=(20-8)+3=12+3=15,$$

and

$$24\div 6\times 2=(24\div 6)\times 2=4\times 2=8.$$

Reading these right to left would give the wrong answer.

Brackets change the answer.

Brackets let you override the usual order by forcing part of the calculation to be done first. Compare

$$2+3^2\times 2=2+9\times 2=20 \text{ with } (2+3)^2\times 2=5^2\times 2=50.$$

The brackets change **what** gets squared, and so change the result completely.

The fraction bar groups.

A **fraction bar** acts like a pair of brackets around the whole top and the whole bottom: you evaluate the numerator and the denominator separately, then divide. So

$$\frac{12+8}{2+3}=\frac{20}{5}=4.$$

When you type this on a calculator you must add the brackets yourself: `(12 + 8) ÷ (2 + 3)`.

Using a scientific calculator.

A scientific calculator already applies the order of operations for you, so `2 + 3 × 4 =` correctly returns 14. But it can only follow what you type, so **you** must insert brackets whenever you need part of a calculation done first — around a sum that is being multiplied, or around the top and bottom of a fraction. Estimating the answer first (for example, "about $2+12$, so roughly 14") is the best way to catch a mis-typed expression.

Worked examples

Example 1 — scaffolded

Evaluate $6+4\times3^2-(10-7)$.

Working	Comment
Brackets first: $(10-7)=3$.	Always deal with brackets before anything else.
Indices next: $3^2=9$, so the expression is $6+4\times9-3$.	Powers come before $\times$, $\div$, $+$ and $-$.
Multiplication: $4\times9=36$, giving $6+36-3$.	$\times$ before $+$ and $-$.
Add and subtract left to right: $6+36-3=39$.	$42-3=39$.

Example 2 — scaffolded

At a uniform shop, Maya buys 4 shirts at $\$25$ each and 4 pairs of socks at $\$5$ each. A 10% discount applies to the **whole** order. Find the amount she pays, and explain what must be typed on a calculator.

Working	Comment
Cost before discount: $4\times25+4\times5=100+20=120$ dollars.	Each $\times$ is done before the $+$.
A 10% discount means she pays 90%: multiply the total by 0.9.	$100\%-10\%=90\%=0.9$.
She must bracket the whole total: $(4\times25+4\times5)\times0.9=120\times0.9=108$.	The bracket forces the sum to be found *before* multiplying by 0.9.
Without the bracket the calculator reads $4\times25+4\times5\times0.9=100+18=118$ — the wrong answer.	Only the last term would be discounted.

$\therefore$ Maya pays $\$108$.

Example 3 — unscaffolded

Evaluate

$$\frac{4^2+8}{3}+\sqrt{25}-2\times3.$$

The fraction bar groups the numerator, so evaluate it first: $4^2+8=16+8=24$, and then $24\div3=8$. The root gives $\sqrt{25}=5$, and the multiplication gives $2\times3=6$. The expression is now $8+5-6$, and working left to right, $8+5-6=7$.

$\therefore$ the value is 7.

Example 4 — unscaffolded

A tradesperson charges a $\$60$ call-out fee plus $\$45$ for each hour worked. On Monday she works 3 hours and on Tuesday she works 2 hours (two separate visits). Find her total charge, and show that grouping the calculation two different ways gives the same result.

Each visit has its own call-out fee, so the total is

$$(60+45\times3)+(60+45\times2)=(60+135)+(60+90)=195+150=345.$$

Grouping the fees and the hourly charges instead,

$$2\times60+45\times(3+2)=120+45\times5=120+225=345.$$

Both arrangements obey the order of operations and give the same total.

$\therefore$ the total charge is $\$345$.

Practice questions

Scaffolded

Q1. Evaluate each expression, stating the operation you do first. (a) $7+2\times5$ (b) $(7+2)\times5$ (c) $20-6\div2$

Q2. Evaluate. (a) 3^2+4^2 (b) 5×2^3 (c) $\sqrt{9}+\sqrt{16}$

Q3. Evaluate. (a) $18\div(3+6)$ (b) $18\div3+6$ (c) $40-(5+3)\times2$

Q4. Priya buys 6 pens at $\$3$ each and 2 notebooks at $\$7$ each. (a) Write a single expression for the total cost. (b) Evaluate it. (c) She also uses a $\$5$ voucher. Write and evaluate an expression for the amount she pays.

Q5. Consider $2+3^2\times2$ and $(2+3)^2\times2$. (a) Evaluate $2+3^2\times2$. (b) Evaluate $(2+3)^2\times2$. (c) Explain why the two answers are different.

Q6. Evaluate each expression (the fraction bar groups the top and the bottom). (a) $\dfrac{12+8}{4}$ (b) $\dfrac{6\times5}{2+1}$ (c) $\dfrac{20-2^2}{2^2}$

Unscaffolded

Q7. Evaluate $50-3\times(4+2^2)$.

Q8. Evaluate $\dfrac{9+3\times7}{2\times3}$.

Q9. Evaluate $100\div5^2+6\times(11-8)$.

Q10. Evaluate $\sqrt{36}+2\times(15-3^2)$.

Q11. Two students evaluate $12-8\div4$. Sam writes $12-8=4$, then $4\div4=1$. Aisha writes $8\div4=2$, then $12-2=10$. State who is correct, and explain the other student's error.

Q12. Insert one pair of brackets into $3+5\times2-1$ so that its value is 15.

Q13. A phone plan charges a $\$15$ monthly fee plus $\$0.20$ for each call. Write a single expression for the cost of a month with 12 calls, then evaluate it.

Q14. An electrician charges a $\$110$ call-out fee plus $\$70$ for each hour worked. Write a single expression for a job lasting 4 hours, then evaluate it.

Q15. Evaluate $\dfrac{(6+4)^2}{4}-3\times2$.

Q16. Insert one pair of brackets into $8+12\div2\times3$ so that its value is 30.

Q17. A pizza shop sells pizzas for $\$12$ and garlic breads for $\$5$. A family orders 3 pizzas and 2 garlic breads and splits the total equally between 5 people. Write a single expression (using brackets) for each person's share, then evaluate it.

Q18. A week's pay is meant to be $38\times25+4\times37$ (normal hours plus overtime). By mistake the payroll system computes $38\times(25+4)\times37$. (a) Evaluate the correct pay. (b) Evaluate the incorrect expression. (c) Explain, using the order of operations, why the bracket changes the result so much.

Answers

Q1. (a) 17. (b) 45. (c) 17. **Q2.** (a) 25. (b) 40. (c) 7. **Q3.** (a) 2. (b) 12. (c) 24. **Q4.** (a) $6 \times 3 + 2 \times 7$. (b) $\$32$. (c) $6 \times 3 + 2 \times 7 - 5 = \$27$. **Q5.** (a) 20. (b) 50. (c) see solution. **Q6.** (a) 5. (b) 10. (c) 4. **Q7.** 26. **Q8.** 5. **Q9.** 22. **Q10.** 18. **Q11.** Aisha is correct (10); see solution. **Q12.** $(3+5) \times 2 - 1$. **Q13.** $15 + 0.20 \times 12 = \$17.40$. **Q14.** $110 + 70 \times 4 = \$390$. **Q15.** 19. **Q16.** $(8+12) \div 2 \times 3$. **Q17.** $(3 \times 12 + 2 \times 5) \div 5 = \$9.20$. **Q18.** (a) $\$1098$. (b) $\$40\,774$. (c) see solution.

Fully-worked solutions

Q1. (a) Multiplication before addition: $7 + 2 \times 5 = 7 + 10 = 17$. (b) Brackets first: $(7+2) \times 5 = 9 \times 5 = 45$. (c) Division before subtraction: $20 - 6 \div 2 = 20 - 3 = 17$.

Q2. (a) Indices first: $3^2 + 4^2 = 9 + 16 = 25$. (b) Index before multiplication: $5 \times 2^3 = 5 \times 8 = 40$. (c) Roots first: $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

Q3. (a) Bracket first: $18 \div (3+6) = 18 \div 9 = 2$. (b) Division before addition: $18 \div 3 + 6 = 6 + 6 = 12$. (c) Bracket, then multiply, then subtract: $40 - (5+3) \times 2 = 40 - 8 \times 2 = 40 - 16 = 24$.

Q4. (a) The total cost is $6 \times 3 + 2 \times 7$. (b) Each product is found before adding: $6 \times 3 + 2 \times 7 = 18 + 14 = 32$, i.e. $\$32$. (c) Subtracting the voucher: $6 \times 3 + 2 \times 7 - 5 = 18 + 14 - 5 = 27$, i.e. $\$27$.

Q5. (a) The index applies only to the 3: $2 + 3^2 \times 2 = 2 + 9 \times 2 = 2 + 18 = 20$. (b) The bracket is squared: $(2+3)^2 \times 2 = 5^2 \times 2 = 25 \times 2 = 50$. (c) In (a) only the 3 is squared, but in (b) the bracket makes the sum $2 + 3 = 5$ be squared instead. Because the brackets change **what** is squared, the two expressions have different values.

Q6. The fraction bar groups the numerator and the denominator. (a) $\frac{12+8}{4} = \frac{20}{4} = 5$. (b) $\frac{6 \times 5}{2+1} = \frac{30}{3} = 10$. (c) $\frac{20 - 2^2}{2^2} = \frac{20-4}{4} = \frac{16}{4} = 4$.

Q7. Work inside the bracket first, indices before addition: $4 + 2^2 = 4 + 4 = 8$. Then $50 - 3 \times 8 = 50 - 24 = 26$.

Q8. Numerator: $9 + 3 \times 7 = 9 + 21 = 30$. Denominator: $2 \times 3 = 6$. So $\frac{30}{6} = 5$.

Q9. Index first: $5^2 = 25$, so $100 \div 25 = 4$. Bracket: $11 - 8 = 3$, so $6 \times 3 = 18$. Then $4 + 18 = 22$.

Q10. Root: $\sqrt{36} = 6$. Inside the bracket, index before subtraction: $15 - 3^2 = 15 - 9 = 6$, so $2 \times 6 = 12$. Then $6 + 12 = 18$.

Q11. The expression $12 - 8 \div 4$ has a division, which must be done **before** the subtraction: $8 \div 4 = 2$, then $12 - 2 = 10$. Aisha is correct. Sam worked strictly left to right, doing the subtraction first; this ignores that division has higher priority than subtraction, so his answer of 1 is wrong.

Q12. Placing the bracket around $3 + 5$ makes that sum be done first: $(3+5) \times 2 - 1 = 8 \times 2 - 1 = 16 - 1 = 15$, as required.

Q13. The cost is the fixed fee plus $\$0.20$ per call: $15 + 0.20 \times 12$. Multiplication first: $0.20 \times 12 = 2.40$, then $15 + 2.40 = 17.40$. The monthly cost is $\$17.40$. (No brackets are needed, because the calculator multiplies before it adds.)

Q14. The charge is the call-out fee plus $\$70$ per hour for 4 hours: $110 + 70 \times 4$. Multiplication first: $70 \times 4 = 280$, then $110 + 280 = 390$. The job costs $\$390$.

Q15. The fraction bar groups the top: $(6+4)^2 = 10^2 = 100$, so $\frac{100}{4} = 25$. Then $3 \times 2 = 6$, and $25 - 6 = 19$.

Q16. Placing the bracket around $8+12$ forces that sum first: $(8+12)\div 2\times 3=20\div 2\times 3$. Working left to right, $20\div 2=10$, then $10\times 3=30$, as required.

Q17. The total order is $3\times 12+2\times 5=36+10=46$ dollars. Splitting it between 5 people needs a bracket around the total: $(3\times 12+2\times 5)\div 5=46\div 5=9.20$. Each person pays $\$9.20$.

Q18. (a) Following the order of operations, each multiplication is done first: $38\times 25+4\times 37=950+148=1098$, i.e. $\$1098$. (b) The bracket forces $25+4=29$ to be found first, and everything is then multiplied: $38\times 29\times 37=1102\times 37=40\,774$, i.e. $\$40\,774$. (c) In the correct expression the + separates two products, so the normal pay and the overtime pay are added. The bracket turns that + into part of a single running multiplication, so instead of *adding* the overtime it *multiplies* everything together — producing a vastly larger, and meaningless, result.

Theory

Ratio, rate and proportion are three closely related ways of comparing quantities. They appear everywhere in daily life — mixing concrete, reading a recipe, comparing prices, working out a travel time — so being fluent with them is one of the most useful skills in this course.

Ratios.

A **ratio** compares two or more quantities **measured in the same unit**. We write it with a colon, such as $3:2$, read "three to two". Order matters: $3:2$ is not the same as $2:3$.

To **simplify** a ratio, divide every part by their highest common factor, just like simplifying a fraction:

$$12:18=(12\div 6):(18\div 6)=2:3.$$

If the quantities are in **different units**, convert them to the same unit first. For example 45 minutes compared with 1 hour is

$$45\,\text{min}:60\,\text{min}=3:4.$$

Two ratios that simplify to the same thing are **equivalent ratios** ($2:3=4:6=20:30$). A ratio can have **three or more parts**, such as a concrete mix of cement, sand and gravel written $1:2:3$.

Dividing a quantity in a given ratio.

To share a quantity in a ratio, add the parts to get the **total number of parts**, find the **value of one part**, then multiply. To share $\$360$ in the ratio $4:5$:

- total parts $=4+5=9$;
- one part $=\$360\div 9=\$40$;
- the shares are $4\times\$40=\$160$ and $5\times\$40=\$200$ (which add back to $\$360$).

Rates.

A **rate** compares two quantities measured in **different units**, such as km per hour, dollars per kilogram, or litres per 100 km. A **unit rate** gives the amount for **one** unit of the second quantity — this is what makes rates easy to compare. Common rates include

$$\text{speed}=\frac{\text{distance}}{\text{time}},\ \text{unit price}=\frac{\text{cost}}{\text{quantity}},\ \text{fuel use}=\frac{\text{litres}}{100\,\text{km}}.$$

To find a **best buy**, reduce each option to the same unit rate (for example, the price per 100 mL) and choose the smallest.

Direct proportion.

Two quantities are in **direct proportion** when they increase or decrease in the same ratio: double one and the other doubles. The quickest tool is the **unitary method** — find the value of **one**, then scale up or down. If 8 m of fabric costs $\$52$, then one metre costs $\$52\div 8=\$6.50$, so 15 m costs $15\times\$6.50=\$97.50$.

Connecting ratios, fractions and percentages.

A two-part ratio can be read as fractions of the whole. If girls and boys are in the ratio $3:2$, the whole is $3+2=5$ parts, so girls are $3/5$ of the class and boys are $2/5=40\,\%$. This lets you move freely between a ratio, a fraction and a percentage.

Worked examples

Example 1 — scaffolded

A concrete mix uses cement, sand and gravel in the ratio $25\,\text{kg} : 50\,\text{kg} : 75\,\text{kg}$. Simplify the ratio, then find how much sand and gravel are needed with 40 kg of cement, and the total mass of the mix.

Working	Comment
Divide each part by the HCF, 25: $25 : 50 : 75 = 1 : 2 : 3$.	Simplify like a fraction, using every part.
With 40 kg of cement, one part $= 40$ kg.	Cement is 1 part, so a part is 40 kg.
Sand $= 2 \times 40 = 80$ kg; gravel $= 3 \times 40 = 120$ kg.	Multiply each part by the value of one part.
Total $= 40 + 80 + 120 = 240$ kg.	Add the three quantities.

Example 2 — scaffolded

A supermarket sells the same rice in two packs: a 2 kg pack for $\$7.60$ and a 5 kg pack for $\$18.50$. Which is the better buy?

Working	Comment
Small pack: $\$7.60 \div 2 = \$3.80$ per kg.	Unit price = cost ÷ quantity.
Large pack: $\$18.50 \div 5 = \$3.70$ per kg.	Same unit for a fair comparison.
$\$3.70 < \$3.80$, so the 5 kg pack is cheaper per kilogram.	Best buy = smallest unit price.
∴ the 5 kg pack is the better buy.	State the decision.

Example 3 — unscaffolded

A cyclist rides 63 km in 2 hours 15 minutes at a steady pace. Find the average speed, the distance covered in 3.5 hours at that speed, and the time taken to ride 42 km.

First write the time as a decimal: 2 h 15 min $= 2.25$ h. Then

$$\text{speed} = \frac{63}{2.25} = 28\,\text{km/h}.$$

At 28 km/h the distance in 3.5 h is $28 \times 3.5 = 98$ km, and the time to ride 42 km is

$$\text{time} = \frac{42}{28} = 1.5\,\text{h} = 1\,\text{h}\,30\,\text{min}.$$

∴ the average speed is 28 km/h, the cyclist covers 98 km in 3.5 h, and 42 km takes 1.5 hours.

Example 4 — unscaffolded

Six identical solar panels together produce 2.7 kW in full sun. Assuming the panels share the load equally, find the power of one panel, the power of 14 panels, and the number of panels needed to produce 9 kW.

This is a direct-proportion problem, so use the unitary method. One panel produces

$$\frac{2.7}{6} = 0.45\,\text{kW}.$$

Then 14 panels produce $14 \times 0.45 = 6.3$ kW, and to reach 9 kW you need

$$\frac{9}{0.45} = 20\,\text{panels}.$$

∴ one panel produces 0.45 kW, 14 panels produce 6.3 kW, and 20 panels are needed for 9 kW.

Practice questions

Scaffolded

Q1. Simplify each ratio to its lowest terms. (a) $15:25$. (b) 40 minutes : 2 hours. (c) 500 mL : 750 mL.

Q2. A prize of $\$420$ is shared between two people in the ratio $3:4$. (a) Find the total number of parts. (b) Find the value of one part. (c) Find the two shares.

Q3. A car travels 600 km using 48 L of petrol. (a) Find the fuel efficiency in kilometres per litre. (b) Find the fuel use in litres per 100 km. (c) If petrol costs $\$1.85$ per litre, find the cost of the fuel used.

Q4. Olive oil is sold in a 500 mL bottle for $\$6.50$ and a 750 mL bottle for $\$9.30$. (a) Find the price per 100 mL of the 500 mL bottle. (b) Find the price per 100 mL of the 750 mL bottle. (c) State which bottle is the better buy.

Q5. A roll of fabric costs $\$90$ for 6 metres. (a) Find the cost of 1 metre. (b) Find the cost of 12 metres. (c) Find the length that can be bought for $\$195$.

Q6. In a class the ratio of girls to boys is $7:3$. (a) What fraction of the class are girls? (b) What percentage of the class are boys? (c) If there are 30 students, how many are girls?

Unscaffolded

Q7. A paint colour mixes blue and white in the ratio $2:5$. A batch uses 8 L of blue. How much white is needed, and what is the total volume of the batch?

Q8. A grandparent shares $\$4500$ between three grandchildren in the ratio $2:3:4$. Find each grandchild's share, and check that the shares add to $\$4500$.

Q9. A train travels 315 km in 3.5 hours at a steady speed. (a) Find the average speed. (b) Find the distance it would cover in 5 hours at that speed. (c) Find the time it would take to travel 210 km.

Q10. A breakfast cereal is sold in three sizes: 375 g for $\$4.50$, 500 g for $\$5.80$, and 750 g for $\$8.40$. By finding the price per 100 g, decide which size is the best buy.

Q11. A casual worker is paid $\$146.25$ for a 9-hour shift. (a) Find the hourly rate of pay. (b) Find the pay for a 34-hour week at this rate. (c) Find the number of hours needed to earn $\$650$.

Q12. Car P uses fuel at 7.5 L per 100 km and Car Q at 6.4 L per 100 km. For an 850 km trip, find the fuel used by each car, and the extra cost of driving Car P if petrol is $\$1.90$ per litre.

Q13. A pancake recipe for 6 people uses 450 g of flour, 300 mL of milk and 4 eggs. Rewrite the recipe for 15 people.

Q14. A map is drawn to a scale of 1 cm : 25 km. (a) Two towns are 7.2 cm apart on the map. Find the actual distance between them. (b) Find the map distance for two cities that are 300 km apart. (c) Find the map distance for a road that is 45 km long.

Q15. A fruit punch mixes orange juice and pineapple juice in the ratio $5:3$. (a) How much pineapple juice should be mixed with 750 mL of orange juice? (b) If only 240 mL of pineapple juice is available, how much orange juice should be used to keep the same ratio?

Q16. A tap fills a 240 L tank in 16 minutes at a steady rate. (a) Find the flow rate in litres per minute. (b) Find the time to fill a 600 L tank at the same rate. (c) Find the volume of water delivered in 25 minutes.

Q17. A household budgets its monthly income in the ratio rent : food : savings : other $= 6:3:2:4$. The monthly income is $\$4500$. (a) Find the amount budgeted for each category. (b) Find the percentage of income that is saved, correct to one decimal place.

Q18. In a factory, 4 machines produce 1500 items in 5 hours. Each machine works at the same steady rate. (a) Find the number of items one machine produces in one hour. (b) Find the number of items 6 machines produce in 8 hours. (c) Find the number of machines needed to produce 2250 items in 3 hours.

Answers

Q1. (a) $3:5$. (b) $1:3$. (c) $2:3$. **Q2.** (a) 7 parts. (b) $\$60$. (c) $\$180$ and $\$240$. **Q3.** (a) 12.5 km/L. (b) 8 L/100 km. (c) $\$88.80$. **Q4.** (a) $\$1.30$ per 100 mL. (b) $\$1.24$ per 100 mL. (c) the 750 mL bottle. **Q5.** (a) $\$15$. (b) $\$180$. (c) 13 m. **Q6.** (a) $7/10$. (b) 30 %. (c) 21 girls. **Q7.** white $=20$ L; total $=28$ L. **Q8.** $\$1000$, $\$1500$, $\$2000$ (sum $\$4500$). **Q9.** (a) 90 km/h. (b) 450 km. (c) 2 h 20 min. **Q10.** 375 g $=\$1.20$/100 g; 500 g $=\$1.16$/100 g; 750 g $=\$1.12$/100 g — the 750 g pack is the best buy. **Q11.** (a) $\$16.25$/h. (b) $\$552.50$. (c) 40 hours. **Q12.** Car P 63.75 L; Car Q 54.4 L; extra cost of Car P $\approx \$17.77$. **Q13.** 1125 g flour, 750 mL milk, 10 eggs. **Q14.** (a) 180 km. (b) 12 cm. (c) 1.8 cm. **Q15.** (a) 450 mL. (b) 400 mL. **Q16.** (a) 15 L/min. (b) 40 min. (c) 375 L. **Q17.** (a) rent $\$1800$, food $\$900$, savings $\$600$, other $\$1200$. (b) 13.3 %. **Q18.** (a) 75 items. (b) 3600 items. (c) 10 machines.

Fully-worked solutions

Q1. (a) The HCF of 15 and 25 is 5: $15:25=3:5$. (b) Convert to the same unit: 2 hours $=120$ min, so $40:120=1:3$ (dividing by 40). (c) $500:750=2:3$ (dividing by 250).

Q2. (a) Total parts $=3+4=7$. (b) One part $=\$420\div 7=\$60$. (c) The shares are $3\times\$60=\$180$ and $4\times\$60=\$240$; check $180+240=420$. ✓

Q3. (a) $600\div 48=12.5$ km/L. (b) $\frac{48}{600}\times 100=8$ L per 100 km. (c) Fuel cost $=48\times\$1.85=\$88.80$.

Q4. (a) $\$6.50\div 5=\$1.30$ per 100 mL (since 500 mL is 5 lots of 100 mL). (b) $\$9.30\div 7.5=\$1.24$ per 100 mL. (c) $\$1.24<\$1.30$, so the 750 mL bottle is the better buy.

Q5. (a) $\$90\div 6=\$15$ per metre. (b) $12\times\$15=\$180$. (c) $\$195\div\$15=13$ m.

Q6. The whole is $7+3=10$ parts. (a) Girls are $7/10$ of the class. (b) Boys are $3/10=3/10\times 100\%=30\%$. (c) Girls $=7/10\times 30=21$.

Q7. The blue is 2 parts $=8$ L, so one part $=4$ L. White is 5 parts $=5\times 4=20$ L. Total volume $=8+20=28$ L.

Q8. Total parts $=2+3+4=9$; one part $=\$4500\div 9=\$500$. The shares are $2\times\$500=\$1000$, $3\times\$500=\$1500$ and $4\times\$500=\$2000$. Check: $1000+1500+2000=4500$. ✓

Q9. (a) Speed $=\frac{315}{3.5}=90$ km/h. (b) Distance $=90\times 5=450$ km. (c) Time $=\frac{210}{90}=2\,1/3$ h $=2$ h 20 min (since $1/3$ h $=20$ min).

Q10. Price per 100 g: 375 g pack $\frac{\$4.50}{3.75}=\$1.20$; 500 g pack $\frac{\$5.80}{5}=\$1.16$; 750 g pack $\frac{\$8.40}{7.5}=\$1.12$. The lowest unit price is $\$1.12$, so the 750 g pack is the best buy.

Q11. (a) Hourly rate $=\frac{\$146.25}{9}=\$16.25$. (b) Weekly pay $=34\times\$16.25=\$552.50$. (c) Hours $=\frac{\$650}{\$16.25}=40$ hours.

Q12. Car P uses $7.5\times\frac{850}{100}=7.5\times 8.5=63.75$ L; Car Q uses $6.4\times 8.5=54.4$ L. Car P uses $63.75-54.4=9.35$ L more, costing $9.35\times\$1.90=\$17.765\approx\$17.77$ extra.

Q13. The scale factor is $\frac{15}{6}=2.5$. Flour $=450\times 2.5=1125$ g; milk $=300\times 2.5=750$ mL; eggs $=4\times 2.5=10$.

Q14. Each 1 cm represents 25 km. (a) $7.2\times 25=180$ km. (b) $300\div 25=12$ cm. (c) $45\div 25=1.8$ cm.

Q15. (a) The orange is 5 parts $=750$ mL, so one part $=150$ mL; pineapple is 3 parts $=3\times 150=450$ mL. (b) The pineapple is 3 parts $=240$ mL, so one part $=80$ mL; orange is 5 parts $=5\times 80=400$ mL.

Q16. (a) Flow rate $=\frac{240}{16}=15$ L/min. (b) Time $=\frac{600}{15}=40$ min. (c) Volume $=15\times 25=375$ L.

Q17. Total parts $=6+3+2+4=15$; one part $=\frac{\$4500}{15}=\$300$. (a) Rent $=6\times\$300=\$1800$; food $=3\times\$300=\$900$; savings $=2\times\$300=\$600$; other $=4\times\$300=\$1200$. (b) Savings are $\frac{2}{15}=0.1333\ldots=13.3\%$ (to 1 d.p.).

Q18. (a) The 4 machines make 1500 items in 5 hours, so one machine in one hour makes $\frac{1500}{4\times 5}=75$ items. (b) 6 machines in 8 hours make $75\times 6\times 8=3600$ items. (c) In 3 hours one machine makes $75\times 3=225$ items, so the number of machines needed for 2250 items is $\frac{2250}{225}=10$.

Theory

Many everyday quantities are built from a number multiplied by itself: the **area** of a square, the **volume** of a cube, a population that **doubles**, or the number of bytes on a hard drive. **Powers** and **roots** are the shorthand for this kind of repeated multiplication.

Index (power) notation.

Writing b^n is a short way of multiplying the **base** b by itself n times:

$$b^n = b \times b \times \cdots \times b \left(n \, \text{factors} \right).$$

The small raised number n is the **index** (also called the **power** or **exponent**). For example $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. We read 2^5 as “two to the power of five”.

Two powers are so common they have their own names:

- **squared** means to the power of 2 — $7^2 = 7 \times 7 = 49$ (“seven squared”);
- **cubed** means to the power of 3 — $4^3 = 4 \times 4 \times 4 = 64$ (“four cubed”).

Squares and cubes in measurement.

A square with side length s has **area** s^2, which is why area is measured in **square** units. A cube with edge length s has **volume** s^3, measured in **cubic** units. So squaring turns a length into an area, and cubing turns a length into a volume.

Powers of 10.

Powers of 10 connect directly to place value: 10^n is 1 followed by n zeros.

Power	10^1	10^2	10^3	10^6	10^9
Value	10	100	1000	1 000 000	1 000 000 00(
Name	ten	hundred	thousand	million	billion

Powers of the same base can be combined by **adding** the indices — for instance $10^2 \times 10^3 = 10^5$, because $100 \times 1000 = 100\,000$. This is handy when comparing very large or very small quantities.

Square roots and cube roots.

A **root** reverses a power. The **square root** of a number is the value that, when squared, gives that number: since $5^2 = 25$, the square root of 25 is 5, written $\sqrt{25} = 5$. The **cube root** reverses cubing: since $4^3 = 64$, the cube root of 64 is 4, written $\sqrt[3]{64} = 4$.

Roots undo the measurement powers above: a square root finds the **side of a square** from its area, and a cube root finds the **edge of a cube** from its volume.

Perfect squares and estimating.

The **perfect squares** are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$ Their square roots are whole numbers. Most numbers are not perfect squares, so their roots are decimals that we **estimate** by trapping them between two perfect squares. For example $\sqrt{50}$ lies between $\sqrt{49} = 7$ and $\sqrt{64} = 8$, and because 50 is very close to 49 the root is a little above 7 (a calculator gives 7.07).

Using a scientific calculator.

For larger powers and for roots, use your calculator:

- the **power key** (marked x^y, y^x or $\wedge$) raises any base to any index;
- the **square key** (x^2) squares a number, and the **square-root key** finds a square root;
- the **cube-root key** (or the "x-th root" key set to 3) finds a cube root.

Always check the result is reasonable — for instance a square root must be smaller than the number itself (for numbers above 1), and a side length must be positive.

Worked examples

Example 1 — scaffolded

A square courtyard has sides of 8 m and a cube-shaped water tank has edges of 3 m. (a) Evaluate 8^2 to find the area of the courtyard. (b) Evaluate 3^3 to find the volume of the tank. (c) Evaluate 2^5.

Working	Comment
(a) $8^2 = 8 \times 8 = 64$, so the area is 64 m².	Squaring a side gives an **area** (square units).
(b) $3^3 = 3 \times 3 \times 3 = 27$, so the volume is 27 m³.	Cubing an edge gives a **volume** (cubic units).
(c) $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.	Multiply the base 2 by itself five times, or use the power key.

Example 2 — scaffolded

(a) A square vegetable patch has an area of 196 m². Find its side length.
(b) A cube-shaped box has a volume of 64 cm³. Find the length of one edge.
(c) Evaluate $\sqrt{900}$ and $\sqrt[3]{1000}$.

Working	Comment
(a) Side = $\sqrt{196} = 14$ m, because $14^2 = 196$.	A square root finds the side of a square from its area.
(b) Edge = $\sqrt[3]{64} = 4$ cm, because $4^3 = 64$.	A cube root finds the edge of a cube from its volume.
(c) $\sqrt{900} = 30$ (since $30^2 = 900$) and $\sqrt[3]{1000} = 10$ (since $10^3 = 1000$).	Use the square-root and cube-root keys, then check by squaring/cubing.

Example 3 — unscaffolded

In a laboratory a single cell divides into two every hour. Write the number of cells after n hours as a power of 2, find the number of cells after 8 hours, and state how many times as many cells there are after 8 hours compared with after 4 hours.

Starting from 1 cell, the count doubles each hour, so after n hours there are 2^n cells. After 8 hours there are

$$2^8 = 256\,\text{cells},$$

and after 4 hours there are $2^4 = 16$ cells. Comparing the two,

$$\frac{2^8}{2^4} = \frac{256}{16} = 16 = 2^4.$$

∴ there are 256 cells after 8 hours, which is 16 times as many as after 4 hours.

Example 4 — unscaffolded

A square sports field is to be built with an area of 500 m². Between which two whole numbers does its side length lie, and what is the side length correct to one decimal place?

The side length is $\sqrt{500}$. The nearest perfect squares are $22^2 = 484$ and $23^2 = 529$, and since

$$484 < 500 < 529,$$

the side length lies between 22 m and 23 m. Using the square-root key, $\sqrt{500} = 22.36\ldots$, which rounds to 22.4 m.

$\therefore$ the side is between 22 m and 23 m, and equals 22.4 m to one decimal place.

Practice questions

Scaffolded

Q1. Evaluate. (a) 6^2 (b) 4^3 (c) 2^6

Q2. Powers of ten. (a) Evaluate 10^2. (b) Evaluate 10^5. (c) Write $1\,000\,000$ as a power of 10.

Q3. Find each square root. (a) $\sqrt{64}$ (b) $\sqrt{121}$ (c) $\sqrt{400}$

Q4. Find each cube root. (a) $\sqrt[3]{8}$ (b) $\sqrt[3]{125}$ (c) $\sqrt[3]{1000}$

Q5. Measurement. (a) A square tile has side 15 cm; find its area. (b) A cubic box has edge 5 cm; find its volume. (c) A square lawn has an area of 100 m²; find its side length.

Q6. Estimating roots. (a) Between which two consecutive whole numbers does $\sqrt{30}$ lie? (b) Use a calculator to write $\sqrt{30}$ correct to one decimal place. (c) Between which two consecutive whole numbers does $\sqrt[3]{50}$ lie?

Unscaffolded

Q7. Evaluate 3^4, 5^3 and 2^{10}.

Q8. Write each product in index form. (a) $7 \times 7 \times 7$ (b) $2 \times 2 \times 2 \times 2 \times 2$ (c) $10 \times 10 \times 10 \times 10$

Q9. A square swimming-pool cover has an area of 156.25 m². Find the side length of the cover.

Q10. A cube-shaped storage container has a volume of $27\,000$ cm³. Find the length of one edge.

Q11. A water lily doubles the area it covers each day. On day 0 it covers 1 m². Write the area covered after d days as a power of 2, and find the area covered on day 7.

Q12. A sheet of paper 0.1 mm thick is folded in half repeatedly, doubling its thickness at each fold. (a) Write the thickness after n folds. (b) Find the thickness after 7 folds. (c) Find the thickness after 10 folds.

Q13. Estimate $\sqrt{200}$ to the nearest whole number, then use a calculator to give it correct to two decimal places.

Q14. A hard drive stores about 10^{12} bytes and one photo is about 10^6 bytes. Using powers of 10, estimate how many photos the drive can hold.

Q15. (a) Evaluate 12^2. (b) Hence write down $\sqrt{144}$. (c) Explain the relationship between squaring a number and taking its square root.

Q16. A square garden bed has an area of 40 m². A landscaper needs its side length to order edging. (a) Between which two whole numbers does the side length lie? (b) Find the side length correct to two decimal places. (c) Find the total length of edging needed (the perimeter), to the nearest metre.

Q17. Compare 2^5 and 5^2. Which is larger, and by how much?

Q18. A cube has a volume of 343 cm³. (a) Find the edge length. (b) Find the area of one face. (c) Find the total surface area of all six faces.

Answers

Q1. (a) 36. (b) 64. (c) 64. **Q2.** (a) 100. (b) $100\,000$. (c) 10^6. **Q3.** (a) 8. (b) 11. (c) 20. **Q4.** (a) 2. (b) 5. (c) 10. **Q5.** (a) 225 cm². (b) 125 cm³. (c) 10 m. **Q6.** (a) 5 and 6. (b) 5.5. (c) 3 and 4. **Q7.** 81, 125, 1024. **Q8.** (a) 7^3. (b) 2^5. (c) 10^4. **Q9.** 12.5 m. **Q10.** 30 cm. **Q11.** area $= 2^d$; on day 7 it is $2^7 = 128$ m². **Q12.** (a) 0.1×2^n mm. (b) 12.8 mm. (c) 102.4 mm. **Q13.** nearest whole number 14; $\sqrt{200} = 14.14$. **Q14.** $10^6 = 1\,000\,000$ photos. **Q15.** (a) 144. (b) 12. (c) they are inverse (opposite) operations. **Q16.** (a) 6 and 7. (b) 6.32 m. (c) 25 m. **Q17.** $2^5 = 32$ is larger than $5^2 = 25$, by 7. **Q18.** (a) 7 cm. (b) 49 cm². (c) 294 cm².

Fully-worked solutions

Q1. (a) $6^2 = 6 \times 6 = 36$. (b) $4^3 = 4 \times 4 \times 4 = 64$. (c) $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

Q2. (a) $10^2 = 100$. (b) $10^5 = 100\,000$ (1 followed by five zeros). (c) $1\,000\,000$ is 1 followed by six zeros, so $1\,000\,000 = 10^6$.

Q3. (a) $\sqrt{64} = 8$ since $8^2 = 64$. (b) $\sqrt{121} = 11$ since $11^2 = 121$. (c) $\sqrt{400} = 20$ since $20^2 = 400$.

Q4. (a) $\sqrt[3]{8} = 2$ since $2^3 = 8$. (b) $\sqrt[3]{125} = 5$ since $5^3 = 125$. (c) $\sqrt[3]{1000} = 10$ since $10^3 = 1000$.

Q5. (a) Area of a square $= s^2 = 15^2 = 225$ cm². (b) Volume of a cube $= s^3 = 5^3 = 125$ cm³. (c) Side $= \sqrt{100} = 10$ m.

Q6. (a) $5^2 = 25$ and $6^2 = 36$, and $25 < 30 < 36$, so $\sqrt{30}$ lies between 5 and 6. (b) The square-root key gives $\sqrt{30} = 5.477\ldots \approx 5.5$. (c) $3^3 = 27$ and $4^3 = 64$, and $27 < 50 < 64$, so $\sqrt[3]{50}$ lies between 3 and 4.

Q7. $3^4 = 3 \times 3 \times 3 \times 3 = 81$; $5^3 = 5 \times 5 \times 5 = 125$; $2^{10} = 1024$ (using the power key).

Q8. (a) Three factors of 7: 7^3. (b) Five factors of 2: 2^5. (c) Four factors of 10: 10^4.

Q9. The cover is square, so its side is $\sqrt{156.25}$. Using the square-root key, $\sqrt{156.25} = 12.5$ m (check: $12.5^2 = 156.25$).

Q10. The container is a cube, so one edge is $\sqrt[3]{27\,000}$. Since $30^3 = 27\,000$, the edge is 30 cm.

Q11. Doubling each day from 1 m² gives 2^d m² after d days. On day 7 the area is $2^7 = 128$ m².

Q12. (a) Each fold doubles the thickness, so after n folds it is 0.1×2^n mm. (b) After 7 folds: $0.1 \times 2^7 = 0.1 \times 128 = 12.8$ mm. (c) After 10 folds: $0.1 \times 2^{10} = 0.1 \times 1024 = 102.4$ mm.

Q13. $14^2 = 196$ and $15^2 = 225$, and 200 is close to 196, so $\sqrt{200}$ is just above 14; to the nearest whole number it is 14. The square-root key gives $\sqrt{200} = 14.14$ (to two decimal places).

Q14. The number of photos is about

$$\frac{10^{12}}{10^6} = 10^{12-6} = 10^6 = 1\,000\,000,$$

so the drive holds roughly one million photos.

Q15. (a) $12^2 = 144$. (b) Since $12^2 = 144$, $\sqrt{144} = 12$. (c) Squaring and square-rooting are **inverse operations**: squaring a number and then taking the square root returns the original number (for positive numbers), and vice versa.

Q16. (a) $6^2 = 36$ and $7^2 = 49$, and $36 < 40 < 49$, so the side lies between 6 m and 7 m. (b) $\sqrt{40} = 6.32$ m (to two decimal places). (c) The perimeter is $4 \times 6.32 = 25.3$ m, which is 25 m to the nearest metre.

Q17. $2^5 = 32$ and $5^2 = 25$. Since $32 > 25$, 2^5 is larger, by $32 - 25 = 7$.

Q18. (a) The edge is $\sqrt[3]{343}=7$ cm since $7^3=343$. (b) One face is a square of side 7 cm, so its area is $7^2=49$ cm². (c) A cube has six faces, so the total surface area is $6\times 49=294$ cm².